Rashid et al. *Journal of Inequalities and Applications* (2020) 2020:177
https://doi.org/10.1186/s13660-020-02445-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

New generalized reverse Minkowski and related integral inequalities involving generalized fractional conformable integrals

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Abstract

This paper gives some novel generalizations by considering the generalized conformable fractional integrals operator for reverse Minkowski type and reverse Hölder type inequalities. Furthermore, novel consequences connected with this inequality, together with statements and confirmation of various variants for the advocated generalized conformable fractional integral operator, are elaborated. Moreover, our derived results are provided to show comparisons of convergence between old and modified operators towards a function under different parameters and conditions. The numerical approximations of our consequence have several utilities in applied sciences and fractional integro-differential equations.

MSC: 26D15; 26D20; 26D07

Keywords: Integral inequality; Minkowski inequality; Reverse Minkowski inequality; Conformable integrals; Generalized conformable fractional integral operators

1 Introduction

Fractional calculus, generally referred to as the calculus of non-integer order, was a trademark outgrowth of traditional definitions of calculus integral and derivative. The concept of fractional calculus has provoked a host of researchers and was comprehensively studied in the literature for the last few decades. A continuous effort has been made on an enormous scale and everybody has been stimulated by its different aspects. In the present century, the exceptional idea has been described by several mathematicians with a slightly distinct technique in different time scales; see, for instance, the Liouville, Riemann, Grunwald, Letnikov, Hadamard, Weyl, Riesz, Marchaud, Kober and Caputo fractional integrals (see [1–11]). Most of these researchers first of all added fractional integrals, on the concept of which the associated fractional derivative and other associated results had been produced. Recently, Khalil et al. [2] and Abdeljawad [1] introduced fractional operators known as fractional conformable derivatives and integrals. Jarad et al. [12] established the fractional conformable integral operators. Meanwhile in [13], Anderson and Ulness introduced the concept of local derivatives for upgrading the concept of the fractional conformable derivative. The exponential and Mittag-Leffler functions are used as kernels by several researchers for developing new fractional techniques. In [14], Khan et al. established a new class of generalized conformable fractional integral operators. Such generalizations encourage future studies requiring extra thoughts to merge the fractional operators and achieve the variants regarding such fractional operators.

Conformable derivatives are nonlocal fractional derivatives. They can be called fractional since we can take derivatives up to arbitrary order. However, since in the community of fractional calculus, nonlocal fractional derivatives only are to be called fractional, we prefer to replace conformable fractional by conformable (as a type of local fractional). Conformable derivatives and other types of local fractional derivatives or modified conformable derivatives in [13] can gain in importance by the ability to use them to generate more generalized nonlocal fractional derivatives with singular kernels (see [15–22]).

Integral inequalities have potential application in several areas of science: technology, mathematics, chemistry, plasma physics, among others; especially we point out initial value problems, the stability of linear transformation, integral differential equations, and impulse equations [23–33]. Variants regarding fractional integral operators are of use in significant strategies amongst researchers and accumulate fertile functional applications in various areas of science; see [34–45]. On account of their potential results to be utilized for the presence of nontrivial and positive solutions of a distinct kind of fractional differential equations, our findings concerning fractional integrals are appreciable and essential.

An enormous heft of present literature comprises generalizations of several variants by fractional integral operators and their applications [46–52]. We state some of them, that is, the variants of Minkowski, Hardy, Opial, Hermite–Hadamard, Grüss, Lyenger, Wrtinger, Ostrowski, Čebyšev and Pólya–Szegö [53–59]. Such applications of fractional integral operators compelled us to show the generalization of the reverse Minkowski inequality [43, 44, 53] involving generalized conformable fractional integrals operators.

The article is composed thus: in Sect. 2 we demonstrate the notations and primary definitions of our newly introduced operator generalized conformable fractional integrals. Also, we present the results concerning the reverse Minkowski inequality. In Sect. 3, we advocate essential consequences such as the reverse Minkowski inequality via the generalized conformable fractional integral operators. In Sect. 4, we show the associated variants using this fractional integral.

2 Preliminaries

This section is dedicated to some recognized definitions and results associated with the generalized conformable fractional integral operators and their generalization related to the generalized conformable fractional integral operators. Set et al. in [60] proved the Hermite–Hadamard, and reverse Minkowski inequalities for Riemann–Liouville fractional integrals. Additionally, Hardy's type and reverse Minkowski inequalities were supplied by Bougoffa in [38]. The subsequent consequences concerning the reverse Minkowski inequalities are of significance for the classical integrals.

Theorem 2.1 ([60]) *For $p \geq 1$ and let there be two positive functions f_1 and f_2 on $[0,\infty)$. If $0 < \theta_1 \leq \frac{f_1(\eta)}{f_2(\eta)} \leq \theta_2$, $y \in [r_1, r_2]$, then*

$$\left(\int_{r_1}^{r_2} f_1^p(y)\,dy\right)^{\frac{1}{p}} + \left(\int_{r_1}^{r_2} f_2^p(y)\,dy\right)^{\frac{1}{p}} \leq \frac{1+\theta_2(\theta_1+2)}{(\theta_1+1)(\theta_2+1)}\left(\int_{r_1}^{r_2} (f_1+f_2)^p(y)\,dy\right)^{\frac{1}{p}}.$$

Theorem 2.2 ([60]) *For $p \geq 1$ and let there be two positive functions f_1 and f_2 on $[0,\infty)$. If $0 < \theta_1 \leq \frac{f_1(\eta)}{f_2(\eta)} \leq \theta_2$, $y \in [r_1, r_2]$, then*

$$\begin{aligned}&\left(\int_{r_1}^{r_2} f_1^p(y)\,dy\right)^{\frac{2}{p}} + \left(\int_{r_1}^{r_2} f_2^p(y)\,dy\right)^{\frac{2}{p}} \\ &\quad\geq \left(\frac{(1+\theta_2)(\theta_1+1)}{\theta_2} - 2\right)\left(\int_{r_1}^{r_2} f_1^p(y)\,dy\right)^{\frac{1}{p}}\left(\int_{r_1}^{r_2} f_2^p(y)\,dy\right)^{\frac{1}{p}}.\end{aligned}$$

In [44], Dahmani used the Riemann–Liouville fractional integral operators to prove the subsequent reverse Minkowski inequalities.

