# Radii problems for some classes of analytic functions associated with Legendre polynomials of odd degree 

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#### Abstract

The aim of the present paper is to study radii problems for two general classes including various known subclasses of analytic functions associated with the normalized form of Legendre polynomials of odd degree. We also obtain some special cases of the main results presented here with some useful examples.

MSC: 30C10; 30C15; 30C45 Keywords: Legendre polynomials; Starlike functions; Convex functions; Radii problems; Strongly starlike functions


## 1 Introduction and preliminaries

Let $\mathbb{U}(r):=\{z \in \mathbb{C}:|z|<r\}$ be the disk in the complex plane $\mathbb{C}$ centered at the origin, with radius $r>0$, and denote by $\mathbb{U}:=\mathbb{U}(1)$ the unit disk. We denote by $\mathcal{A}$ the class of analytic functions in the unit disk $\mathbb{U}$ normalized by $f(0)=f^{\prime}(0)-1=0$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions.

We denote by $\mathcal{S}^{*}(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order $\alpha$ in $\mathbb{U}$, that is,

$$
\mathcal{S}^{*}(\alpha):=\left\{f \in \mathcal{A}: \operatorname{Re} \frac{z f^{\prime}(z)}{f(z)}>\alpha, z \in \mathbb{U}\right\}, \quad 0 \leq \alpha<1 .
$$

Also, let us denote by $\widetilde{\mathcal{S}}^{*}(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are strongly starlike of order $\alpha$ in $\mathbb{U}$, that is,

$$
\widetilde{\mathcal{S}}^{*}(\alpha):=\left\{f \in \mathcal{A}:\left|\arg \frac{z f^{\prime}(z)}{f(z)}\right|<\frac{\alpha \pi}{2}, z \in \mathbb{U}\right\}, \quad 0 \leq \alpha<1 .
$$

Thus, in particular, $\mathcal{S}^{*}:=\mathcal{S}^{*}(0)=\widetilde{\mathcal{S}}^{*}(1)$ represents the class of starlike functions in the open unit disk $\mathbb{U}$.

The real numbers

$$
r_{\alpha}^{*}(f):=\sup \left\{r>0: \operatorname{Re} \frac{z f^{\prime}(z)}{f(z)}>\alpha, z \in \mathbb{U}(r)\right\}
$$

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and

$$
\tilde{r}_{\alpha}^{*}(f):=\sup \left\{r>0:\left|\arg \frac{z f^{\prime}(z)}{f(z)}\right|<\frac{\alpha \pi}{2}, z \in \mathbb{U}(r)\right\}
$$

are called the radius of starlikeness of order $\alpha$ and the radius of strong starlikeness of order $\alpha$ of the function $f$, respectively. In particular, $r^{*}(f):=r_{0}^{*}(f)=r_{1}^{\tilde{*}}(f)$ is called the radius of starlikeness of the function $f$.

Recently, Darus et al. [8] considered the general class $k-\mathcal{U C S T}(\alpha)$ defined as follows.

Definition 1 Let $f \in \mathcal{A}$. Then $f \in k-\mathcal{U C S T}(\alpha)$ if

$$
\operatorname{Re} \frac{\left(z f^{\prime}(z)\right)^{\prime}}{f^{\prime}(z)}>k\left|(1-\alpha) \frac{z f^{\prime}(z)}{f(z)}+\alpha \frac{\left(z f^{\prime}(z)\right)^{\prime}}{f^{\prime}(z)}-1\right|, \quad z \in \mathbb{U}
$$

where $k \geq 0$ and $0 \leq \alpha \leq 1$.

## Remark 1

(i) For $k=0$ we get the class $0-\mathcal{U C S T}(\alpha)=: \mathcal{K}$, which includes the well-known class of convex functions, that is,

$$
\operatorname{Re}\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)>0, \quad z \in \mathbb{U}
$$

(ii) For $\alpha=1$ we obtain the class $k-\mathcal{U C S T}(1)=: k-\mathcal{U C V}$ (see [11]), which includes the class of $k$-uniformly convex functions, that is,

$$
\operatorname{Re}\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)>k\left|\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right|, \quad z \in \mathbb{U}
$$

(iii) For $k=1$ and $\alpha=1$ we get the class $1-\mathcal{U C S T}(1)=: \mathcal{U C V}$ (see [13]), which includes the class of uniformly convex functions, that is,

$$
\operatorname{Re}\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)>\left|\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right|, \quad z \in \mathbb{U} .
$$

(iv) For $\alpha=0$ we have the class $k-\mathcal{U C S T}(0)=: k-\mathcal{M} \mathcal{N}$ (see [14]), which represents the functions $f \in \mathcal{A}$ satisfying

$$
\operatorname{Re}\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)>k\left|\frac{z f^{\prime}(z)}{f(z)}-1\right|, \quad z \in \mathbb{U}
$$

(v) For $k=1$ and $\alpha=0$ we get the class $1-\mathcal{U C S T}(0)=: \mathcal{M N}$ (see [15]), which represents the functions $f \in \mathcal{A}$ satisfying

$$
\operatorname{Re}\left(1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right)>\left|\frac{z f^{\prime}(z)}{f(z)}-1\right|, \quad z \in \mathbb{U}
$$

