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Vector-valued multilinear singular integrals with nonsmooth kernels and commutators on generalized weighted Morrey space

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Abstract

In this paper, we prove weighted norm inequalities for vector-valued multilinear singular integrals with nonsmooth kernels and commutators on generalized weighted Morrey space.

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1 Introduction

Let T be a multilinear operator defined on the m-fold product of Schwarz spaces and taking values in the space of tempered distributions,

$$T:\mathcal{S}(R^n)\times\cdots\times\mathcal{S}(R^n)\to\mathcal{S}'(R^n).$$

In [1] the multilinear operator T satisfying the following conditions was studied:

(1) There exists a function K defined off the diagonal $x=y_1=\cdots=y_m$ in $(R^n)^{m+1}$ such that

$$T(f_1,\dots,f_m)(x)=\int_{(R^n)^m}K(x,y_1,\dots,y_m)f_1(y_1)\cdots f_m(y_m)\,dy_1\cdots dy_m \tag{1.1}$$

for all $x\notin\bigcap_{j=1}^m \operatorname{supp} f_j$.

(2) There exists $C>0$ such that

$$\bigl|K(y_0,y_1,\dots,y_m)\bigr|\le\frac{C}{(\sum_{k,l=0}^m|y_k-y_l|)^{mn}}. \tag{1.2}$$

(3) For some $\epsilon>0$, there exists $C>0$ such that

$$\bigl|K(y_0,y_1,\dots,y_j,\dots,y_m)-K\bigl(y_0,y_1,\dots,y_j',\dots,y_m\bigr)\bigr|\le\frac{C|y_j-y_j'|^\epsilon}{(\sum_{k,l=0}^m|y_k-y_l|)^{mn+\epsilon}}, \tag{1.3}$$

provided that $0\le j\le m$ and $|y_j-y_j'|\le\frac{1}{2}\max_{0\le k\le m}|y_j-y_k|$.

(4) There exist $1 \leq q_1, \dots, q_m < \infty$ such that

$$T : L^{q_1} \times \cdots \times L^{q_m} \to L^q$$

is bounded, where $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$.

In [1] it is proved that

$$T : L^{q_1} \times \cdots \times L^{q_m} \to L^q,$$

where $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ and $1 < q_j < \infty$ for all $j = 1, \dots, m$, and

$$T : L^{q_1} \times \cdots \times L^{q_m} \to L^{q,\infty},$$

where $1 \leq q_1, \dots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. In particular,

$$T : L^1 \times \cdots \times L^1 \to L^{\frac{1}{m},\infty}.$$

Let $\overrightarrow{b} = (b_1, \dots, b_m) \in (\mathrm{BMO})^m$ be a locally integrable vector function. The commutator of $\overrightarrow{b}$ and m-linear Calderón–Zygmund operator T, denoted $T_{\sum \overrightarrow{b}}$, was introduced by Pérez and Torres [2] and defined as

$$T_{\sum \overrightarrow{b}}(f_1, \dots, f_m) = \sum_{j=1}^{m} T_{b_j}^{j}(f_1, \dots, f_m),$$

where

$$T_{b_j}^{j}(\overrightarrow{f}) = b_j T(f_1, \dots, f_j, \dots, f_m) - T(f_1, \dots, b_j f_j, \dots, f_m).$$

The iterated commutator $T_{\prod \overrightarrow{b}}$ is defined by

$$T_{\prod \overrightarrow{b}}(f_1, \dots, f_m) = \bigl[b_1, \dots, \bigl[b_{m-1}, [b_m, T]_m\bigr]_{m-1} \cdots\bigr]_1.$$

To clarify the notation, if T is associated in the usual way with a kernel K satisfying (1.1)–(1.3), then, formally,

$$T_{\sum \overrightarrow{b}}(\overrightarrow{f})(x) = \int_{(R^n)^m} \sum_{j=1}^{m} \bigl(b_j(x) - b_j(y_j)\bigr) K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m$$

and

$$T_{\prod \overrightarrow{b}}(\overrightarrow{f})(x) = \int_{(R^n)^m} \prod_{j=1}^{m} \bigl(b_j - b_j(y_j)\bigr) K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m.$$

The theory of multiple weight associated with m-linear Calderón–Zygmund operators was developed by Lerner et al. [3]. Let $1 < p_j < \infty$ for $j = 1, \dots, m$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and

$\vec{p} = (p_1, \dots, p_m)$. They showed that if $\vec{w} \in A_{\vec{p}}$ (see the definition in the next section), then

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

If $1 \le p_j < \infty$ for $j = 1, \dots, m$ and at least one $p_j = 1$, then they also proved that

$$\|T(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Let $1 < p_j < \infty$, $j = 1, \dots, m$, and $1 < p < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, Pérez and Torres proved that if $\vec{b} \in (\mathrm{BMO})^m$, then

$$\|T_{\sum \vec{b}}(\vec{f})\|_{L^p} \le C \sum_{j=1}^m \|b_j\|_{\mathrm{BMO}} \prod_{j=1}^m \|f_j\|_{L^{p_j}}.$$

In [3] the weighted L^p-version of bounds is also obtained: for all $\vec{w} \in A_{\vec{p}}$,

$$\|T_{\sum \vec{b}}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le C \sum_{j=1}^m \|b_j\|_{\mathrm{BMO}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

As for $T_{\sum \vec{b}}$, a strong-type bound for $T_{\prod \vec{b}}$ was also established by Pérez et al. [4].

The vector-valued multilinear operator T_γ associated with the operator T was first studied by Grafakos and Martell [5]. For $\gamma > 0$, the vector-valued multilinear operator T_γ is defined by

$$\begin{aligned} T_\gamma(\vec{f})(x) &= \bigl|T(f_1, \dots, f_m)(x)\bigr|_\gamma \\ &= \left(\sum_{k=1}^\infty \bigl|T(f_{1k}, \dots, f_{mk})(x)\bigr|^\gamma\right)^{1/\gamma}, \end{aligned}$$

where $f_i = \{f_{ik}\}_{k=1}^\infty$ for $i = 1, \dots, m$. Let $\frac{1}{m} < p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_i}$ with $1 < p_1, \dots, p_m < \infty$, $\frac{1}{m} < \gamma < \infty$, and $\frac{1}{\gamma} = \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_i}$ with $1 < \gamma_1, \dots, \gamma_m < \infty$. Grafakos and Martell proved that

$$\|T_\gamma(\vec{f})\|_{L^p(R^n)} \le C \prod_{j=1}^m \bigl\||f_j|_{\gamma_j}\bigr\|_{L^{p_j}(R^n)}. \tag{1.4}$$

Later, Cruz-Uribe et al. [6] proved that if $\frac{1}{m} \le p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_i}$ with $1 < p_1, \dots, p_m < \infty$, $\frac{1}{m} < \gamma < \infty$, and $\frac{1}{\gamma} = \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_i}$ with $1 < \gamma_1, \dots, \gamma_m < \infty$, then

$$\|T_\gamma(\vec{f})\|_{L^{p,\infty}(R^n)} \le C \prod_{j=1}^m \bigl\||f_j|_{\gamma_j}\bigr\|_{L^{p_j}(R^n)}. \tag{1.5}$$

They also obtained the weighted L^p-versions of (1.4) and (1.5), but their results are not the multiple weighted estimates obtained by Lerner et al. [3].