Theorem 2.3 ([44]) *Let $\varsigma > 0$ and $p \geq 1$, and let there be two positive functions f_1 and f_2 on $[0,\infty)$ such that, for all $y > 0$, $\mathcal{K}_{r_1^+}^{\varsigma} f_1^p(y) < \infty$, $\mathcal{K}_{r_1^+}^{\varsigma} f_2^p(y) < \infty$. If $0 < \theta_1 \leq \frac{f_1(\eta)}{f_2(\eta)} \leq \theta_2$, $\eta \in [r_1, y]$, then the following inequality holds*:

$$\left(\mathcal{K}_{r_1^+}^{\varsigma} f_1^p(y)\right)^{\frac{1}{p}} + \left(\mathcal{K}_{r_1^+}^{\varsigma} f_2^p(y)\right)^{\frac{1}{p}} \leq \frac{1+\theta_2(\theta_1+2)}{(\theta_1+1)(\theta_2+1)}\left(\mathcal{K}_{r_1^+}^{\varsigma}(f_1+f_2)^p(y)\right)^{\frac{1}{p}}.$$

Theorem 2.4 ([44]) *Let $\varsigma > 0$ and $p \geq 1$, and let there be two positive functions f_1 and f_2 on $[0,\infty)$ such that, for all $y > 0$, $\mathcal{K}_{r_1^+}^{\varsigma} f_1^p(y) < \infty$, $\mathcal{K}_{r_1^+}^{\varsigma} f_2^p(y) < \infty$. If $0 < \theta_1 \leq \frac{f_1(\eta)}{f_2(\eta)} \leq \theta_2$, $\eta \in [r_1, y]$, then the following inequality holds*:

$$\left(\mathcal{K}_{r_1^+}^{\varsigma} f_1^p(y)\right)^{\frac{2}{p}} + \left(\mathcal{K}_{r_1^+}^{\varsigma} f_2^p(y)\right)^{\frac{2}{p}} \geq \left(\frac{(1+\theta_2)(\theta_1+1)}{\theta_2} - 2\right)\left(\mathcal{K}_{r_1^+}^{\varsigma} f_1^p(y)\right)^{\frac{1}{p}}\left(\mathcal{K}_{r_1^+}^{\varsigma} f_2^p(y)\right)^{\frac{1}{p}}.$$

Recall the definition of the generalized conformable fractional integral which is mainly due to [14].

Definition 2.5 ([14]) Let f be a conformable integrable function on the interval $[r_1, r_2] \subseteq [0,\infty)$. The right-sided and left-sided generalized conformable fractional integrals ${}_{\varrho}^{\tau}\mathcal{K}_{r_1^+}^{\varsigma}$ and ${}_{\varrho}^{\tau}\mathcal{K}_{r_2^-}^{\varsigma}$ of order $\varsigma > 0$ are defined by

$${}_{\varrho}^{\tau}\mathcal{K}_{r_1^+}^{\varsigma} f(y) = \frac{1}{\Gamma(\varsigma)}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\varsigma-1}\frac{f(\eta)}{\eta^{1-\tau-\varrho}}\,d\eta, \quad y > r_1, \tag{2.1}$$

and

$${}_{\varrho}^{\tau}\mathcal{K}_{r_2^-}^{\varsigma} f(y) = \frac{1}{\Gamma(\varsigma)}\int_{y}^{r_2}\left(\frac{\eta^{\tau+\varrho}-y^{\tau+\varrho}}{\tau+\varrho}\right)^{\varsigma-1}\frac{f(\eta)}{\eta^{1-\tau-\varrho}}\,d\eta, \quad y < r_2, \tag{2.2}$$

where $\varsigma \in \mathbb{C}$, $\Re(\varsigma) > 0$, $\varrho \in (0,1]$, $\tau \in \mathcal{R}$ with $\tau + \varrho \neq 0$, and Γ is the well-known gamma function.

Remark 2.6 In Eqs. (2.1) and (2.2):

(i) If $\tau = 0$, then we attain the subsequent Riemann–Liouville type fractional conformable integral operators; see [12]:

$$
{}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f(y) = \frac{1}{\Gamma(\varsigma)} \int_{r_1}^{y} \left(\frac{y^{\varrho} - \eta^{\varrho}}{\varrho} \right)^{\varsigma - 1} \frac{f(\eta)}{\eta^{1-\varrho}} \, d\eta, \quad y > r_1, \tag{2.3}
$$

and

$$
{}_{\varrho}\mathcal{K}^{\varsigma}_{r_2^-} f(y) = \frac{1}{\Gamma(\varsigma)} \int_{y}^{r_2} \left(\frac{\eta^{\varrho} - y^{\varrho}}{\varrho} \right)^{\varsigma - 1} \frac{f(\eta)}{\eta^{1-\varrho}} \, d\eta, \quad y < r_2, \tag{2.4}
$$

where $\varsigma \in \mathbb{C}$, $\Re(\varsigma) > 0$, $\varrho \in (0, 1]$.

(ii) If $\tau = 0$ and $\varrho = 1$, then we attain the subsequent Riemann–Liouville type fractional integral operators; see [10, 15]:

$$
\mathcal{K}^{\varsigma}_{r_1^+} f(y) = \frac{1}{\Gamma(\varsigma)} \int_{r_1}^{y} (y - \eta)^{\varsigma - 1} f(\eta) \, d\eta, \quad y > r_1, \tag{2.5}
$$

and

$$
\mathcal{K}^{\varsigma}_{r_2^-} f(y) = \frac{1}{\Gamma(\varsigma)} \int_{y}^{r_2} (\eta - y)^{\varsigma - 1} f(\eta) \, d\eta, \quad y < r_2, \tag{2.6}
$$

where $\varsigma \in \mathbb{C}$, $\Re(\varsigma) > 0$.