The real number

$$
r_{\alpha}^{k-\mathcal{U C S T}}(f):=\sup \left\{r>0: \operatorname{Re} \frac{\left(z f^{\prime}(z)\right)^{\prime}}{f^{\prime}(z)}>k\left|(1-\alpha) \frac{z f^{\prime}(z)}{f(z)}+\alpha\left(\frac{\left(z f^{\prime}(z)\right)^{\prime}}{f^{\prime}(z)}\right)-1\right|, z \in \mathbb{U}(r)\right\}
$$

is called the $k-\mathcal{U C S T}(\alpha)$ radius of the function $f$. We note that

$$
\begin{aligned}
& r_{\alpha}^{0-\mathcal{U C S T}}(f)=: r^{c}(f), \quad r_{1}^{k-\mathcal{U C S T}}(f)=: r^{u c v}(f), \quad r_{1}^{1-\mathcal{U C S T}}(f)=: r^{u c}(f), \\
& r_{0}^{k-\mathcal{U C S T}}(f)=: r^{k-\mathcal{M N}}(f), \quad r_{0}^{1-\mathcal{U C S T}}(f)=: r^{\mathcal{M N}}(f) .
\end{aligned}
$$

We recall that the Legendre polynomials are solutions of the Legendre differential equation

$$
\frac{\mathrm{d}}{\mathrm{~d} z}\left[\left(1-z^{2}\right) \frac{\mathrm{d}}{\mathrm{~d} z} P_{n}(z)\right]+n(n+1) P_{n}(z)=0, \quad n \in \mathbb{N}:=\{1,2, \ldots\}
$$

while the Rodrigues formula for the Legendre polynomials is

$$
P_{n}(z)=\frac{1}{2^{n} n!} \frac{\mathrm{d}^{n}}{\mathrm{~d} z^{n}}\left[\left(z^{2}-1\right)^{n}\right], \quad n \in \mathbb{N} .
$$

The Legendre polynomials are symmetric or antisymmetric, that is,

$$
P_{n}(-z)=(-1)^{n} P_{n}(z), \quad n \in \mathbb{N} .
$$

Since we will study different radius properties for the Legendre polynomials of odd degree, let us consider the following normalized form of the Legendre $P_{2 n-1}$ polynomials, that is, $\mathcal{P}_{2 n-1}$ given by

$$
\mathcal{P}_{2 n-1}(z):=\frac{P_{2 n-1}(z)}{P_{2 n-1}^{\prime}(0)}=z+a_{3} z^{3}+\cdots+a_{2 n-1} z^{2 n-1}
$$

It is well known that the Rodrigues formula implies that the Legendre polynomials of odd degree have only real roots, and the roots of $P_{2 n-1}(z)$ are $0=z_{0}<z_{1}<\cdots<z_{n-1}$ and $-z_{1},-z_{2}, \ldots,-z_{n-1}$, while the product representation of the polynomial $\mathcal{P}_{2 n-1}$ is

$$
\begin{equation*}
\mathcal{P}_{2 n-1}(z)=a_{2 n-1} z\left(z^{2}-z_{1}^{2}\right)\left(z^{2}-z_{2}^{2}\right) \cdots\left(z^{2}-z_{n-1}^{2}\right) \tag{1}
\end{equation*}
$$

Bulut and Engel [7] have obtained the radius of starlikeness, convexity, and uniform convexity (see $[10,12,13]$ ) of the normalized form of the Legendre polynomial of odd degree. In the recent years, several authors determined the radius of starlikeness, convexity, and uniform convexity for some special functions, that is a relative new direction in the geometric function theory (see, for example, [1-6]).
In the present paper we obtain the radius of strong starlikeness and other related radius of the normalized form of Legendre polynomials of odd degree, and the technique of the proofs used in our paper is similar to that of several papers [1, 3-6]. Further, our results are well supported by some examples.

In order to prove our main results, we require the following lemmas.

Lemma 1 ([7, Lemma 1.1]) If $|z| \leq r<\gamma$, where $\gamma>0$, then we have

$$
\begin{equation*}
\operatorname{Re} \frac{z}{\gamma-z} \leq \frac{r}{\gamma-r}, \tag{2}
\end{equation*}
$$

$$
\begin{align*}
& \left|\frac{z}{\gamma-z}\right| \leq \frac{r}{\gamma-r},  \tag{3}\\
& \left|\frac{z}{(\gamma-z)^{2}}\right| \leq \frac{r}{(\gamma-r)^{2}} . \tag{4}
\end{align*}
$$

Lemma 2 ([9, Lemma 3.1]) If

$$
R_{a} \leq(\operatorname{Re} a) \sin (\pi \gamma / 2)-(\operatorname{Im} a) \cos (\pi \gamma / 2), \quad \text { with } \operatorname{Im} a \geq 0
$$

then the disk $|w-a| \leq R_{a}$ is contained in the sector $|\arg w| \leq \pi \gamma / 2$, where $0<\gamma \leq 1$.