For the sequence $\{\vec{f_k}\}_{k=1}^{\infty} = (f_{1k},\ldots,f_{mk})_{k=1}^{\infty}$, the vector-valued versions of the commutators $T_{\sum\vec{b},\gamma}$ and $T_{\prod\vec{b},\gamma}$ are defined by

$$T_{\sum\vec{b},\gamma}(\vec{f})(x) = \left|T_{\sum\vec{b},\gamma}(\vec{f})\right|_{\gamma}(x) = \left(\sum_{k=1}^{\infty}\left|T_{\sum\vec{b}}(f_{1k},\ldots,f_{mk})(x)\right|^{\gamma}\right)^{1/\gamma}$$

and

$$T_{\prod\vec{b},\gamma}(\vec{f})(x) = \left|T_{\prod\vec{b},\gamma}(\vec{f})\right|_{\gamma}(x) = \left(\sum_{k=1}^{\infty}\left|T_{\prod\vec{b}}(f_{1k},\ldots,f_{mk})(x)\right|^{\gamma}\right)^{1/\gamma}.$$

In 2008, Tang [7] established weighted norm inequalities for the commutators of a vector-valued multilinear operator, but his results are not the multiple weighted estimates obtained by Lerner et al. [3].

In this paper, we consider T associated with the kernel satisfying a weaker regularity conditions introduced in [8, 9]. Let $\{A_t\}_{t>0}$ be a class of integral operators that play the role of an approximation of the identity. We always assume that the operators A_t are associated with kernels $a_t(x,y)$ in the sense that, for all $f \in \bigcup_{p\in[1,\infty]} L^p$ and $x \in R^n$,

$$A_t f(x) = \int_{R^n} a_t(x,y)f(y)\,dy$$

and that the kernels $a_t(x,y)$ satisfy the condition

$$\left|a_t(x,y)\right| \le h_t(x,y) := t^{-n/s}h\left(\frac{|x-y|}{t^{1/s}}\right), \tag{1.6}$$

where s is a positive fixed constant, and h is a positive bounded decreasing function such that for some $\eta > 0$,

$$\lim_{r\to\infty} r^{n+\eta}h\left(r^s\right) = 0. \tag{1.7}$$

Recall that the jth transpose $T^{*,j}$ of the m-linear operator T is defined as

$$\left\langle T^{*,j}(f_1,\ldots,f_m),g\right\rangle = \left\langle T(f_1,\ldots,f_{j-1},g,f_{j+1},\ldots,f_m),f_j\right\rangle$$

for all $f_1,\ldots,f_m,g$ in $\mathcal{S}(R^n)$. Note that the kernel $K^{*,j}$ of $T^{*,j}$ is related to the kernel K of T via the identity

$$K^{*,j}(x,y_1,\ldots,y_{j-1},y_j,y_{j+1},\ldots,y_m) = K(y_j,y_1,\ldots,y_{j-1},x,y_{j+1},\ldots,y_m).$$

If an m-linear operator T maps a product of Banach spaces $X_1 \times \cdots \times X_m$ to another Banach space X, then the transpose $T^{*,j}$ maps $X_1 \times \cdots \times X_{j-1} \times X \times X_{j+1} \times \cdots \times X_m$ to X_j. Moreover, the norms of T and $T^{*,j}$ is equal. To maintain uniform notation, we occasionally denote T by $T^{*,0}$ and K by $K^{*,0}$.

Assumption 1 For each $i \in \{1,\dots,m\}$, there exists an operator $\{A_t^{(i)}\}_{t>0}$ with kernels $a_t^{(i)}(x,y)$ that satisfy conditions (1.6)–(1.7) with constants s, η, and that for every $j \in \{0,1,\dots,m\}$, there exist kernels $K_t^{*,j,(i)}(x,y_1,\dots,y_m)$ such that

$$\begin{aligned}\langle T^{*,j}(f_1,\dots,A_t^{(i)}f_i,\dots,f_m),f\rangle &= \int_{R^n}\int_{(R^n)^m} K_t^{*,j,(i)}(x,y_1,\dots,y_m) \\ &\quad \times f_1(y_1)\cdots f_m(y_m)g(x)\,dy_1\cdots dy_m\,dx \qquad (1.8)\end{aligned}$$

for all $f_1,\dots,f_m$ in $\mathcal{S}(R^n)$ with $\bigcap_{k=1}^m \operatorname{supp}(f_k)\cap\operatorname{supp}(g)=\emptyset$. Moreover, there exist a function $\phi \in C(R)$ with $\operatorname{supp}\phi \subset [-1,1]$ and constants $\epsilon > 0$ and A such that for all $j \in \{0,1,\dots,m\}$ and $i \in \{1,\dots,m\}$, we have

$$\begin{aligned}\bigl|K^{*,j}(x,y_1,\dots,y_m) - K^{*,j,(i)}(x,y_1,\dots,y_m)\bigr| &\le \frac{A}{(\sum_{k=1}^m |x-y_k|)^{mn}} \sum_{k=1,k\neq i}^m \phi\biggl(\frac{|y_i-y_k|}{t^{1/s}}\biggr) \\ &\quad + \frac{At^{\epsilon/s}}{(\sum_{k=1}^m |x-y_k|)^{mn+\epsilon}} \qquad (1.9)\end{aligned}$$

whenever $t^{1/s} \le |x-y_i|/2$.

If T satisfies Assumption 1, then we will say that T is an m-linear operator with generalized Calderón–Zygmund kernel K. We denote the set of functions K satisfying (1.8) and (1.9) with parameters m, A, s, η, and ϵ by m-GCZK(A,s,η,ϵ). We say that T is of class m-GCZO(A,s,η,ϵ) if T has an associated kernel K in m-GCZK(A,s,η,ϵ).

Assumption 2 There exist operators $\{B_t\}_{t>0}$ with kernels $b_t(x,y)$ that satisfy conditions (1.6) and (1.7) with constants s and η. Let

$$K_t^{(0)}(x,y_1,\dots,y_m) = \int_{R^n} K(z,y_1,\dots,y_m)b_t(x,z)\,dz.$$

We assume that the kernels $K_t^{(0)}(x,y_1,\dots,y_m)$ satisfy the following estimates: there exist a function $\phi \in C(R)$ with $\operatorname{supp}\phi \subset [-1,1]$ and constants $\epsilon > 0$ and A such that

$$\bigl|K_t^{(0)}(x,y_1,\dots,y_m)\bigr| \le \frac{A}{(\sum_{k=1}^m |x-y_k|)^{mn}}$$

whenever $2t^{1/s} \le \min_{1\le j\le m}|x-y_j|$, and

$$\begin{aligned}\bigl|K(x,y_1,\dots,y_m) - K_t^{(0)}(x',y_1,\dots,y_m)\bigr| &\le \frac{A}{(\sum_{k=1}^m |x-y_k|)^{mn}} \sum_{k=1,k\neq i}^m \phi\biggl(\frac{|y_i-y_k|}{t^{1/s}}\biggr) \\ &\quad + \frac{At^{\epsilon/s}}{(\sum_{k=1}^m |x-y_k|)^{mn+\epsilon}}\end{aligned}$$

whenever $2|x-x'| \le t^{1/s}$ and $2t^{1/s} \le \max_{1\le j\le m}|x-y_i|$.

Throughout this paper, we always assume that the m-linear operator T satisfies the following assumption.

Assumption 3 There exist $p_1,\dots,p_m \in [1,\infty)$ and $p \in (0,\infty)$ with $1/p = \sum_{j=1}^m 1/p_j$ such that T maps $L^{p_1}(R^n) \times \cdots \times L^{p_m}(R^n)$ to $L^p(R^n)$.

When T is of class m-GCZO(A,s,η,ϵ) and satisfies Assumption 3, Duong et al. [9] proved that the multilinear singular integral operator T is bounded from $L^{p_1}(w) \times \cdots \times L^{p_m}(w)$ to $L^p(w)$, where $w \in A_{p_0}$ with $p_0 = \min(p_1,\dots,p_m) > 1$. Grafakos et al. [10] obtained that T maps $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{w}})$ $(L^{p,\infty}(\nu_{\vec{w}}))$ for $\vec{w} \in A_{\vec{p}}$. For the boundedness of commutator generated by a BMO function, Anh and Duong [11] established that $T_{\sum \vec{b}}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{w}})$ for $\vec{w} \in \prod_{j=1}^m A_{p_j}$ with $p_j > 1, j = 1,\dots,m$. Chen and Wu [12] proved that $T_{\sum \vec{b}}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{w}})$ for $w \in A_{\vec{p}}$, $\vec{b} \in \mathrm{BMO}^m$.