3 Reverse Minkowski inequalities via generalized conformable fractional integral operators

This section comprises our principal involvement of establishing the proof of the reverse Minkowski inequalities via generalized conformable fractional integral operators defined in (2.1) and (2.2) and an associated theorem insinuated as the reverse Minkowski inequalities.

Theorem 3.1 *For $\varsigma > 0$, $\varrho \in (0, 1]$, $\tau \in \mathcal{R}$ and $\varrho + \tau \neq 0$ with $p \geq 1$ and let there be two positive functions f_1, f_2 on $[0, \infty)$ such that, for all $y > r_1$, ${}^{\tau}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^p(y) < \infty$ and ${}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_2^p(y) < \infty$. If $0 < \theta_1 \leq \frac{f_1(\eta)}{f_2(\eta)} \leq \theta_2$ for $\theta_1, \theta_2 \in \mathcal{R}^+$ and for all $x \in [r_1, y]$, then*

$$
\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^p(y) \right)^{\frac{1}{p}} + \left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_2^p(y) \right)^{\frac{1}{p}} \leq \frac{1 + \theta_2(\theta_1 + 2)}{(\theta_1 + 1)(\theta_2 + 1)} \left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} (f_1 + f_2)^p(y) \right)^{\frac{1}{p}}. \tag{3.1}
$$

Proof By the suppositions mentioned in Theorem 3.1, $\frac{f_1(\eta)}{f_2(\eta)} \leq \theta_2$, $r_1 \leq \eta \leq y$, we have

$$
(M + 1)^p f_1^p(\eta) \leq M^p \big(f_1(\eta) + f_2(\eta) \big)^p. \tag{3.2}
$$

If we multiply both sides of (3.2) with $\frac{1}{\Gamma(\varsigma)\varsigma^{1-\tau-\varrho}}(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho})^{\tau-1}$ and then integrate the subsequent inequality with respect to η from r_1 to y, we obtain

$$\begin{aligned}&\frac{(M+1)^p}{\Gamma(\varsigma)}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\tau-1}\frac{f_1^p(\eta)}{\varsigma^{1-\tau-\varrho}}\,d\eta\\&\quad\le\frac{M^p}{\Gamma(\varsigma)}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\tau-1}\frac{(f_1(\eta)+f_2(\eta))^p}{\varsigma^{1-\tau-\varrho}}\,d\eta.\end{aligned}\tag{3.3}$$

Similarly,

$$\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_1^p(y)\right)^{\frac{1}{p}}\le\frac{\theta_2}{\theta_2+1}\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}(f_1+f_2)^p(y)\right)^{\frac{1}{p}}.\tag{3.4}$$

In contrast, as $mf_2(\eta)\le f_1(\eta)$, it follows that

$$\left(1+\frac{1}{\theta_1}\right)^p f_2^p(\eta)\le\left(\frac{1}{\theta_1}\right)^p\left(f_1(\eta)+f_2(\eta)\right)^p.\tag{3.5}$$

Again, if we multiply both sides of (3.5) with $\frac{1}{\Gamma(\varsigma)\varsigma^{1-\tau-\varrho}}(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho})^{\tau-1}$ and then integrate the subsequent inequality with respect to η from r_1 to y, we obtain

$$\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_2^p(y)\right)^{\frac{1}{p}}\le\frac{1}{\theta_1+1}\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}(f_1+f_2)^p(y)\right)^{\frac{1}{p}}.\tag{3.6}$$

Thus adding (3.4) and (3.6) yields the desired inequality. □

Inequality (3.1) is referred to as the reverse Minkowski inequality via generalized conformable fractional integrals.

Theorem 3.2 *For $\varsigma>0$, $\varrho\in(0,1]$, $\tau\in\mathcal{R}$ and $\varrho+\tau\neq 0$ with $p\ge 1$ let there be two positive functions f_1, f_2 on $[0,\infty)$ such that, for all $y>r_1$, ${}_{\varrho}^{\tau}\mathcal{K}_{r_1^+}^{\varsigma}f_2^p(y)<\infty$ and ${}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_1^p(y)<\infty$. If $0<\theta_1\le\frac{f_1(\eta)}{f_2(\eta)}\le\theta_2$ for $\theta_1,\theta_2\in\mathcal{R}^+$ and for all $\eta\in[r_1,y]$, then*

$$\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_1^p(y)\right)^{\frac{2}{p}}+\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_2^p(y)\right)^{\frac{2}{p}}\le\left(\frac{(\theta_1+1)(\theta_2+1)}{\theta_2}-2\right)\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_1^p(y)\right)^{\frac{1}{p}}\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_1^p(y)\right)^{\frac{1}{p}}.\tag{3.7}$$

Proof The product of inequalities (3.4) and (3.6) yields

$$\left(\frac{(\theta_1+1)(\theta_2+1)}{\theta_2}-2\right)\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_1^p(y)\right)^{\frac{1}{p}}\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_2^p(y)\right)^{\frac{1}{p}}\le\left[\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}(f_1+f_2)^p(y)\right)^{\frac{1}{p}}\right]^2.\tag{3.8}$$

Now, utilizing the Minkowski inequality to the right hand side of (3.8), one obtains

$$\begin{aligned}&\left[\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}(f_1+f_2)^p(y)\right)^{\frac{1}{p}}\right]^2\\&\quad\le\left[{}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_1^p(y))^{\frac{1}{p}}+{}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_2^p(y))^{\frac{1}{p}}\right]^2\\&\quad\le\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_1^p(y)\right)^{\frac{2}{p}}+\left({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_2^p(y)\right)^{\frac{2}{p}}+2\left[{}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_1^p(y))^{\frac{1}{p}}\right]\left[{}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_2^p(y))^{\frac{1}{p}}\right].\end{aligned}\tag{3.9}$$

Thus, from inequalities (3.8) and (3.9), we obtain the inequality (3.7). □

4 Certain associated inequalities via generalized conformable fractional integral operators (GCFI)

This section is dedicated to deriving certain associated variants regarding GCFI operator.