Lemma 3 ([9, Lemma 3.2]) For $|z| \leq r<1$ and $\left|z_{k}\right|=R>r$, we have

$$
\left|\frac{z}{z-z_{k}}+\frac{r^{2}}{R^{2}-r^{2}}\right| \leq \frac{R r}{R^{2}-r^{2}}
$$

## 2 Main results

Using the first of the above lemmas, we obtain the $k-\mathcal{U C S T}(\alpha)$ radius of $\mathcal{P}_{2 n-1}$ as follows.
Theorem 1 The radius of $k-\mathcal{U C S T}(\alpha)$ of $\mathcal{P}_{2 n-1}$ is $r_{\alpha}^{k-\mathcal{U C S T}}\left(\mathcal{P}_{2 n-1}\right)=r_{1}$, where $r_{1}$ denotes the smallest positive root of the equation

$$
k(1-\alpha)\left(\frac{r \mathcal{P}_{2 n-1}^{\prime}(r)}{\mathcal{P}_{2 n-1}(r)}-1\right)+(1+k \alpha) \frac{r \mathcal{P}_{2 n-1}^{\prime \prime}(r)}{\mathcal{P}_{2 n-1}^{\prime}(r)}+1=0 .
$$

Proof Differentiating logarithmically the product representation (1) of the $\mathcal{P}_{2 n-1}$ polynomial, we have

$$
\begin{equation*}
\frac{z \mathcal{P}_{2 n-1}^{\prime}(z)}{\mathcal{P}_{2 n-1}(z)}=1-\sum_{k=1}^{n-1} \frac{2 z^{2}}{z_{k}^{2}-z^{2}}, \quad z \in \mathbb{C} \backslash\left\{ \pm z_{1}, \pm z_{2}, \ldots, \pm z_{n-1}\right\} \tag{5}
\end{equation*}
$$

where $z_{k}$, with $k \in\{1,2, \ldots, n-1\}$, is the $k$ th positive zero of the normalized Legendre polynomial of odd degree. The logarithmical differentiation of the above equality leads to

$$
1+\frac{z \mathcal{P}_{2 n-1}^{\prime \prime}(z)}{\mathcal{P}_{2 n-1}^{\prime}(z)}=\frac{z \mathcal{P}_{2 n-1}^{\prime}(z)}{\mathcal{P}_{2 n-1}(z)}-\frac{\sum_{k=1}^{n-1} \frac{4 z^{2} z_{k}^{2}}{\left(z_{k}^{2}-z^{2}\right)^{2}}}{1-\sum_{k=1}^{n-1} \frac{2 z^{2}}{z_{k}^{2}-z^{2}}}, \quad z \in \mathbb{C} \backslash\left\{ \pm z_{1}, \pm z_{2}, \ldots, \pm z_{n-1}\right\}
$$

that is,

$$
\begin{equation*}
1+\frac{z \mathcal{P}_{2 n-1}^{\prime \prime}(z)}{\mathcal{P}_{2 n-1}^{\prime}(z)}=1-\sum_{k=1}^{n-1} \frac{2 z^{2}}{z_{k}^{2}-z^{2}}-\frac{\sum_{k=1}^{n-1} \frac{4 z^{2} z_{k}^{2}}{\left(z_{k}^{2}-z^{2}\right)^{2}}}{1-\sum_{k=1}^{n-1} \frac{2 z^{2}}{z_{k}^{2}-z^{2}}}, \quad z \in \mathbb{C} \backslash\left\{ \pm z_{1}, \pm z_{2}, \ldots, \pm z_{n-1}\right\} \tag{6}
\end{equation*}
$$

If $|z| \leq r<z_{1}$, then it follows that $|z| \leq r<z_{k}$ for all $k \in\{1,2, \ldots, n-1\}$. Hence, replacing $z$ by $z^{2}$ and $\gamma$ by $z_{k}^{2}$ for all $k \in\{1,2, \ldots, n-1\}$ in inequalities (2) and (4) of Lemma 1 , we get

$$
\operatorname{Re} \frac{z^{2}}{z_{k}^{2}-z^{2}} \leq \frac{r^{2}}{z_{k}^{2}-r^{2}}
$$

and

$$
\left|\frac{z^{2}}{\left(z_{k}^{2}-z^{2}\right)^{2}}\right| \leq \frac{r^{2}}{\left(z_{k}^{2}-r^{2}\right)^{2}},
$$

respectively. Now, for $|z| \leq r<z_{1}$ from (6) and two above inequalities, we deduce the following inequalities:

$$
\begin{align*}
\operatorname{Re}\left(1+\frac{z \mathcal{P}_{2 n-1}^{\prime \prime}(z)}{\mathcal{P}_{2 n-1}^{\prime}(z)}\right) & \geq 1-\sum_{k=1}^{n-1} \operatorname{Re} \frac{2 z^{2}}{z_{k}^{2}-z^{2}}-\frac{\sum_{k=1}^{n-1}\left|\frac{4 z^{2} z_{k}^{2}}{\left(z_{k}^{2}-z^{2}\right)^{2}}\right|}{\left|1-\sum_{k=1}^{n-1} \frac{z^{2}}{z_{k}^{2}-z^{2}}\right|} \\
& \geq 1-\sum_{k=1}^{n-1} \frac{2 r^{2}}{z_{k}^{2}-r^{2}}-\frac{\sum_{k=1}^{n-1} \frac{4 r^{2} z_{k}^{2}}{\left(z_{k}^{2}-r^{2}\right)^{2}}}{1-\sum_{k=1}^{n-1}\left|\frac{2 z^{2}}{z_{k}^{2}-z^{2}}\right|} \\
& \geq 1-\sum_{k=1}^{n-1} \frac{2 r^{2}}{z_{k}^{2}-r^{2}}-\frac{\sum_{k=1}^{n-1} \frac{4 r^{2} z_{k}^{2}}{\left(z_{k}^{2}-r^{2}\right)^{2}}}{1-\sum_{k=1}^{n-1} \frac{2 r^{2}}{z_{k}^{2}-r^{2}}} \\
& =1+\frac{r \mathcal{P}_{2 n-1}^{\prime \prime}(r)}{\mathcal{P}_{2 n-1}^{\prime}(r)} . \tag{7}
\end{align*}
$$