On the other hand, for the vector-valued Calderón–Zygmund operator T_γ in m-GCZO(A,s,η,ϵ) satisfying Assumption 3. Chen et al. [13] proved that T_γ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{w}})$ $(L^{p,\infty}(\nu_{\vec{w}}))$ for $w \in A_{\vec{p}}$. They also obtained the boundedness of commutators $T_{\sum \vec{b},\gamma}$ and $T_{\prod \vec{b},\gamma}$ from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu_{\vec{w}})$ for $w \in A_{\vec{p}}$, $\vec{b} \in \mathrm{BMO}^m$. He and Zhou [14] extended the results of Chen et al. to weighted Morrey spaces. They proved that T_γ is bounded from $L^{p_1,\theta}(w_1) \times \cdots \times L^{p_m,\theta}(w_m)$ to $L^{p,\theta}(\nu_{\vec{w}})$ $(WL^{p,\theta}(\nu_{\vec{w}}))$ for $\vec{w} \in \prod_{j=1}^m A_{p_j}$ with $p_j > 1$, $j = 1,\dots,m$, where $0 < \theta < 1$. They also obtained the boundedness of the lth-order iterated BMO commutator $T_{\prod \vec{b}_\sigma,\gamma}$ in weighted Morrey spaces.

The generalized weighted Morrey space $(L^p(w),L^q)^\alpha$ was introduced by Feuto [15]. Moreover, he showed that the Calderón–Zygmund operators, Marcinkiewicz operators, the maximal operators associated with Bochner–Riesz operators, and their commutators are bounded on $(L^p(w),L^q)^\alpha$.

Inspired by the works mentioned, in this paper, we prove weighted norm inequalities for vector-valued multilinear singular integrals with nonsmooth kernels and commutators on generalized weighted Morrey spaces. We state our main results as follows.

Theorem 1.1 *Let T be a multilinear operator in m-GCZO(A,s,η,ϵ) with kernel K satisfying Assumption 2. Let $p \le \alpha < q \le \infty$, $p_1,\dots,p_m \in [1,\infty)$ with $1/p = \sum_{j=1}^m 1/p_j$, and $\gamma_1,\dots,\gamma_m \in (1,\infty)$ with $1/\gamma = \sum_{j=1}^m 1/\gamma_j$. Then for $\vec{w} = (w_1,\dots,w_m) \in A_{\vec{p}}$, we have:*

(i) *when all $p_j > 1$, there exists a constant C such that*

$$\bigl\|T_\gamma(\vec{f})\bigr\|_{(L^p(\nu_{\vec{w}}),L^q)^\alpha} \le C\prod_{j=1}^m \bigl\||f_j|_{\gamma_j}\bigr\|_{(L^{p_j}(w_j),L^{qp_j/p})^{\alpha p_j/p}};$$

(ii) *when some $p_j = 1$, there exists a constant C such that*

$$\bigl\|T_\gamma(\vec{f})\bigr\|_{(L^{p,\infty}(\nu_{\vec{w}}),L^q)^\alpha} \le C\prod_{j=1}^m \bigl\||f_j|_{\gamma_j}\bigr\|_{(L^{p_j}(w_j),L^{qp_j/p})^{\alpha p_j/p}}.$$

Theorem 1.2 *Let T be a multilinear operator in m-GCZO(A,s,η,ϵ) with kernel K satisfying Assumption 2. Let $p \le \alpha < q \le \infty$, $p_1,\dots,p_m \in (1,\infty)$ with $1/p = \sum_{j=1}^m 1/p_j$, and $\gamma_1,\dots,\gamma_m \in (1,\infty)$ with $1/\gamma = \sum_{j=1}^m 1/\gamma_j$. If $\vec{w} = (w_1,\dots,w_m) \in A_{\vec{p}}$ and $\vec{b} \in \mathrm{BMO}^m$, then* *there exists a constant C such that*

$$\left\| T_{\sum \vec{b},\gamma}(\vec{f})\right\|_{(L^p(\nu_{\vec{w}}),L^q)^\alpha} \le C\sum_{j=1}^m \|b_j\|_{\mathrm{BMO}}\prod_{i=1}^m \left\| |f_i|_{\gamma_i}\right\|_{(L^{p_i}(w_i),L^{qp_i/p})^{\alpha p_i/p}}.$$

Theorem 1.3 *Let T be a multilinear operator in m-GCZO(A,s,η,ϵ) with kernel K satisfying Assumption 2. Let $p \le \alpha < q \le \infty$, $p_1,\dots,p_m \in (1,\infty)$ with $1/p = \sum_{j=1}^m 1/p_j$, and $\gamma_1,\dots,\gamma_m \in (1,\infty)$ with $1/\gamma = \sum_{j=1}^m 1/\gamma_j$. If $\vec{w} = (w_1,\dots,w_m) \in A_{\vec{p}}$ and $\vec{b} \in \mathrm{BMO}^m$, then there exists a constant C such that*

$$\left\| T_{\prod \vec{b},\gamma}(\vec{f})\right\|_{(L^p(\nu_{\vec{w}}),L^q)^\alpha} \le C\prod_{j=1}^m \|b_j\|_{\mathrm{BMO}}\prod_{i=1}^m \left\| |f_i|_{\gamma_i}\right\|_{(L^{p_i}(w_i),L^{qp_i/p})^{\alpha p_i/p}}.$$

2 Some preliminaries and notations

For a measurable set E, we define $|E|$ as the Lebesgue measure of E, and χ_E as the characteristic function of E; $Q(x,r)$ denotes the cube centered at x with the sidelength r, $aQ(x,r) = Q(x,ar)$, and $\vec{p} = (p_1,\dots,p_m)$. For any number $r > 0$, $r\vec{p} = (rp_1,\dots,rp_m)$. For a locally integrable function f, f_Q denotes the average $f_Q = \frac{1}{|Q|}\int_Q f(x)\,dx$. The letter C will denote a constant not necessarily the same at each occurrence.

By a weight we always mean a positive locally integrable function. We say that a weight w belongs to the class A_p for $1 < p < \infty$ if there is a constant C such that for all cubes Q,

$$\left(\frac{1}{|Q|}\int_Q w(y)\,dy\right)\left(\frac{1}{|Q|}\int_Q w(y)^{\frac{1}{p-1}}\,dy\right)^{p-1} \le C.$$

In particular case, when $p = 1$, it is understood as

$$\left(\frac{1}{|Q|}\int_Q w(y)\,dy\right) \le C\inf_{x\in Q} w(x).$$

If $w \in A_p$, then there exist positive constants δ and C such that

$$\frac{w(E)}{w(Q)} \le C\left(\frac{|E|}{|Q|}\right)^\delta \tag{2.1}$$

for any measurable subset E of a ball Q. Since the classes A_p increase with respect to p, we write $A_\infty = \bigcup_{p\ge 1} A_p$.

Definition 2.1 (Multiple weights [3]) Let $\vec{p} = (p_1,\dots,p_m)$ and $1/p = 1/p_1 + \cdots + 1/p_m$ with $1 \le p_1,\dots,p_m < \infty$. Given $\vec{w} = (w_1,\dots,w_m)$ with each w_j being nonnegative measurable, set

$$\nu_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}.$$

We say that $\vec{w}$ satisfies the $A_{\vec{p}}$ condition and write $\vec{w} \in A_{\vec{p}}$ if

$$\sup_Q\left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}(x)\,dx\right)^{1/p}\prod_{j=1}^m\left(\frac{1}{|Q|}\int_Q w_j(x)^{1-p_j'}\,dx\right)^{1/p_j'} < \infty,$$

where the supremum is taken over all cubes $Q \subset R^n$, and the term $(\frac{1}{|Q|}\int_Q w_j(x)^{1-p_j'}\,dx)^{1/p_j'}$ is understood as $(\inf_Q w_j)^{-1}$ when $p_j = 1$.