Theorem 4.1 *For $\varsigma > 0$, $\varrho \in (0,1]$, $\tau \in \mathcal{R}$, $\varrho + \tau \neq 0$ with $p,q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, suppose that there are two positive functions f_1, f_2 on $[0,\infty)$ such that, for all $y > r_1$, ${}^{\tau}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^p(y) < \infty$ and ${}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_2^p(y) < \infty$. If $0 < \theta_1 \leq \frac{f_1(\eta)}{f_2(\eta)} \leq \theta_2$ for $\theta_1, \theta_2 \in \mathcal{R}^+$ and for all $\eta \in [r_1, y]$, then*

$$\bigl({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^p(y)\bigr)^{\frac{1}{p}} \bigl({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_2^q(y)\bigr)^{\frac{1}{q}} \leq \biggl(\frac{\theta_2}{\theta_1}\biggr)^{\frac{1}{pq}} \bigl({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^{\frac{1}{p}}(y) f_2^{\frac{1}{q}}(y)\bigr). \tag{4.1}$$

Proof Under the given suppositions $\frac{f_1(\eta)}{f_2(\eta)} \leq \theta_2$, $r_1 \leq \eta \leq y$, therefore we have

$$f_2^{\frac{1}{q}}(\eta) \geq \theta_2^{\frac{-1}{q}} f_1^{\frac{1}{q}}(\eta). \tag{4.2}$$

Taking the product of both sides of (4.2) by $f_1^{\frac{1}{p}}(\eta)$, it follows that

$$f_1^{\frac{1}{p}}(\eta) f_2^{\frac{1}{q}}(\eta) \geq \theta_2^{\frac{-1}{q}} f_1(\eta). \tag{4.3}$$

If we multiply both sides of (4.3) with $\frac{1}{\Gamma(\varsigma)\varsigma^{1-\tau-\varrho}}(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho})^{\tau-1}$ and then integrate the subsequent inequality with respect to η from r_1 to y, we obtain

$$\begin{aligned} &\frac{\theta_2^{\frac{-1}{q}}}{\Gamma(\varsigma)} \int_{r_1}^{y} \biggl(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\biggr)^{\tau-1} \frac{f_1(\eta)\,d\eta}{\varsigma^{1-\tau-\varrho}} \\ &\quad \leq \frac{1}{\Gamma(\varsigma)} \int_{r_1}^{y} \biggl(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\biggr)^{\tau-1} \frac{f_1^{\frac{1}{p}}(\eta) f_2^{\frac{1}{q}}(\eta)\,d\eta}{\varsigma^{1-\tau-\varrho}}. \end{aligned} \tag{4.4}$$

Consequently, we have

$$\theta_2^{-\frac{1}{pq}} \bigl({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^p(y)\bigr)^{\frac{1}{p}} \leq \bigl({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^{\frac{1}{p}}(y) f_2^{\frac{1}{q}}(y)\bigr)^{\frac{1}{p}}. \tag{4.5}$$

In contrast, as $\theta_1 f_2(\eta) \leq f_1(\eta)$, we have

$$\theta_1^{\frac{1}{p}} f_2^{\frac{1}{p}}(\eta) \leq f_1^{\frac{1}{p}}(\eta). \tag{4.6}$$

Again, if we multiply both sides of (4.6) by $f_2^{\frac{1}{q}}(\eta)$ and invoke the relation $\frac{1}{p} + \frac{1}{q} = 1$, it yields

$$\theta_1^{\frac{1}{p}} f_2(\eta) \leq f_1^{\frac{1}{p}}(\eta) f_2^{\frac{1}{q}}(\eta). \tag{4.7}$$

If we multiply both sides of (4.7) with $\frac{1}{\Gamma(\varsigma)\varsigma^{1-\tau-\varrho}}(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho})^{\tau-1}$ and then integrate the subsequent inequality with respect to η from r_1 to y, we obtain

$$\theta_1^{\frac{1}{pq}} \bigl({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_2(y)\bigr)^{\frac{1}{q}} \leq \bigl({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^{\frac{1}{p}}(y) f_2^{\frac{1}{q}}(y)\bigr)^{\frac{1}{q}}. \tag{4.8}$$

Multiplying (4.5) and (4.8), the required inequality (4.1) can be concluded. □

Theorem 4.2 *For $\varsigma > 0$, $\varrho \in (0,1]$, $\tau \in \mathcal{R}$, $\varrho + \tau \neq 0$ with $p,q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that there are two positive functions f_1, f_2 on $[0,\infty)$ such that, for all $y > r_1$, ${}^{\tau}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^p(y) < \infty$ and ${}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_2^p(y) < \infty$. If $0 < \theta_1 \leq \frac{f_1(\eta)}{f_2(\eta)} \leq \theta_2$ for $\theta_1, \theta_2 \in \mathcal{R}^+$ and for all $\eta \in [r_1, y]$, then*

$$\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1(y)f_2(y)\right) \leq \frac{2^{p-1}\theta_2^p}{p(\theta_2+1)^p}\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+}\left(f_1^p + f_2^p\right)(y)\right) + \frac{2^{q-1}}{p(\theta_1+1)^p}\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+}\left(f_1^q + f_2^q\right)(y)\right). \quad (4.9)$$

Proof By the given assumption $\frac{f_1(\eta)}{f_2(\eta)} < \theta_2$, we have

$$(\theta_2+1)^p f_1^p(\eta) \leq \theta_2^p (f_1+f_2)^p(\eta). \quad (4.10)$$

If we multiply both sides of (4.10) with $\frac{1}{\Gamma(\varsigma)\varsigma^{1-\tau-\varrho}}(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho})^{\tau-1}$ and then integrate the subsequent inequality with respect to η from r_1 to y, we obtain

$$\begin{aligned}&\frac{(\theta_2+1)^p}{\Gamma(\varsigma)}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\tau-1}\frac{f_1^p(\eta)\,d\eta}{\varsigma^{1-\tau-\varrho}}\\&\quad\leq \frac{\theta_2^p}{\Gamma(\varsigma)}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\tau-1}\frac{(f_1+f_2)^p(\eta)\,d\eta}{\varsigma^{1-\tau-\varrho}}.\end{aligned} \quad (4.11)$$