On the other hand, replacing $z$ by $z^{2}$ and $\gamma$ by $z_{k}$ for all $k \in\{1,2, \ldots, n-1\}$ in inequality (3) of Lemma 1 and using relations (5) and (6), we obtain

$$
\begin{align*}
& \left|(1-\alpha) \frac{z \mathcal{P}_{2 n-1}^{\prime}(z)}{\mathcal{P}_{2 n-1}(z)}+\alpha\left(1+\frac{z \mathcal{P}_{2 n-1}^{\prime \prime}(z)}{\mathcal{P}_{2 n-1}^{\prime}(z)}\right)-1\right| \\
& \quad=\left|\frac{z \mathcal{P}_{2 n-1}^{\prime}(z)}{\mathcal{P}_{2 n-1}(z)}-\alpha \frac{\sum_{k=1}^{n-1} \frac{4 z^{2} z_{k}^{2}}{\left(z_{k}^{2}-z^{2}\right)^{2}}}{1-\sum_{k=1}^{n-1} \frac{2 z^{2}}{z_{k}^{2}-z^{2}}}-1\right| \\
& \quad=\left\lvert\,-\sum_{k=1}^{n-1} \frac{2 z^{2}}{z_{k}^{2}-z^{2}}-\alpha \frac{\left.\sum_{k=1}^{n-1} \frac{4 z^{2} z_{k}^{2}}{1-\sum_{k=1}^{n-1} \frac{2 z^{2}}{z_{k}^{2}-z^{2}}}\left|\leq \sum_{k=1}^{n-1}\right| \frac{2 z^{2}}{z_{k}^{2}-z^{2}} \right\rvert\,+\alpha \frac{\sum_{k=1}^{n-1}\left|\frac{4 z^{2} z_{k}^{2}}{\left(z_{k}^{2}-z^{2}\right)^{2}}\right|}{\left|1-\sum_{k=1}^{n-1} \frac{2 z^{2}}{z_{k}^{2}-z^{2}}\right|}}{} \begin{array}{l}
\leq \sum_{k=1}^{n-1} \frac{2 r^{2}}{z_{k}^{2}-r^{2}}+\alpha \frac{\sum_{k=1}^{n-1} \frac{4 r^{2} z_{k}^{2}}{1-\sum_{k=1}^{n-1}\left|\frac{2 z^{2}}{z_{k}^{2}-z^{2}}\right|} \leq \sum_{k=1}^{n-1} \frac{2 r^{2}}{z_{k}^{2}-r^{2}}+\alpha \frac{\sum_{k=1}^{n-1} \frac{4 r^{2} z_{k}^{2}}{\left(z_{k}^{2}-r^{2}\right)^{2}}}{1-\sum_{k=1}^{n-1} \frac{2 r^{2}}{z_{k}^{2}-r^{2}}}}{\quad=(\alpha-1) \frac{r \mathcal{P}_{2 n-1}^{\prime}(r)}{\mathcal{P}_{2 n-1}(r)}-\alpha\left(1+\frac{r \mathcal{P}_{2 n-1}^{\prime \prime}(r)}{\mathcal{P}_{2 n-1}^{\prime}(r)}\right)+1}
\end{array}{ }^{1}\right.
\end{align*}
$$

for $|z| \leq r<z_{1}$.
Therefore, from (7) and (8), for any $k \geq 0$ we have

$$
\begin{aligned}
& \operatorname{Re}\left(1+\frac{z \mathcal{P}_{2 n-1}^{\prime \prime}(z)}{\mathcal{P}_{2 n-1}^{\prime}(z)}\right)-k\left|(1-\alpha) \frac{z \mathcal{P}_{2 n-1}^{\prime}(z)}{\mathcal{P}_{2 n-1}(z)}+\alpha\left(1+\frac{z \mathcal{P}_{2 n-1}^{\prime \prime}(z)}{\mathcal{P}_{2 n-1}^{\prime}(z)}\right)-1\right| \\
& \quad \geq 1+\frac{r \mathcal{P}_{2 n-1}^{\prime \prime}(r)}{\mathcal{P}_{2 n-1}^{\prime}(r)}-k\left[(\alpha-1) \frac{r \mathcal{P}_{2 n-1}^{\prime}(r)}{\mathcal{P}_{2 n-1}(r)}-\alpha\left(1+\frac{r \mathcal{P}_{2 n-1}^{\prime \prime}(r)}{\mathcal{P}_{2 n-1}^{\prime}(r)}\right)+1\right] \\
& \quad=k(1-\alpha) \frac{r \mathcal{P}_{2 n-1}^{\prime}(r)}{\mathcal{P}_{2 n-1}(r)}+(1+k \alpha) \frac{r \mathcal{P}_{2 n-1}^{\prime \prime}(r)}{\mathcal{P}_{2 n-1}^{\prime}(r)}+k(\alpha-1)+1
\end{aligned}
$$

$$
=1-(k+1) \sum_{k=1}^{n-1} \frac{2 r^{2}}{z_{k}^{2}-r^{2}}-(1+k \alpha) \frac{\sum_{k=1}^{n-1} \frac{4 r^{2} z_{k}^{2}}{\left(z_{k}^{2}-r^{2}\right)^{2}}}{1-\sum_{k=1}^{n-1} \frac{2 r^{2}}{z_{k}^{2}-r^{2}}}
$$