Lemma 2.2 ([3]) *Let* $1 \le p_1, \dots, p_m < \infty$ *and* $\vec{w} = (w_1, \dots, w_m)$. *Then the following statements are equivalent*:

(i) $\vec{w} \in A_{\vec{p}}$;

(ii) $w_j^{1-p_j'} \in A_{mp_j'}$, $j = 1, \dots, m$, *and* $v_{\vec{w}} \in A_{mp}$,

where $w_j^{1-p_j'}$ *is understood as* $w_j^{1/m} \in A_1$ *in the case* $p_j = 1$.

Lemma 2.3 ([3]) *Let* $1 \le p_1, \dots, p_m < \infty$ *and* $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{p}}$. *Then there exists* $r > 1$ *such that* $\vec{w} \in A_{\vec{p}/r}$.

To prove the results for commutators, we recall the definition and some basic properties of BMO function spaces. We say a locally integrable function b is in BMO if

$$\|b\|_{\mathrm{BMO}} := \sup_Q \frac{1}{|Q|}\int_Q \left|b(y) - b_Q\right| dy < \infty.$$

For $b \in \mathrm{BMO}$, $1 < p < \infty$, we have $\|b\|_{\mathrm{BMO}} = \|b\|_{\mathrm{BMO}^p}$, where

$$\|b\|_{\mathrm{BMO}^p} := \sup_Q \left(\frac{1}{|Q|}\int_Q \left|b(x) - b_Q\right|^p dx\right)^{1/p},$$

and for all cubes Q, if $w \in A_\infty$, then by (2.1) and the John–Nirenberg inequality we have

$$\left(\frac{1}{w(Q)}\int_Q \left|b(x) - b_Q\right|^p w(x)\,dx\right)^{1/p} \le C\|b\|_{\mathrm{BMO}}. \tag{2.2}$$

For all nonnegative integers k, by simple calculation we get

$$|b_{2^{k+1}Q} - b_Q| \le C(k+1)\|b\|_{\mathrm{BMO}}. \tag{2.3}$$

Definition 2.4 ([15] (Generalized weighted Morrey space)) Let $1 \le p \le \alpha \le q \le \infty$, and let w be a weight. The space $(L^p(w), L^q)^\alpha := (L^p(w), L^q)^\alpha(R^n)$ is defined as the set of all measurable functions f satisfying $\|f\|_{(L^p(w),L^q)^\alpha} < \infty$, where

$$\|f\|_{(L^p(w),L^q)^\alpha} := \sup_{r>0} \|f\|_{(L^p(w),L^q)^\alpha,r} < \infty$$

with

$$\|f\|_{(L^p(w),L^q)^\alpha,r} := \left[\int_{R^n} \left(\left(w\big(B(y,r)\big)\right)^{1/\alpha - 1/p - 1/q} \|f\chi_{B(y,r)}\|_{L^p(w)}\right)^q dy\right]^{1/q}$$

for $r > 0$, with the usual modification when $q = \infty$. When $w = 1$, the space $(L^p, L^q)^\alpha$ was introduced in [16]. For $p < \alpha$ and $q = \infty$, the space $(L^p(w), L^q)^\alpha$ is the weighted Morrey space $L^{q,\theta}(w)$ with $\theta = 1/p - 1/\alpha$ defined by Komori and Shirai [17].

The weak space $(L^{p,\infty}(w), L^q)^\alpha$ is defined with

$$\|f\|_{(L^{p,\infty}(w),L^q)^\alpha,r} := \left[\int_{R^n} \left(\left(w\big(B(y,r)\big)\right)^{1/\alpha-1/p-1/q} \|f\chi_{B(y,r)}\|_{L^{p,\infty}(w)}\right)^q dy\right]^{1/q}.$$

When $p = 1$, the space $(L^{1,\infty}(w), L^q)^\alpha$ was introduced in [15].

The following results were obtained by Chen et al.

Theorem A ([13]) *Let T be a multilinear operator in m-GCZO(A,s,η,ϵ) with kernel K satisfying Assumption 2. Let $1 \le p_1,\dots,p_m < \infty$ with $1/p = \sum_{j=1}^m 1/p_j$. Then for $\vec{w} = (w_1,\dots,w_m) \in A_{\vec{p}}$, we have:*

(i) *If $1 < p_j < \infty$, $j = 1,\dots,m$, then*

$$\|T_\gamma(\vec{f})\|_{L^p(v_{\vec{w}})} \le C\prod_{j=1}^m \||f_j|_{\gamma_j}\|_{L^{p_j}(w_j)}.$$

(ii) *If $1 \le p_j < \infty$, $j = 1,\dots,m$, and at least one $p_j = 1$, then*

$$\|T_\gamma(\vec{f})\|_{L^{p,\infty}(v_{\vec{w}})} \le C\prod_{j=1}^m \||f_j|_{\gamma_j}\|_{L^{p_j}(w_j)}.$$

Theorem B ([13]) *Let T be a multilinear operator in m-GCZO(A,s,η,ϵ) with kernel K satisfying Assumption 2. Let $1 < p_1,\dots,p_m < \infty$ with $1/p = \sum_{j=1}^m 1/p_j$. If $\vec{w} = (w_1,\dots,w_m) \in A_{\vec{p}}$ and $\vec{b} \in \mathrm{BMO}^m$, then there exists a constant C such that*

$$\|T_{\sum\vec{b},\gamma}(\vec{f})\|_{L^p(v_{\vec{w}})} \le C\sum_{j=1}^m \|b_j\|_{\mathrm{BMO}} \prod_{i=1}^m \||f_i|_{\gamma_i}\|_{L^{p_i}(w_i)}.$$

Theorem C ([13]) *Let T be a multilinear operator in m-GCZO(A,s,η,ϵ) with kernel K satisfying Assumption 2. Let $1 < p_1,\dots,p_m < \infty$ with $1/p = \sum_{j=1}^m 1/p_j$. If $\vec{w} = (w_1,\dots,w_m) \in A_{\vec{p}}$ and $\vec{b} \in \mathrm{BMO}^m$, then there exists a constant C such that*

$$\|T_{\prod\vec{b},\gamma}(\vec{f})\|_{L^p(v_{\vec{w}})} \le C\prod_{j=1}^m \|b_j\|_{\mathrm{BMO}} \prod_{i=1}^m \||f_i|_{\gamma_i}\|_{L^{p_i}(w_i)}.$$