It follows that

$$\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^p(y)\right) \leq \frac{\theta_2^p}{(\theta_2+1)^p}\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+}(f_1+f_2)^p(y)\right). \quad (4.12)$$

In contrast, using $0 < \theta_1 \leq \frac{f_1(\eta)}{f_2(\eta)}$, $r_1 < \eta < y$, we have

$$(\theta_1+1)^q f_2^q(\eta) \leq (f_1+f_2)^q(\eta). \quad (4.13)$$

Again, if we multiply both sides of (4.13) with $\frac{1}{\Gamma(\varsigma)\varsigma^{1-\tau-\varrho}}(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho})^{\tau-1}$ and then integrate the subsequent inequality with respect to η from r_1 to y, we obtain

$$\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_2^q(y)\right) \leq \frac{1}{(\theta_1+1)^q}\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+}(f_1+f_2)^q(y)\right). \quad (4.14)$$

Now, taking into account Young's inequality,

$$f_1(\eta)f_2(\eta) \leq \frac{f_1^p(\eta)}{p} + \frac{f_2^q(\eta)}{q}. \quad (4.15)$$

Now, if we multiply both sides of (4.15) with $\frac{1}{\Gamma(\varsigma)\varsigma^{1-\tau-\varrho}}(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho})^{\tau-1}$ and then integrate the subsequent inequality with respect to η from r_1 to y, we obtain

$$\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+}(f_1f_2)(y)\right) \leq \frac{1}{p}\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^p(y)\right) + \frac{1}{q}\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_2^q(y)\right). \quad (4.16)$$

With the aid of (4.12) and (4.14) with (4.16), one obtains

$$\begin{aligned}
&\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}(f_1f_2)(y)\big)\\
&\quad\leq \frac{1}{p}\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_1^p(y)\big)+\frac{1}{q}\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_2^q(y)\big)\\
&\quad\leq \frac{\theta_2^p}{p(\theta_2+1)^p}\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}(f_1+f_2)^p(y)\big)+\frac{1}{q(\theta_1+1)^q}\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}(f_1+f_2)^q(y)\big).
\end{aligned}\tag{4.17}$$

Using the inequality $(\mu+\nu)^s \leq 2^{s-1}(\mu^s+\nu^s)$, $s>1$, $\mu,\nu>0$, one can obtain

$$\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}(f_1+f_2)^p(y)\big)\leq 2^{p-1}\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}\big(f_1^p+f_2^p\big)(y)\big)\tag{4.18}$$

and

$$\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}(f_1+f_2)^q(y)\big)\leq 2^{q-1}\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}\big(f_1^q+f_2^q\big)(y)\big).\tag{4.19}$$

Hence, the proof of (4.9) can be concluded from (4.17), (4.18), and (4.19) collectively. □

Theorem 4.3 *For $\varsigma>0$, $\varrho\in(0,1]$, $\tau\in\mathcal{R}$, $\varrho+\tau\neq 0$ with $p\geq 1$ and let there be two positive functions f_1, f_2 on $[0,\infty)$ such that, for all $y>r_1$, ${}_{\varrho}^{\tau}\mathcal{K}_{r_1^+}^{\varsigma}f_1^p(y)<\infty$ and ${}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}f_2^p(y)<\infty$. If $0<\lambda<\theta_1\leq\frac{f_1(\eta)}{f_2(\eta)}\leq\theta_2$ for $\theta_1,\theta_2\in\mathcal{R}^+$ and for all $\eta\in[r_1,y]$, then*

$$\begin{aligned}
\frac{\theta_2+1}{\theta_2-\lambda}\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}\big(f_1(y)-\lambda f_2(y)\big)\big)&\leq\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}(f_1)^p(y)\big)^{\frac{1}{p}}+\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}(f_2)^p(y)\big)^{\frac{1}{p}}\\
&\leq\frac{\theta_1+1}{\theta_1-\lambda}\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}\big(f_1(y)-\lambda f_2(y)\big)\big)^{\frac{1}{p}}.
\end{aligned}\tag{4.20}$$

Proof Under the given supposition $0<\lambda<\theta_1\leq\frac{f_1^p(\eta)}{f_2^p(\eta)}\leq\theta_2$, we have

$$\begin{aligned}
\theta_1\lambda\leq\theta_2\lambda\quad&\Rightarrow\quad\theta_1\lambda+\theta_1\leq\theta_1\lambda+\theta_2\leq\theta_2\lambda+\theta_2\\
&\Rightarrow\quad(\theta_2+1)(\theta_1-\lambda)\leq(\theta_1+1)(\theta_2-\lambda).
\end{aligned}$$

It follows that

$$\frac{\theta_2+1}{\theta_2-\lambda}\leq\frac{\theta_1+1}{\theta_1-\lambda}.$$

Also, we have

$$\theta_1-\lambda\leq\frac{f_1(\eta)-\lambda f_2(\eta)}{f_2(\eta)}\leq\theta_2-\lambda,$$

implying

$$\frac{(f_1(\eta)-\lambda f_2(\eta))^p}{(\theta_2-\lambda)^p}\leq f_2^p(\eta)\leq\frac{(f_1(\eta)-\lambda f_2(\eta))^p}{(\theta_1-\lambda)^p}.\tag{4.21}$$

Furthermore, we have

$$\frac{1}{\theta_2} \le \frac{f_2(\eta)}{f_1(\eta)} \le \frac{1}{\theta_1} \quad \Rightarrow \quad \frac{\theta_1 - \lambda}{\lambda\theta_1} \le \frac{f_1(\eta) - \lambda f_2(\eta)}{\lambda f_1(\eta)} \le \frac{\theta_2 - \lambda}{\theta_2 \lambda}.$$

It follows that

$$\left(\frac{\theta_2}{\theta_2 - \lambda}\right)^p \big(f_1(\eta) - \lambda f_2(\eta)\big)^p \le f_1^p(\eta) \le \left(\frac{\theta_1}{\theta_1 - \lambda}\right)^p \big(f_1(\eta) - \lambda f_2(\eta)\big)^p. \tag{4.22}$$