for $|z| \leq r<z_{1}$.
Let $\omega: I \rightarrow \mathbb{R}$, where $I$ is the open interval $\left(0, z_{1}\right)$ which is subset of $\mathbb{R}$ be the function defined by

$$
\begin{aligned}
\omega(r) & :=k(1-\alpha) \frac{r \mathcal{P}_{2 n-1}^{\prime}(r)}{\mathcal{P}_{2 n-1}(r)}+(1+k \alpha) \frac{r \mathcal{P}_{2 n-1}^{\prime \prime}(r)}{\mathcal{P}_{2 n-1}^{\prime}(r)}+k(\alpha-1)+1 \\
& =1-(k+1) \sum_{k=1}^{n-1} \frac{2 r^{2}}{z_{k}^{2}-r^{2}}-(1+k \alpha) \frac{\sum_{k=1}^{n-1} \frac{4 r^{2} z_{k}^{2}}{\left(z_{k}^{2}-r^{2}\right)^{2}}}{1-\sum_{k=1}^{n-1} \frac{2 r^{2}}{z_{k}^{2}-r^{2}}} .
\end{aligned}
$$

Since $\lim _{r \searrow 0} \omega(r)=1, \lim _{r \nearrow z_{1}} \omega(r)=-\infty$, and the function $\omega$ is continuous, it follows that the equation $\omega(r)=0$ has at least a root in $\left(0, z_{1}\right)$. Thus, if $r_{1}$ is the smallest positive root of the equation $\omega(r)=0$, then we have

$$
\operatorname{Re}\left(1+\frac{z \mathcal{P}_{2 n-1}^{\prime \prime}(z)}{\mathcal{P}_{2 n-1}^{\prime}(z)}\right) \geq k\left|(1-\alpha) \frac{z \mathcal{P}_{2 n-1}^{\prime}(z)}{\mathcal{P}_{2 n-1}(z)}+\alpha\left(1+\frac{z \mathcal{P}_{2 n-1}^{\prime \prime}(z)}{\mathcal{P}_{2 n-1}^{\prime}(z)}\right)-1\right|
$$

for $|z|<r_{1}$, and

$$
\inf _{|z|<r_{1}}\left\{\operatorname{Re}\left(1+\frac{z \mathcal{P}_{2 n-1}^{\prime \prime}(z)}{\mathcal{P}_{2 n-1}^{\prime}(z)}\right)-k\left|(1-\alpha) \frac{z \mathcal{P}_{2 n-1}^{\prime}(z)}{\mathcal{P}_{2 n-1}(z)}+\alpha\left(1+\frac{z \mathcal{P}_{2 n-1}^{\prime \prime}(z)}{\mathcal{P}_{2 n-1}^{\prime}(z)}\right)-1\right|\right\}=0 .
$$

It follows that $r_{\alpha}^{k-\mathcal{U C S T}}\left(\mathcal{P}_{2 n-1}\right)=r_{1}$ is the radius of $k-\mathcal{U C S T}(\alpha)$ of the normalized Legendre polynomial $\mathcal{P}_{2 n-1}$, and hence this completes our proof.

Choosing $k=0$ in Theorem 1, we obtain the next result which was given by Bulut and Engel for $\beta=0$ in [7, Theorem 2.2].

Corollary 1 The radius of convexity of $\mathcal{P}_{2 n-1}$ is $r^{c}\left(\mathcal{P}_{2 n-1}\right)=r_{2}$, where $r_{2}$ denotes the smallest positive root of the equation

$$
r \mathcal{P}_{2 n-1}^{\prime \prime}(r)+\mathcal{P}_{2 n-1}^{\prime}(r)=0
$$

Example 1 For $n=2$ we have

$$
\mathcal{P}_{3}(z):=\frac{P_{3}(z)}{P_{3}^{\prime}(0)}=\frac{\frac{1}{2}\left(5 z^{3}-3 z\right)}{-\frac{3}{2}}=-\frac{1}{3}\left(5 z^{3}-3 z\right) .
$$

Like we see from Fig. 1 (a) the domain $\mathcal{P}_{3}(\mathbb{U})$ is not convex; moreover, the function $\mathcal{P}_{3}$ is not univalent in $\mathbb{U}$. From Corollary 1 it follows that the radius of convexity of $\mathcal{P}_{3}$ is $r^{c}\left(\mathcal{P}_{3}\right)=1 / \sqrt{15} \simeq 0.2581988897 \ldots$, where $1 / \sqrt{15}$ denotes the smallest positive root of the equation

$$
r \mathcal{P}_{3}^{\prime \prime}(r)+\mathcal{P}_{3}^{\prime}(r)=-15 r^{2}+1=0
$$

According to the above result, the domain $\mathcal{P}_{3}\left(\mathbb{U}\left(r^{c}\left(\mathcal{P}_{3}\right)\right)\right)$ shown in Fig. 1(b) is convex.

(a) The image of $\mathbb{U}$ under $\mathcal{P}_{3}(z)=$ $-\frac{1}{3}\left(5 z^{3}-3 z\right)$

(b) The image of $\mathbb{U}\left(r^{c}\left(\mathcal{P}_{3}\right)\right)$ under $\mathcal{P}_{3}(z)=-\frac{1}{3}\left(5 z^{3}-3 z\right)$

Figure 1 The image $\mathcal{P}_{3}(z)$

Letting $\alpha=1$ in Theorem 1, we obtain the next special case.