3 Proof of the main results

Proof of Theorem 1.1 (i) Let $\{f_{1k},\dots,f_{mk}\}_{k=1}^\infty$ be any smooth vector-valued functions. For any $Q = Q(y,r) \in R^n$, We split each $\vec{f_k} = \vec{f_k}^0 + \vec{f_k}^\infty$, where $\{\vec{f_k}^0\}_{k=1}^\infty = \{\vec{f_k}\chi_{Q^*}\}_{k=1}^\infty = \{f_{1k}\chi_{Q^*},\dots,f_{mk}\chi_{Q^*}\}_{k=1}^\infty$ and $Q^* = 8Q$. Then

$$\begin{aligned}\prod_{j=1}^m f_{jk}(y_j) &= \prod_{j=1}^m \big(f_{jk}^0(y_j) + f_{jk}^\infty(y_j)\big) = \sum_{\alpha_1,\dots,\alpha_m} f_{1k}^{\alpha_1}(y_1)\cdots f_{mk}^{\alpha_m}(y_m)\\ &= \prod_{j=1}^m f_{jk}^0(y_j) + \sum_{\alpha_1,\dots,\alpha_m\in\{0,\infty\}} f_{1k}^{\alpha_1}(y_1)\cdots f_{mk}^{\alpha_m}(y_m) + \prod_{j=1}^m f_{jk}^\infty(y_j),\end{aligned}$$

where $\alpha_1,\dots,\alpha_m$ are not all equal to 0 or ∞ at the same time. Hence, for $x\in Q(y,r)$, we have

$$\begin{aligned}\bigl|T_\gamma(f_1,\dots,f_m)(x)\bigr| &= \biggl|T_\gamma\bigl(f_1^0,\dots,f_m^0\bigr)(x)\\&\quad+\sum_{\alpha_1,\dots,\alpha_m\in\{0,\infty\}}T_\gamma\bigl(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\bigr)(x)+T_\gamma\bigl(f_1^\infty,\dots,f_m^\infty\bigr)\biggr|\\&\le\bigl|T_\gamma\bigl(f_1^0,\dots,f_m^0\bigr)(x)\bigr|\\&\quad+\biggl|\sum_{\alpha_1,\dots,\alpha_m\in\{0,\infty\}}T_\gamma\bigl(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\bigr)(x)\biggr|+\bigl|T_\gamma\bigl(f_1^\infty,\dots,f_m^\infty\bigr)\bigr|\\&:=I+II+III.\end{aligned}$$

We first estimate III. Taking $t=(2r)^s$, since $x,z\in Q$ and $y_j\in R^n\setminus Q^*$, for all $j=1,\dots,m$, we have

$$|y_j-z|\ge|y_j-r|-|z-r|>7r>2t^{1/s}.$$

Hence $\phi(|y_j-z|/t^{1/s})=0$. By Assumption 2 we have

$$\bigl|K(x,y_1,\dots,y_m)-K_t^{(0)}(x,y_1,\dots,y_m)\bigr|\le\frac{At^{\epsilon/s}}{(\sum_{k=1}^m|x-y_k|)^{mn+\epsilon}}.$$

Then for any $x\in Q$, by Assumption 2, we have

$$\begin{aligned}III&\le\Biggl(\sum_{k=1}^\infty\biggl(\int_{(R^n\setminus Q^*)^m}\bigl|K(x,\vec{y})-K_t^{(0)}(x,\vec{y})\bigr|\prod_{j=1}^m\bigl|f_{jk}(y_j)\bigr|\,d\vec{y}\biggr)^\gamma\Biggr)^{1/\gamma}\\&\quad+\Biggl(\sum_{k=1}^\infty\biggl(\int_{(R^n\setminus Q^*)^m}\bigl|K_t^{(0)}(x,\vec{y})\bigr|\prod_{j=1}^m\bigl|f_{jk}(y_j)\bigr|\,d\vec{y}\biggr)^\gamma\Biggr)^{1/\gamma}\\&\le C\Biggl(\sum_{k=1}^\infty\biggl(\sum_{\nu=1}^\infty\int_{(8^{\nu+1}Q\setminus8^\nu Q)^m}\frac{At^{\epsilon/s}}{(\sum_{k=1}^m|x-y_k|)^{mn+\epsilon}}\prod_{j=1}^m\bigl|f_{jk}(y_j)\bigr|\,d\vec{y}\biggr)^\gamma\Biggr)^{1/\gamma}\\&\quad+\Biggl(\sum_{k=1}^\infty\biggl(\sum_{\nu=1}^\infty\int_{(8^{\nu+1}Q\setminus8^\nu Q)^m}\frac{A}{(\sum_{k=1}^m|x-y_k|)^{mn}}\prod_{j=1}^m\bigl|f_{jk}(y_j)\bigr|\,d\vec{y}\biggr)^\gamma\Biggr)^{1/\gamma}\\&\le C\Biggl(\sum_{k=1}^\infty\biggl(\sum_{\nu=1}^\infty\biggl(\frac{|Q^*|^{\epsilon/s}}{(8^{\nu+1}|Q|^{1/n})^{mn+\epsilon}}+\frac{1}{(8^{\nu+1}|Q|^{1/n})^{mn}}\biggr)\int_{(8^{\nu+1}Q)^m}\prod_{j=1}^m\bigl|f_{jk}(y_j)\bigr|\,d\vec{y}\biggr)^\gamma\Biggr)^{1/\gamma}\\&\le C\sum_{\nu=1}^\infty\frac{1}{|8^{\nu+1}Q|^m}\prod_{j=1}^m\Biggl(\sum_{k=1}^\infty\biggl(\int_{8^{\nu+1}Q}\bigl|f_{jk}(y_j)\bigr|\,dy_j\biggr)^{\gamma_j}\Biggr)^{1/\gamma_j}\\&\le C\sum_{\nu=1}^\infty\prod_{j=1}^m\frac{1}{|8^{\nu+1}Q|}\int_{8^{\nu+1}Q}|f_j|_{\gamma_j}(y_j)\,dy_j,\end{aligned}$$

and the Hölder inequality gives

$$\int_{8^{\nu+1}Q} |f_j|_{s_j}(y_j)\,dy_j \le C\biggl(\int_{8^{\nu+1}Q} \bigl(|f_j|_{s_j}(y_j)\bigr)^{p_j} w_j(y_j)\,dy_j\biggr)^{1/p_j} \biggl(\int_{8^{\nu+1}Q} w_j(y_j)^{1-p_j'}\,dy_j\biggr)^{1/p_j'}.$$

By the definition of $A_{\overrightarrow{p}}$ we obtain

$$III \le C\sum_{\nu=1}^{\infty} \frac{1}{(\int_{8^{\nu+1}Q} \nu_{\overrightarrow{w}})^{1/p}} \prod_{j=1}^{m} \bigl\| |f_j|_{\gamma_j} \chi_{8^{\nu+1}Q} \bigr\|_{L^{p_j}(w_j)}.$$

For II, without loss of generality, we assume that $\alpha_i = \infty$ for $i = 1,\dots,l$ and $\alpha_j = 0$ for $j = l+1,\dots,m$. For $x \in Q(y,r)$, by Assumption 2 we get

$$\begin{aligned}
&\bigl|T_\gamma\bigl(f_{1k}^{\infty},\dots,f_{lk}^{\infty},f_{(l+1)k}^{0},\dots,f_{mk}^{0}\bigr)(x)\bigr| \\
&\quad\le \Biggl(\sum_{k=1}^{\infty}\biggl(\int_{(R^n\setminus Q^*)^m} \bigl|K(x,\overrightarrow{y}) - K_t^{(0)}(x,\overrightarrow{y})\bigr| \prod_{j=1}^{m}\bigl|f_{jk}(y_j)\bigr|\,d\overrightarrow{y}\biggr)^{\gamma}\Biggr)^{1/\gamma} \\
&\qquad + \Biggl(\sum_{k=1}^{\infty}\biggl(\int_{(R^n\setminus Q^*)^m} \bigl|K_t^{(0)}(x,\overrightarrow{y})\bigr| \prod_{j=1}^{m}\bigl|f_{jk}(y_j)\bigr|\,d\overrightarrow{y}\biggr)^{\gamma}\Biggr)^{1/\gamma} \\
&\quad\le C\Biggl(\sum_{k=1}^{\infty}\Biggl(\prod_{j=l+1}^{m}\int_{Q^*}\bigl|f_{jk}(y_j)\bigr|\,dy_j\biggl(\int_{(R^n\setminus Q^*)^l}\biggl(\frac{At^{\epsilon/s}}{(\sum_{j=1}^{l}|x-y_j|)^{mn+\epsilon}} + \frac{A}{(\sum_{j=1}^{l}|x-y_j|)^{mn}}\biggr)\biggr) \\
&\qquad \times \prod_{j=1}^{l}\bigl|f_{jk}(y_j)\bigr|\,dy_j\Biggr)^{\gamma}\Biggr)^{1/\gamma} \\
&\quad\le C\sum_{\nu=1}^{\infty}\biggl(\frac{|Q^*|^{\epsilon/s}}{(8^{\nu+1}|Q|^{1/n})^{mn+\epsilon}} + \frac{1}{(8^{\nu+1}|Q|^{1/n})^{mn}}\biggr) \\
&\qquad \times \prod_{j=l+1}^{m}\Biggl(\sum_{k=1}^{\infty}\biggl(\int_{Q^*}\bigl|f_{jk}(y_j)\bigr|\,dy_j\biggr)^{\gamma_j}\Biggr)^{1/\gamma_j} \prod_{j=1}^{l}\Biggl(\sum_{k=1}^{\infty}\biggl(\int_{8^{\nu+1}Q}\bigl|f_{jk}(y_j)\bigr|\,dy_j\biggr)^{\gamma_j}\Biggr)^{1/\gamma_j} \\
&\quad\le C\sum_{\nu=1}^{\infty}\frac{1}{|8^{\nu+1}Q|^m}\prod_{j=l+1}^{m}\int_{Q^*}|f_j|_{\gamma_j}(y_j)\,dy_j \prod_{j=1}^{l}\int_{8^{\nu+1}Q}|f_j|_{\gamma_j}(y_j)\,dy_j \\
&\quad\le C\sum_{\nu=1}^{\infty}\prod_{j=1}^{m}\frac{1}{|8^{\nu+1}Q|}\int_{8^{\nu+1}Q}|f_j|_{\gamma_j}(y_j)\,dy_j,
\end{aligned}$$