If we multiply both sides of (4.22) with $\frac{1}{\Gamma(\varsigma)\varsigma^{1-\tau-\varrho}}(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho})^{\tau-1}$ and then integrate the subsequent inequality with respect to η from r_1 to y, we obtain

$$\begin{aligned}
&\frac{1}{(\theta_2 - \lambda)^p \Gamma(\varsigma)} \int_{r_1}^{y} \left(\frac{y^{\tau+\varrho} - \eta^{\tau+\varrho}}{\tau + \varrho}\right)^{\tau-1} \frac{(f_1(\eta) - \lambda f_2(\eta))^p \, d\eta}{\varsigma^{1-\tau-\varrho}} \\
&\quad \le \frac{1}{\Gamma(\varsigma)} \int_{r_1}^{y} \left(\frac{y^{\tau+\varrho} - \eta^{\tau+\varrho}}{\tau + \varrho}\right)^{\tau-1} \frac{(f_2^p(\eta)) \, d\eta}{\varsigma^{1-\tau-\varrho}} \\
&\quad \le \frac{1}{(\theta_1 - \lambda)^p \Gamma(\varsigma)} \int_{r_1}^{y} \left(\frac{y^{\tau+\varrho} - \eta^{\tau+\varrho}}{\tau + \varrho}\right)^{\tau-1} \frac{(f_1(\eta) - \lambda f_2(\eta))^p \, d\eta}{\varsigma^{1-\tau-\varrho}}.
\end{aligned}$$

Accordingly, it can be written as

$$\begin{aligned}
\frac{1}{\theta_2 - \lambda} \big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma} \big(f_1(y) - \lambda f_2(y)\big)^p\big)^{\frac{1}{p}} &\le \big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma} (f_2)^p(y)\big)^{\frac{1}{p}} \\
&\le \frac{1}{\theta_1 - \lambda} \big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma} \big(f_1(y) - \lambda f_2(y)\big)^p\big)^{\frac{1}{p}}.
\end{aligned} \tag{4.23}$$

Adopting the same technique with (4.22), one obtains

$$\begin{aligned}
\frac{\theta_2}{\theta_2 - \lambda} \big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma} \big(f_1(y) - \lambda f_2(y)\big)^p\big)^{\frac{1}{p}} &\le \big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma} f_1^p(y)\big)^{\frac{1}{p}} \\
&\le \frac{\theta_1}{\theta_1 - \lambda} \big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma} \big(f_1(y) - \lambda f_2(y)\big)^p\big)^{\frac{1}{p}}.
\end{aligned} \tag{4.24}$$

Hence, by adding inequalities (4.23) and (4.24), we attain the inequality (4.20). □

Theorem 4.4 *For $\varsigma > 0$, $\varrho \in (0, 1]$, $\tau \in \mathcal{R}$, $\varrho + \tau \neq 0$ with $p \ge 1$ and let there are two positive functions f_1, f_2 on $[0, \infty)$ such that, for all $y > r_1$, ${}_{\varrho}^{\tau}\mathcal{K}_{r_1^+}^{\varsigma} f_1^p(y) < \infty$ and ${}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma} f_2^p(y) < \infty$. If $0 < \hbar \le f_1(\eta) \le \mathcal{H}$ and $0 < \mathfrak{m} \le f_2(\eta) \le \mathcal{M}$ for $\theta_1, \theta_2 \in \mathcal{R}^+$ and for all $\eta \in [r_1, y]$, then*

$$\big({}_{\varrho}^{y}\mathcal{K}_{r_1}^{y} f_1^p(\eta)\big)^{\frac{1}{p}} + \big({}_{\varrho}^{y}\mathcal{K}_{r_1}^{y} f_2^p(\eta)\big)^{\frac{1}{p}} \le \frac{\mathcal{H}(\hbar + \mathcal{M}) + \mathcal{M}(\mathcal{H} + \mathfrak{m})}{(\mathfrak{m} + \mathcal{H})(\hbar + \mathcal{M})} \big({}_{\varrho}^{y}\mathcal{K}_{r_1}^{y} (f_1 + f_2)^p(\eta)\big)^{\frac{1}{p}}. \tag{4.25}$$

Proof Under the given suppositions, observe that

$$\frac{1}{\mathcal{M}} \le \frac{1}{f_2(\eta)} \le \frac{1}{\mathfrak{m}}. \tag{4.26}$$

Conducting the product between (4.26) and $0 \le \hbar \le f_1(\eta) \le \mathcal{H}$, we have

$$\frac{\hbar}{\mathcal{M}} \le \frac{f_1(\eta)}{f_2(\eta)} \le \frac{\mathcal{H}}{\mathfrak{m}}. \tag{4.27}$$

From (4.27), we obtain

$$f_2^p(\eta) \leq \left(\frac{\mathcal{M}}{\hbar + \mathcal{M}}\right)^p \bigl(f_1(\eta) + f_2(\eta)\bigr)^p \tag{4.28}$$

and

$$f_1^p(\eta) \leq \left(\frac{\mathcal{H}}{\mathfrak{m} + \mathcal{H}}\right)^p \bigl(f_1(\eta) + f_2(\eta)\bigr)^p. \tag{4.29}$$

If we multiply both sides of (4.28) with $\frac{1}{\Gamma(\varsigma)\varsigma^{1-\tau-\varrho}}(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho})^{\tau-1}$ and then integrate the subsequent inequality with respect to η from r_1 to y, we obtain

$$\begin{aligned}&\frac{1}{\Gamma(\varsigma)}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\tau-1}\frac{f_2^p(\eta)\,d\eta}{\varsigma^{1-\tau-\varrho}}\\&\quad\leq \frac{\mathcal{M}^p}{(\hbar+\mathcal{M})^p\Gamma(\varsigma)}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\tau-1}\frac{(f_1(y)+f_2(y))^p(\eta)\,d\eta}{\varsigma^{1-\tau-\varrho}}.\end{aligned} \tag{4.30}$$