Corollary 2 The radius of $k$-uniformly convexity of $\mathcal{P}_{2 n-1}$ is $r^{u c v}\left(\mathcal{P}_{2 n-1}\right)=r_{3}$, where $r_{3}$ denotes the smallest positive root of the equation

$$
(1+k) \frac{r \mathcal{P}_{2 n-1}^{\prime \prime}(r)}{\mathcal{P}_{2 n-1}^{\prime}(r)}+1=0
$$

Setting $k=1$ in Corollary 2, we obtain the following result which was given by Bulut and Engel in [7, Theorem 2.3].

Example 2 The radius of uniform convexity of $\mathcal{P}_{2 n-1}$ is $r^{u c}\left(\mathcal{P}_{2 n-1}\right)=r_{4}$, where $r_{4}$ denotes the smallest positive root of the equation

$$
1+2 \frac{r \mathcal{P}_{2 n-1}^{\prime \prime}(r)}{\mathcal{P}_{2 n-1}^{\prime}(r)}=0 .
$$

Example 3 For $n=3$ we have

$$
\mathcal{P}_{5}(z):=\frac{P_{5}(z)}{P_{5}^{\prime}(0)}=\frac{\frac{1}{8}\left(63 z^{5}-70 z^{3}+15 z\right)}{\frac{15}{8}}=\frac{1}{15}\left(63 z^{5}-70 z^{3}+15 z\right) .
$$

Like we see from Fig. 2(a) the function $\mathcal{P}_{5}$ is not univalent in $\mathbb{U}$. From Example 2 it follows that the radius of uniform convexity of $\mathcal{P}_{5}$ is $r^{u c}\left(\mathcal{P}_{5}\right)=\sqrt{735-42 \sqrt{259}} / 63 \simeq$ $0.1219993521 \ldots$, where $\sqrt{735-42 \sqrt{259}} / 63$ denotes the smallest positive root of the equation

$$
\mathcal{P}_{5}^{\prime}(r)+2 r \mathcal{P}_{5}^{\prime \prime}(r)=\frac{1}{15}\left(2835 z^{4}-1050 z^{2}+15\right)=0
$$

According to the above result, the domain $\mathcal{P}_{5}\left(\mathbb{U}\left(r^{u c}\left(\mathcal{P}_{5}\right)\right)\right)$ is uniformly convex, and it is plotted in Fig. 2(b).

Letting $\alpha=0$ in Theorem 1, we deduce the next result.


Figure 2 The image $\mathcal{P}_{5}(z)$

Corollary 3 The radius of $k-\mathcal{M N}$ of $\mathcal{P}_{2 n-1}$ is $r^{k-\mathcal{M N}}\left(\mathcal{P}_{2 n-1}\right)=r_{5}$, where $r_{5}$ denotes the smallest positive root of the equation

$$
k\left(\frac{r \mathcal{P}_{2 n-1}^{\prime}(r)}{\mathcal{P}_{2 n-1}(r)}-1\right)+\frac{r \mathcal{P}_{2 n-1}^{\prime \prime}(r)}{\mathcal{P}_{2 n-1}^{\prime}(r)}+1=0 .
$$

Letting $k=1$ in Corollary 3, we obtain the following special case.
Example 4 The radius of $\mathcal{M} \mathcal{N}$ of $\mathcal{P}_{2 n-1}$ is $r^{\mathcal{M N}}\left(\mathcal{P}_{2 n-1}\right)=r_{6}$, where $r_{6}$ denotes the smallest positive root of the equation

$$
\frac{r \mathcal{P}_{2 n-1}^{\prime}(r)}{\mathcal{P}_{2 n-1}(r)}+\frac{r \mathcal{P}_{2 n-1}^{\prime \prime}(r)}{\mathcal{P}_{2 n-1}^{\prime}(r)}=0 .
$$

In the following theorem we obtain the radius of strong starlikeness of order $\alpha$ of $\mathcal{P}_{2 n-1}$.
Theorem 2 The radius of strong starlikeness of order $\alpha$ of $\mathcal{P}_{2 n-1}$ is $\tilde{r}_{\alpha}^{*}\left(\mathcal{P}_{2 n-1}\right)=r_{1}^{*}$, where $r_{1}^{*}$ is the smallest positive root of the equation

$$
\sum_{k=1}^{n-1} \frac{2 r^{2}\left(z_{k}^{2}+r^{2} \sin \frac{\alpha \pi}{2}\right)}{z_{k}^{4}-r^{4}}-\sin \frac{\pi \alpha}{2}=0, \quad 0<\alpha \leq 1 .
$$

Proof If $|z| \leq r<z_{1}$, then it follows that $|z| \leq r<z_{k}$ for all $k \in\{1,2, \ldots, n-1\}$, where $z_{k}$, with $k \in\{1,2, \ldots, n-1\}$, is the $k$ th positive zero of the normalized Legendre polynomial of odd degree. Hence, replacing $z$ by $z^{2}$ and $R$ by $z_{k}^{2}$ for all $k \in\{1,2, \ldots, n-1\}$ in the inequality of Lemma 3, we get

$$
\left|\frac{z^{2}}{z^{2}-z_{k}^{2}}+\frac{r^{2}}{z_{k}^{4}-r^{4}}\right| \leq \frac{z_{k}^{2} r^{2}}{z_{k}^{4}-r^{4}}, \quad|z| \leq r<z_{1}, k \in\{1,2, \ldots, n-1\} .
$$