and by the Hölder inequality and the definition of $A_{\overrightarrow{p}}$ we have

$$II \le C\sum_{\nu=1}^{\infty} \frac{1}{(\int_{8^{\nu+1}Q} \nu_{\overrightarrow{w}})^{1/p}} \prod_{j=1}^{m} \bigl\| |f_j|_{\gamma_j} \chi_{8^{\nu+1}Q} \bigr\|_{L^{p_j}(w_j)}.$$

Combining the estimates of II and III, we obtain

$$\begin{aligned}\left|T_\gamma(\vec{f})(x)\right| &\le C\left|T_\gamma(\vec{f}^{\,0})\right| \\ &\quad + C\sum_{\nu=1}^{\infty}\frac{1}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}}\prod_{j=1}^{m}\left\||f_j|_{\gamma_j}\chi_{8^{\nu+1}Q}\right\|_{L^{p_j}(w_j)}. \qquad (3.1)\end{aligned}$$

Taking the $L^p(\nu_{\vec{w}})$ norms on the cube $Q(y,r)$ of both sides of (3.1), by Theorem A(i) we get

$$\begin{aligned}\left\|T_\gamma(\vec{f})\chi_{Q(x,r)}\right\|_{L^p(\nu_{\vec{w}})} &\le C\prod_{j=1}^{m}\left\||f_j|_{\gamma_j}\chi_{8^{\nu+1}Q}\right\|_{L^{p_j}(w_j)} \\ &\quad + C\sum_{\nu=1}^{\infty}\frac{(\int_Q\nu_{\vec{w}})^{1/p}}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}}\prod_{j=1}^{m}\left\||f_j|_{\gamma_j}\chi_{8^{\nu+1}Q}\right\|_{L^{p_j}(w_j)}. \qquad (3.2)\end{aligned}$$

Multiplying both sides of (3.2) by $\nu_{\vec{w}}(Q)^{1/\alpha-1/q-1/p}$, by Lemmas 2.2 and 2.3 we get

$$\begin{aligned}&\nu_{\vec{w}}(Q)^{1/\alpha-1/q-1/p}\left\|T_\gamma(\vec{f})\chi_{Q(x,r)}\right\|_{L^p(\nu_{\vec{w}})} \\ &\quad\le C\sum_{\nu=0}^{\infty}\frac{1}{8^{nk\delta(1/\alpha-1/q)}}\nu_{\vec{w}}\left(8^{\nu+1}Q\right)^{1/\alpha-1/q-1/p}\prod_{j=1}^{m}\left\||f_j|_{\gamma_j}\chi_{8^{\nu+1}Q}\right\|_{L^{p_j}(w_j)}\end{aligned}$$

For $\sum_{j=1}^{m}p/p_j=1$, by the Hölder inequality

$$\begin{aligned}&\left\|\nu_{\vec{w}}(Q)^{1/\alpha-1/q-1/p}\left\|T_\gamma(\vec{f})\chi_{Q(x,r)}\right\|_{L^p(\nu_{\vec{w}})}\right\|_{L^q(R^n)} \\ &\quad\le C\sum_{\nu=0}^{\infty}\frac{1}{8^{nk\delta(1/\alpha-1/q)}}\prod_{j=1}^{m}\left\|w_j\left(8^{\nu+1}Q\right)^{p/\alpha p_j-1/p_i-p/qp_j}\left\||f_j|_{\gamma_j}\chi_{8^{\nu+1}Q}\right\|_{L^{p_j}(w_j)}\right\|_{L^{qp_j/p}(R^n)}.\end{aligned}$$

Note that $\sum_{\nu=0}^{\infty}\frac{1}{8^{nk\delta(1/\alpha-1/q)}}$ converges. Hence

$$\left\|T_\gamma(\vec{f})\right\|_{(L^p(\nu_{\vec{w}}),L^q)^\alpha}\le C\prod_{j=1}^{m}\left\||f_j|_{\gamma_j}\right\|_{(L^{p_j}(w_j),L^{qp_j/p})^{\alpha p_j/p}}.$$

(ii) For any $\lambda>0$, by (3.1) and Theorem A(ii) we have

$$\begin{aligned}&\lambda\nu_{\vec{w}}\left(x\in Q(y,r):\left|T_\gamma(\vec{f})(x)\right|>\lambda\right)^{1/p} \\ &\quad\le C\prod_{j=1}^{m}\left\||f_j|_{\gamma_j}\chi_{8^{\nu+1}Q}\right\|_{L^{p_j}(w_j)}+C\sum_{\nu=1}^{\infty}\frac{(\int_Q\nu_{\vec{w}})^{1/p}}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}}\prod_{j=1}^{m}\left\||f_j|_{\gamma_j}\chi_{8^{\nu+1}Q}\right\|_{L^{p_j}(w_j)}.\end{aligned}$$

Hence

$$\begin{aligned}&\left\|T_\gamma(\vec{f})\chi_{Q(y,r)}\right\|_{L^{p,\infty}(\nu_{\vec{w}})} \\ &\quad\le C\prod_{j=1}^{m}\left\||f_j|_{\gamma_j}\chi_{8^{\nu+1}Q}\right\|_{L^{p_j}(w_j)}+C\sum_{\nu=1}^{\infty}\frac{(\int_Q\nu_{\vec{w}})^{1/p}}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}}\prod_{j=1}^{m}\left\||f_j|_{\gamma_j}\chi_{8^{\nu+1}Q}\right\|_{L^{p_j}(w_j)}.\end{aligned}$$

Multiplying both sides of this inequality by $\nu_{\vec{w}}(Q)^{1/\alpha-1/q-1/p}$ and applying a similar method to (i), we complete the proof of Theorem 1.1. □