Accordingly,

$$\bigl({}^{y}_{\varrho}\mathcal{K}^{y}_{r_1} f_2^p(y)\bigr)^{\frac{1}{p}} \leq \frac{\mathcal{M}}{\hbar + \mathcal{M}}\bigl({}^{y}_{\varrho}\mathcal{K}^{y}_{r_1}(f_1 + f_2)^p(y)\bigr)^{\frac{1}{p}}. \tag{4.31}$$

Adopting the same technique as (4.29), one obtains

$$\bigl({}^{y}_{\varrho}\mathcal{K}^{y}_{r_1} f_1^p(y)\bigr)^{\frac{1}{p}} \leq \frac{\mathcal{H}}{\mathfrak{m} + \mathcal{H}}\bigl({}^{y}_{\varrho}\mathcal{K}^{y}_{r_1}(f_1 + f_2)^p(y)\bigr)^{\frac{1}{p}}. \tag{4.32}$$

Hence, by adding (4.31) and (4.32), we obtain the inequality (4.25). □

Theorem 4.5 *For* $\varsigma > 0$, $\varrho \in (0,1]$, $\tau \in \mathcal{R}$, $\varrho + \tau \neq 0$ *with* $p \geq 1$ *let there be two positive functions* f_1, f_2 *on* $[0,\infty)$ *such that, for all* $y > r_1$, ${}^{\tau}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^p(y) < \infty$ *and* ${}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+} f_1^p(y) < \infty$. *If* $0 < \hbar \leq f_1(\eta) \leq \mathcal{H}$ *and* $0 < \mathfrak{m} \leq f_2(\eta) \leq \mathcal{M}$ *for* $\theta_1, \theta_2 \in \mathcal{R}^+$ *for all* $\eta \in [r_1, y]$, *then*

$$\begin{aligned}\frac{1}{\theta_2}\bigl({}^{y}_{\varrho}\mathcal{K}^{y}_{r_1} f_1(y)f_2(y)\bigr) &\leq \frac{1}{(\theta_1+1)(\theta_2+1)}\bigl({}^{y}_{\varrho}\mathcal{K}^{y}_{r_1}\bigl(f_1(y)+f_2(y)\bigr)\bigr)^2\\&\leq \frac{1}{\theta_1}\bigl({}^{y}_{\varrho}\mathcal{K}^{y}_{r_1} f_1(y)f_2(y)\bigr).\end{aligned} \tag{4.33}$$

Proof Under the given suppositions, $0 < \theta_1 \leq \frac{f_1(\eta)}{f_2(\eta)} \leq \theta_2$, it follows that

$$f_2(\eta)(\theta_1 + 1) \leq f_1(\eta) + f_2(\eta) \leq f_2(\eta)(\theta_2 + 1). \tag{4.34}$$

Additionally, we have $\frac{1}{\theta_2} \leq \frac{f_2(\eta)}{f_1(\eta)} \leq \frac{1}{\theta_1}$, which yields

$$\left(\frac{\theta_2 + 1}{\theta_2}\right) f_1(\eta) \leq f_1(\eta) + f_2(\eta) \leq \left(\frac{\theta_1 + 1}{\theta_1}\right) f_1(\eta). \tag{4.35}$$

The product of (4.34) and (4.35) gives

$$\frac{f_1(\eta)f_2(\eta)}{\theta_2} \leq \frac{(f_1(\eta) + f_2(\eta))^2}{(\theta_1 + 1)(\theta_2 + 1)} \leq \frac{f_1(\eta)f_2(\eta)}{\theta_1}. \tag{4.36}$$

Now, if we multiply both sides of (4.36) with $\frac{1}{\Gamma(\varsigma)\varsigma^{1-\tau-\varrho}}(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho})^{\tau-1}$ and then integrate the subsequent inequality with respect to η from r_1 to y, we obtain

$$\begin{aligned}&\frac{1}{\theta_2\Gamma(\varsigma)}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\tau-1}\frac{f_1(\eta)f_2(\eta)\,d\eta}{\varsigma^{1-\tau-\varrho}}\\&\quad\leq\frac{1}{\Gamma(\varsigma)((\theta_1+1)(\theta_2+1))}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\tau-1}\frac{(f_1(\eta)+f_2(\eta))^2\,d\eta}{\varsigma^{1-\tau-\varrho}}\\&\quad\leq\frac{1}{\theta_1\Gamma(\varsigma)}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\tau-1}\frac{f_1(\eta)f_2(\eta)\,d\eta}{\varsigma^{1-\tau-\varrho}}.\end{aligned}$$

One observes that

$$\frac{1}{\theta_2}\left({}^{y}_{\varrho}\mathcal{K}^{y}_{r_1}f_1(y)f_2(y)\right)\leq\frac{1}{(\theta_1+1)(\theta_2+1)}\left({}^{y}_{\varrho}\mathcal{K}^{y}_{r_1}\left(f_1(y)+f_2(y)\right)\right)^2\leq\frac{1}{\theta_1}\left({}^{y}_{\varrho}\mathcal{K}^{y}_{r_1}f_1(y)f_2(y)\right),$$

which is the desired result. □

Theorem 4.6 *For $\varsigma>0$, $\varrho\in(0,1]$, $\tau\in\mathcal{R}$, $\varrho+\tau\neq0$ with $p\geq1$ and let there are two positive functions f_1, f_2 on $[0,\infty)$ such that, for all $y>r_1$, ${}^{\tau}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+}f_1^p(y)<\infty$ and ${}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+}f_2^p(y)<\infty$. If $0<\hbar\leq f_1(\eta)\leq\mathcal{H}$ and $0<\theta_1\leq f_2(\eta)\leq\theta_2$ for $\theta_1,\theta_2\in\mathcal{R}^+$ and for all $\eta\in[r_1,y]$, then*

$$\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+}f_1^p(y)\right)^{\frac{1}{p}}+\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+}f_2^p(y)\right)^{\frac{1}{p}}\leq2\left({}^{y}_{\varrho}\mathcal{K}^{\varsigma}_{r_1^+}\mathcal{U}^p\left(f_1(y),f_2(y)\right)\right)^{\frac{1}{p}},\tag{4.37}$$

where $\mathcal{U}(f_1(\eta),f_2(\eta))=\max\{\theta_2[(1+\frac{\theta_2}{\theta_1})f_1(y)-\theta_2f_2(y)],\frac{(\theta_1+\theta_2)f_2(y)-f_1(y)}{\theta_1}\}$.