Using the above inequalities, from relation (5) we get

$$
\left|\frac{z \mathcal{P}_{2 n-1}^{\prime}(z)}{\mathcal{P}_{2 n-1}(z)}-\left(1-\sum_{k=1}^{n-1} \frac{2 r^{4}}{z_{k}^{4}-r^{4}}\right)\right|=\left|1-\sum_{k=1}^{n-1} \frac{2 z^{2}}{z_{k}^{2}-z^{2}}-\left(1-\sum_{k=1}^{n-1} \frac{2 r^{4}}{z_{k}^{4}-r^{4}}\right)\right|
$$

$$
\begin{aligned}
& =\left|\sum_{k=1}^{n-1} \frac{2 z^{2}}{z^{2}-z_{k}^{2}}+\sum_{k=1}^{n-1} \frac{2 r^{4}}{z_{k}^{4}-r^{4}}\right|=\sum_{k=1}^{n-1}\left|\frac{2 z^{2}}{z^{2}-z_{k}^{2}}+\frac{2 r^{4}}{z_{k}^{4}-r^{4}}\right| \\
& \leq \sum_{k=1}^{n-1} \frac{2 z_{k}^{2} r^{2}}{z_{k}^{4}-r^{4}}
\end{aligned}
$$

for $|z| \leq r<z_{1}$.
Denoting

$$
w:=\frac{z \mathcal{P}_{2 n-1}^{\prime}(z)}{\mathcal{P}_{2 n-1}(z)}, \quad a:=1-\sum_{k=1}^{n-1} \frac{2 r^{4}}{z_{k}^{4}-r^{4}} \quad \text { and } \quad R_{a}:=\sum_{k=1}^{n-1} \frac{2 z_{k}^{2} r^{2}}{z_{k}^{4}-r^{4}},
$$

we see that $\operatorname{Im} a=0$, and from Lemma 2 and the above inequality it follows that the disc $|w-a| \leq R_{a}$ is contained in the sector $|\arg w| \leq \pi \alpha / 2$, that is,

$$
\left|\arg \frac{z \mathcal{P}_{2 n-1}^{\prime}(z)}{\mathcal{P}_{2 n-1}(z)}\right| \leq \frac{\pi \alpha}{2},
$$

if we assume that the inequality

$$
\sum_{k=1}^{n-1} \frac{2 z_{k}^{2} r^{2}}{z_{k}^{4}-r^{4}} \leq\left(1-\sum_{k=1}^{n-1} \frac{2 r^{4}}{z_{k}^{4}-r^{4}}\right) \sin \frac{\pi \alpha}{2}
$$

holds.
Let $\psi: I \rightarrow \mathbb{R}$, where $I$ is the open interval $\left(0, z_{1}\right)$ which is subset of $\mathbb{R}$ be defined by

$$
\psi(r):=\sum_{k=1}^{n-1} \frac{2 r^{2}\left(z_{k}^{2}+r^{2} \sin \frac{\alpha \pi}{2}\right)}{z_{k}^{4}-r^{4}}-\sin \frac{\pi \alpha}{2} .
$$

The above inequality implies that $\psi(r) \leq 0$ for $r \in\left(0, z_{1}\right)$. Also, we have $\lim _{r \searrow 0} \psi(r)=$ $-\sin \frac{\pi \alpha}{2}<0$ and $\lim _{r \nearrow z_{1}} \psi(r)=+\infty$. On the other the hand, we have $\psi^{\prime}(r) \geq 0$ for $z \in\left(0, z_{1}\right)$. It follows that the equation $\psi(r)=0$ has a unique root $r_{1}^{*}$ in $\left(0, z_{1}\right)$. Therefore, the radius of strong starlikeness of order $\alpha$ of $\mathcal{P}_{2 n-1}$ is $\tilde{r}_{\alpha}^{*}\left(\mathcal{P}_{2 n-1}\right)=r_{1}^{*}$.

Example 5 For $n=2$ we have

$$
\mathcal{P}_{3}(z):=\frac{P_{3}(z)}{P_{3}^{\prime}(0)}=\frac{\frac{1}{2}\left(5 z^{3}-3 z\right)}{-\frac{3}{2}}=-\frac{1}{3}\left(5 z^{3}-3 z\right)
$$

where the roots of $P_{3}$ are $z_{0}=0$ and $z_{1}= \pm \sqrt{15} / 5$. From Theorem 2 it follows that the radii of strong starlikeness of order $1 / 3,1 / 2$, and $2 / 3$ of $\mathcal{P}_{3}$ are $\widetilde{r_{1 / 3}^{*}}\left(\mathcal{P}_{3}\right)=\sqrt{-10+5 \sqrt{7}} / 5 \simeq$ $0.3593748213 \ldots, r_{1 / 2}^{*}\left(\mathcal{P}_{3}\right)=\sqrt{-5 \sqrt{2}+5 \sqrt{5}} / 5 \simeq 0.4054267912 \ldots$, and $\widetilde{r_{2 / 3}^{*}}\left(\mathcal{P}_{3}\right)=$ $\frac{\sqrt{15 \sqrt{39}-30 \sqrt{3}}}{15} \simeq 0.4305729813 \ldots$, where $\quad \frac{\sqrt{-10+5 \sqrt{7}}}{5}, \sqrt{-5 \sqrt{2}+5 \sqrt{5}} / 5$, and $\sqrt{15 \sqrt{39}-30 \sqrt{3}} / 15$ denote the smallest positive roots of the equations