Proof of Theorem 1.2 It suffices to prove that $T^j_{b_j,\gamma}, b_j \in \mathrm{BMO}$. For $Q = Q(y,r)$, $x \in Q$, we can write

$$\begin{aligned} T^j_{b_j,\gamma}(\vec{f})(x) &= T^j_{b_j,\gamma}(\vec{f}\chi_{Q^*})(x) + \sum_{\alpha_1,\dots,\alpha_m\in\{0,\infty\}} \big(b_j(x)T_\gamma\big(f_{1k}^{\alpha_1},\dots,f_{jk}^{\alpha_j},\dots,f_{mk}^{\alpha_m}\big) \\ &\quad - T_\gamma\big(f_{1k}^{\alpha_1},\dots,b_jf_{jk}^{\alpha_j},\dots,f_{mk}^{\alpha_m}\big)(x)\big) \\ &\quad + b_j(x)T_\gamma\big(f_{1k}^{\infty},\dots,f_{jk}^{\infty},\dots,f_{mk}^{\infty}\big) - T_\gamma\big(f_{1k}^{\infty},\dots,b_jf_{jk}^{\infty},\dots,f_{mk}^{\infty}\big)(x) \\ &= I' + II' + III', \end{aligned}$$

where $\alpha_1,\dots,\alpha_m$ are not all equal to 0 or ∞ at the same time. For III', we have

$$\begin{aligned} \big|III'\big| &\le \big|\big(b_j(x)-b_Q\big)|T_\gamma\big(f_{1k}^{\infty},\dots,f_{jk}^{\infty},\dots,f_{mk}^{\infty}\big)\big| + \big|T_\gamma\big(f_{1k}^{\infty},\dots,(b_j-b_Q)f_{jk}^{\infty},\dots,f_{mk}^{\infty}\big)(x)\big| \\ &\le \big|b_j(x)-b_Q\big| \sum_{\nu=1}^{\infty} \frac{1}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}} \prod_{i=1}^{m} \big\| |f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\big\|_{L^{p_i}(w_i)} \\ &\quad + \big|T_\gamma\big(f_{1k}^{\infty},\dots,(b_j-b_Q+b_{8^{\nu+1}Q}-b_{8^{\nu+1}Q})f_{jk}^{\infty},\dots,f_{mk}^{\infty}\big)(x)\big|. \end{aligned}$$

Similarly to III in (i), we have

$$\begin{aligned} &\big|T_\gamma\big(f_{1k}^{\infty},\dots,(b_j-b_Q+b_{8^{\nu+1}Q}-b_{8^{\nu+1}Q})f_{jk}^{\infty},\dots,f_{mk}^{\infty}\big)(x)\big| \\ &\quad\le \big|T_\gamma\big(f_{1k}^{\infty},\dots,(b_j-b_{8^{\nu+1}Q})f_{jk}^{\infty},\dots,f_{mk}^{\infty}\big)(x)\big| \\ &\qquad + \big|T_\gamma\big(f_{1k}^{\infty},\dots,(b_{8^{\nu+1}Q}-b_Q)f_{jk}^{\infty},\dots,f_{mk}^{\infty}\big)(x)\big| \\ &\quad\le \sum_{\nu=1}^{\infty} \frac{|b_{8^{\nu+1}Q}-b_Q|}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}} \prod_{i=1}^{m} \big\| |f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\big\|_{L^{p_i}(w_i)} \\ &\qquad + \sum_{\nu=1}^{\infty} \frac{1}{|8^{\nu+1}Q|^m} \int_{(8^{\nu+1}Q)^m} \prod_{i=1,i\ne j}^{m} |f_i|_{\gamma_i}(y_i)|f_j|_{\gamma_j}(y_j)\big|b_j(y_j)-b_{8^{\nu+1}Q}\big|\,d\vec{y}. \end{aligned}$$

Since $\vec{w} \in A_{\vec{p}}$, we can select suitable $r > 1$ such that $\vec{w} \in A_{\vec{p}/r}$ by Lemma 2.3, and by the Hölder inequality and Lemma 2.2 we have

$$\begin{aligned} &\sum_{\nu=1}^{\infty} \frac{1}{|8^{\nu+1}Q|^m} \int_{(8^{\nu+1}Q)^m} \prod_{i=1,i\ne j}^{m} |f_i|_{\gamma_i}(y_i)|f_j|_{\gamma_j}(y_j)\big|b_j(y_j)-b_{8^{\nu+1}Q}\big|\,d\vec{y} \\ &\quad\le \sum_{\nu=1}^{\infty} \frac{1}{|8^{\nu+1}Q|^{m/r}} \Bigg(\prod_{i=1,i\ne j}^{m}\int_{8^{\nu+1}Q}|f_i|_{\gamma_i}^r\,dy_i\Bigg)^{1/r} \bigg(\int_{8^{\nu+1}Q}\big(|f_j|_{\gamma_j}(y_j)\big|b_j(y_j)-b_{8^{\nu+1}Q}\big|\big)^r\bigg)^{1/r} \\ &\quad\le \sum_{\nu=1}^{\infty} \frac{1}{|8^{\nu+1}Q|^{m/r}} \prod_{i=1,i\ne j}^{m} \bigg(\int_{8^{\nu+1}Q}|f_i|_{\gamma_i}^{p_i}w_i(y_i)\,dy_i\bigg)^{1/p_i} \bigg(\int_{8^{\nu+1}Q}w_i(y_i)^{-r/(p_i-r)}\bigg)^{(p_i-r)/p_ir} \\ &\qquad \times \bigg(\int_{8^{\nu+1}Q}\big|b_j(y_j)-b_{8^{\nu+1}Q}\big|^{p_jr/(p_j-r)}w_j(y_j)^{-r/(p_j-r)}\,dy_j\bigg)^{(p_i-r)/p_ir} \end{aligned}$$

$$\begin{aligned}&\times\left(\int_{8^{\nu+1}Q}|f_j|_{\gamma_j}^{p_j}w_j(y_j)\,dy_j\right)^{1/p_j}\\&\le C\|b_j\|_{\mathrm{BMO}}\sum_{\nu=1}^{\infty}\frac{1}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}}\prod_{i=1}^{m}\big\||f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\big\|_{L^{p_i}(w_i)}.\end{aligned}$$

Hence we have

$$\begin{aligned}\left|III'\right|&\le\left|b_j(x)-b_Q\right|\sum_{\nu=1}^{\infty}\frac{1}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}}\prod_{i=1}^{m}\big\||f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\big\|_{L^{p_i}(w_i)}\\&\quad+\sum_{\nu=1}^{\infty}\frac{|b_{8^{\nu+1}Q}-b_Q|}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}}\prod_{i=1}^{m}\big\||f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\big\|_{L^{p_i}(w_i)}\\&\quad+\|b_j\|_{\mathrm{BMO}}\sum_{\nu=1}^{\infty}\frac{1}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}}\prod_{i=1}^{m}\big\||f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\big\|_{L^{p_i}(w_i)}.\end{aligned}$$

For II', we now consider $\alpha_i=\infty$ for $i=1,\dots,l$ and $\alpha_j=0$ for $j=l+1,\dots,m$. There are two cases:

$$\begin{aligned}&\big|b_j(x)T_\gamma\big(f_{1k}^{\infty},\dots,f_{jk}^{\infty},\dots,f_{lk}^{\infty},f_{(l+1)k}^{0},\dots,f_{mk}^{0}\big)\\&\quad-T_\gamma\big(f_{1k}^{\infty},\dots,b_jf_{jk}^{\infty},\dots,f_{lk}^{\infty},f_{(l+1)k}^{0},\dots,f_{mk}^{0}\big)(x)\big|\end{aligned}$$

or

$$\begin{aligned}&\big|b_j(x)T_\gamma\big(f_{1k}^{\infty},\dots,f_{lk}^{\infty},f_{(l+1)k}^{0},\dots,f_{jk}^{0},\dots,f_{mk}^{0}\big)\\&\quad-T_\gamma\big(f_{1k}^{\infty},\dots,f_{lk}^{\infty},f_{(l+1)k}^{0},\dots,b_jf_{jk}^{0},\dots,f_{mk}^{0}\big)(x)\big|.\end{aligned}$$