Proof Under the given suppositions $0<\theta_1\leq\frac{f_1(\eta)}{f_2(\eta)}\leq\theta_2$, $r_1\leq\eta\leq y$, we have

$$0<\theta_1\leq\theta_2+\theta_1-\frac{f_1(\eta)}{f_2(\eta)}\tag{4.38}$$

and

$$\theta_2+\theta_1-\frac{f_1(\eta)}{f_2(\eta)}\leq\theta_2.\tag{4.39}$$

From (4.38) and (4.39), one obtains

$$f_2(\eta)<\frac{(\theta_2+\theta_1)f_2(\eta)-f_1(\eta)}{\theta_1}\leq\mathcal{U}\left(f_1(\eta),f_2(\eta)\right),\tag{4.40}$$

where $\mathcal{U}(f_1(\eta),f_2(\eta))=\max\{\theta_2[(1+\frac{\theta_2}{\theta_1})f_1(y)-\theta_2f_2(y)],\frac{(\theta_1+\theta_2)f_2(y)-f_1(y)}{\theta_1}\}$. Also, from the given supposition $0<\frac{1}{\theta_2}\leq\frac{f_2(\eta)}{f_1(\eta)}\leq\frac{1}{\theta_1}$, one has

$$\frac{1}{\theta_2}\leq\frac{1}{\theta_2}+\frac{1}{\theta_1}-\frac{f_2(\eta)}{f_1(\eta)}\tag{4.41}$$

and

$$\frac{1}{\theta_2}+\frac{1}{\theta_1}-\frac{f_2(\eta)}{f_1(\eta)}\leq\frac{1}{\theta_1}.\tag{4.42}$$

From (4.41) and (4.42), we get

$$\frac{1}{\theta_2} \le \frac{(\frac{1}{\theta_1} + \frac{1}{\theta_2})f_1(\eta) - f_2(\eta)}{f_1(\eta)} \le \frac{1}{\theta_1}, \tag{4.43}$$

implying

$$\begin{aligned} f_1(\eta) &\le \theta_2\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)f_1(\eta) - \theta_2 f_2(\eta) \\ &= \frac{\theta_2(\theta_1 + \theta_2)f_1(\eta) - \theta_2^2\theta_1 f_2(\eta)}{\theta_1\theta_2} \\ &= \left(\frac{\theta_2}{\theta_1} + 1\right)f_1(\eta) - \theta_2 f_2(\eta) \\ &\le \theta_2\left[\left(\frac{\theta_2}{\theta_1} + 1\right)f_1(\eta) - \theta_2 f_2(\eta)\right] \\ &\le \mathcal{U}\big(f_1(\eta), f_2(\eta)\big). \end{aligned} \tag{4.44}$$

From (4.40) and (4.44), we have

$$f_1^p(\eta) \le \mathcal{U}^p\big(f_1(\eta), f_2(\eta)\big) \tag{4.45}$$

and

$$f_2^p(\eta) \le \mathcal{U}^p\big(f_1(\eta), f_2(\eta)\big). \tag{4.46}$$

If we multiply both sides of (4.45) with $\frac{1}{\Gamma(\varsigma)\varsigma^{1-\tau-\varrho}}(\frac{y^{\tau+\varrho}-\eta^{\tau+\varrho}}{\tau+\varrho})^{\tau-1}$ and then integrate the subsequent inequality with respect to η from r_1 to y, we obtain

$$\begin{aligned} &\frac{1}{\Gamma(\varsigma)}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho} - \eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\tau-1}\frac{f_1^p(\eta)\,d\eta}{\varsigma^{1-\tau-\varrho}} \\ &\quad\le \frac{1}{\Gamma(\varsigma)}\int_{r_1}^{y}\left(\frac{y^{\tau+\varrho} - \eta^{\tau+\varrho}}{\tau+\varrho}\right)^{\tau-1}\frac{\mathcal{U}^p(f_1(\eta), f_2(\eta))\,d\eta}{\varsigma^{1-\tau-\varrho}}. \end{aligned} \tag{4.47}$$

Accordingly,

$$\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma} f_1^p(y)\big)^{\frac{1}{p}} \le \big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}\mathcal{U}^p\big(f_1(y), f_2(y)\big)\big)^{\frac{1}{p}}. \tag{4.48}$$

Adopting the same technique for (4.46), we have

$$\big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma} f_2^p(y)\big)^{\frac{1}{p}} \le \big({}_{\varrho}^{y}\mathcal{K}_{r_1^+}^{\varsigma}\mathcal{U}^p\big(f_1(y), f_2(y)\big)\big)^{\frac{1}{p}}. \tag{4.49}$$

Hence, by adding (4.48) and (4.49), we obtain the inequality (4.37). □

5 Concluding remarks

This paper begins with a compact evaluation of fractional integrals in the sense of Riemann–Liouville and Riemann–Liouville type conformable fractional integral operators in addition to a new fractional integral operator according to Khan et al. [14]. We generalize the reverse Minkowski inequalities via generalized conformable fractional integrals; specifically, the inequality concerning fractional integrals in the Riemann–Liouville sense is given [44]. The associated significant variants regarding generalized conformable fractional integrals are demonstrated. Numerous variants can be established for the application of several defined fractional integral operators. One of the well-known inequalities is the Chebyshev inequality lately derived in [38]. Finally, this concept can be extended in the form of a $\mathcal{K}$ analogue for deriving similar types of results and these are also helpful for establishing the refinements of several existing results in the literature.

Acknowledgements
The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding
The author T. Abdeljawad would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 5 May 2020 Accepted: 17 June 2020 Published online: 29 June 2020

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