We just consider the following case, the other case being completely analogous:

$$\begin{aligned}&\big|b_j(x)T_\gamma\big(f_{1k}^{\infty},\dots,f_{jk}^{\infty},\dots,f_{lk}^{\infty},f_{(l+1)k}^{0},\dots,f_{mk}^{0}\big)\\&\qquad-T_\gamma\big(f_{1k}^{\infty},\dots,b_jf_{jk}^{\infty},\dots,f_{lk}^{\infty},f_{(l+1)k}^{0},\dots,f_{mk}^{0}\big)(x)\big|\\&\quad\le\big|\big(b_j(x)-b_Q\big)T_\gamma\big(f_{1k}^{\infty},\dots,f_{jk}^{\infty},\dots,f_{lk}^{\infty},f_{(l+1)k}^{0},\dots,f_{mk}^{0}\big)\big|\\&\qquad+\big|T_\gamma\big(f_{1k}^{\infty},\dots,(b_j-b_Q)f_{jk}^{\infty},\dots,f_{lk}^{\infty},f_{(l+1)k}^{0},\dots,f_{mk}^{0}\big)(x)\big|\\&\quad\le\big|b_j(x)-b_Q\big|\sum_{\nu=1}^{\infty}\frac{1}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}}\prod_{i=1}^{m}\big\||f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\big\|_{L^{p_i}(w_i)}\\&\qquad+\sum_{\nu=1}^{\infty}\frac{|b_{8^{\nu+1}Q}-b_Q|}{(\int_{8^{\nu+1}Q}\nu_{\vec{w}})^{1/p}}\prod_{i=1}^{m}\big\||f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\big\|_{L^{p_i}(w_i)}\\&\qquad+\sum_{\nu=1}^{\infty}\frac{1}{|8^{\nu+1}Q|^m}\prod_{i=l+1}^{m}\int_{Q^*}|f_i|_{\gamma_i}(y_i)\,dy_i\\&\qquad\times\int_{(8^{\nu+1}Q)^m}\prod_{i=1,i\neq j}^{l}|f_i|_{\gamma_i}(y_i)|f_j|_{\gamma_j}(y_j)\big|b_j(y_j)-b_{8^{\nu+1}Q}\big|\,d\vec{y}.\end{aligned}$$

Since $\overrightarrow{w} \in A_{\overrightarrow{p}}$, we can select suitable $r > 1$ such that $\overrightarrow{w} \in A_{\overrightarrow{p}/r}$ by Lemma 2.3, and by the Hölder inequality and Lemma 2.2 we have

$$\begin{aligned}
&\sum_{\nu=1}^{\infty} \frac{1}{|8^{\nu+1}\sqrt{n}Q|^m} \prod_{i=l+1}^{m} \int_{Q^*} |f_i|_{\gamma_i}(y_i)\,dy_i \\
&\quad \times \int_{(8^{\nu+1}\sqrt{n}Q)^m} \prod_{i=1,i\neq j}^{l} |f_i|_{\gamma_i}(y_i)|f_j|_{\gamma_j}(y_j)\bigl|b_j(y_j) - b_{8^{\nu+1}\sqrt{n}Q}\bigr|\,d\overrightarrow{y} \\
&\leq \sum_{\nu=1}^{\infty} \frac{1}{|8^{\nu+1}\sqrt{n}Q|^m} \prod_{i=1,i\neq j}^{m} \int_{8^{\nu+1}\sqrt{n}Q} |f_i|_{\gamma_i}(y_i)\,dy_i \\
&\quad \times \int_{8^{\nu+1}\sqrt{n}Q} |f_j|_{\gamma_j}(y_j)\bigl|b_j(y_j) - b_{8^{\nu+1}\sqrt{n}Q}\bigr|\,dy_j \\
&\leq C\|b_j\|_{\mathrm{BMO}} \sum_{\nu=1}^{\infty} \frac{1}{(\int_{8^{\nu+1}\sqrt{n}Q} \nu_{\overrightarrow{w}})^{1/p}} \prod_{i=1}^{m} \bigl\| |f_i|_{\gamma_i}\chi_{8^{\nu+1}\sqrt{n}Q}\bigr\|_{L^{p_i}(w_i)}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\bigl|T^j_{b_j,\gamma}(\overrightarrow{f})(x)\bigr| \leq{}& \bigl|T^j_{b_j,\gamma}(\overrightarrow{f}\chi_{Q^*})(x)\bigr| \\
&+ \bigl|b_j(x) - b_Q\bigr| \sum_{\nu=1}^{\infty} \frac{1}{(\int_{8^{\nu+1}Q} \nu_{\overrightarrow{w}})^{1/p}} \prod_{i=1}^{m} \bigl\| |f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\bigr\|_{L^{p_i}(w_i)} \\
&+ \sum_{\nu=1}^{\infty} \frac{|b_{8^{\nu+1}Q} - b_Q|}{(\int_{8^{\nu+1}Q} \nu_{\overrightarrow{w}})^{1/p}} \prod_{i=1}^{m} \bigl\| |f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\bigr\|_{L^{p_i}(w_i)} \\
&+ \|b_j\|_{\mathrm{BMO}} \sum_{\nu=1}^{\infty} \frac{1}{(\int_{8^{\nu+1}Q} \nu_{\overrightarrow{w}})^{1/p}} \prod_{i=1}^{m} \bigl\| |f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\bigr\|_{L^{p_i}(w_i)}.
\end{aligned}$$

Taking the $L^p(\nu_{\overrightarrow{w}})$ norms on the cube $Q(y,r)$ of both sides of this inequality, by Theorem B and Lemmas 2.2 and 2.3 we have

$$\begin{aligned}
&\bigl\|T^j_{b_j,\gamma}(\overrightarrow{f})\chi_{Q(y,r)}\bigr\|_{L^p(\nu_{\overrightarrow{w}})} \\
&\leq C\|b_j\|_{\mathrm{BMO}} \prod_{i=1}^{m} \bigl\| |f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\bigr\|_{L^{p_i}(w_i)} \\
&\quad + \|b_j\|_{\mathrm{BMO}} \sum_{\nu=1}^{\infty} \frac{k(\int_Q \nu_{\overrightarrow{w}})^{1/p}}{(\int_{8^{\nu+1}Q} \nu_{\overrightarrow{w}})^{1/p}} \prod_{i=1}^{m} \bigl\| |f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\bigr\|_{L^{p_i}(w_i)}.
\end{aligned}$$

Multiplying both sides of this inequality by $\nu_{\overrightarrow{w}}(Q)^{1/\alpha-1/q-1/p}$, by (2.1) and Lemma 2.2 we get

$$\begin{aligned}
&\nu_{\overrightarrow{w}}(Q)^{1/\alpha-1/q-1/p}\bigl\|T^j_{b_j,\gamma}(\overrightarrow{f})\chi_{Q(y,r)}\bigr\|_{L^p(\nu_{\overrightarrow{w}})} \\
&\leq C\sum_{\nu=0}^{\infty} \frac{(k+1)\|b_j\|_{\mathrm{BMO}}}{8^{nk\delta(1/\alpha-1/q)}} \nu_{\overrightarrow{w}}\bigl(8^{\nu+1}Q\bigr)^{1/\alpha-1/q-1/p} \prod_{i=1}^{m} \bigl\| |f_i|_{\gamma_i}\chi_{8^{\nu+1}Q}\bigr\|_{L^{p_i}(w_i)}.
\end{aligned}$$

By a proof similar to that of Theorem 1.1(i) we have

$$\| T_{\sum \vec{b},\gamma}(\vec{f})\|_{(L^p(\nu_{\vec{w}}),L^q)^\alpha} \le C\sum_{j=1}^m \|b_j\|_{\mathrm{BMO}} \prod_{i=1}^m \big\| |f_i|_{\gamma_i}\big\|_{(L^{p_i}(w_i),L^{qp_i/p})^{\alpha p_i/p}}.$$

Thus we complete the proof of Theorem 1.2. □

The proof of Theorem 1.3 also uses very similar arguments, and hence we omit the details.

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