$$
\frac{2 r^{2}\left(\frac{15}{25}+r^{2} \frac{1}{2}\right)}{\frac{(15)^{2}}{5^{4}}-r^{4}}-\frac{1}{2}=-\frac{3}{2} \cdot \frac{25 r^{4}+20 r^{2}-3}{25 r^{4}-9}=0
$$

$$
\frac{2 r^{2}\left(\frac{15}{25}+r^{2} \frac{\sqrt{2}}{2}\right)}{\frac{(15)^{2}}{5^{4}}-r^{4}}-\frac{\sqrt{2}}{2}=-\frac{3}{2} \cdot \frac{25 \sqrt{2} r^{4}+20 r^{2}-3 \sqrt{2}}{25 r^{4}-9}=0
$$

and

$$
\frac{2 r^{2}\left(\frac{15}{25}+r^{2} \frac{\sqrt{3}}{2}\right)}{\frac{(15)^{2}}{5^{4}}-r^{4}}-\frac{\sqrt{3}}{2}=-\frac{3}{2} \cdot \frac{25 \sqrt{3} r^{4}+20 r^{2}-3 \sqrt{3}}{25 r^{4}-9}=0
$$

respectively. According to the above result, the domains $\mathcal{P}_{3}\left(\mathbb{U}\left(\widetilde{r_{1 / 3}^{*}}\left(\mathcal{P}_{3}\right)\right)\right), \mathcal{P}_{3}\left(\mathbb{U}\left(\widetilde{r_{1 / 2}^{*}}\left(\mathcal{P}_{3}\right)\right)\right)$, and $\mathcal{P}_{3}\left(\mathbb{U}\left(\widetilde{r_{2 / 3}^{*}}\left(\mathcal{P}_{3}\right)\right)\right)$ are strongly starlike of order $1 / 3,1 / 2$, and $2 / 3$, respectively.

Example 6 For $n=3$ we have

$$
\mathcal{P}_{5}(z):=\frac{P_{5}(z)}{P_{5}^{\prime}(0)}=\frac{\frac{1}{8}\left(63 z^{5}-70 z^{3}+15 z\right)}{\frac{15}{8}}=\frac{1}{15}\left(63 z^{5}-70 z^{3}+15 z\right),
$$

where the roots of $P_{5}$ are $z_{0}=0, z_{1}= \pm \sqrt{245-14 \sqrt{70}} / 21$, and $z_{2}= \pm \sqrt{245+14 \sqrt{70}} / 21$. From Theorem 2 it follows that the radii of strong starlikeness of order $1 / 3,1 / 2$, and $2 / 3$ of $\mathcal{P}_{5}$ are $\widetilde{r_{1 / 3}^{*}}\left(\mathcal{P}_{5}\right) \simeq 0.2212264225 \ldots, \widetilde{r_{1 / 2}^{*}}\left(\mathcal{P}_{5}\right) \simeq 0.2537535993 \ldots$, and $\widetilde{r_{2 / 3}^{*}}\left(\mathcal{P}_{5}\right) \simeq$ $0.2724589258 \ldots$, where $\widetilde{r_{1 / 3}^{*}}\left(\mathcal{P}_{5}\right), \widetilde{r_{1 / 2}^{*}}\left(\mathcal{P}_{5}\right)$, and $\widetilde{r_{2 / 3}^{*}}\left(\mathcal{P}_{5}\right)$ denote the smallest positive roots of the equations

$$
\begin{aligned}
& \frac{1,607,445 r^{8}+1,428,840 r^{6}-731,430 r^{4}-340,200 r^{2}+18,225}{\left(1134 r^{4}+40 \sqrt{70}-430\right)\left(-567 r^{4}+20 \sqrt{70}+215\right)}=0, \\
& \frac{1,607,445 \sqrt{2} r^{8}+1,428,840 r^{6}-731,430 \sqrt{2} r^{4}-340,200 r^{2}+18,225 \sqrt{2}}{\left(1134 r^{4}+40 \sqrt{70}-430\right)\left(-567 r^{4}+20 \sqrt{70}+215\right)}=0,
\end{aligned}
$$

and

$$
\frac{1,607,445 \sqrt{3} r^{8}+1,428,840 r^{6}-731,430 \sqrt{3} r^{4}-340,200 r^{2}+18,225 \sqrt{3}}{\left(1134 r^{4}+40 \sqrt{70}-430\right)\left(-567 r^{4}+20 \sqrt{70}+215\right)}=0,
$$

respectively. According to the above result, the domains $\mathcal{P}_{5}\left(\mathbb{U}\left(\widetilde{r_{1 / 3}^{*}}\left(\mathcal{P}_{5}\right)\right)\right), \mathcal{P}_{5}\left(\mathbb{U}\left(\widetilde{r_{1 / 2}^{*}}\left(\mathcal{P}_{5}\right)\right)\right)$, and $\mathcal{P}_{5}\left(\mathbb{U}\left(\widetilde{r_{2 / 3}^{*}}\left(\mathcal{P}_{5}\right)\right)\right)$ are strongly starlike of order $1 / 3,1 / 2$, and $2 / 3$, respectively.

Letting $\alpha=1$ in the above theorem, we get the following corollary.

Corollary 4 The radius of starlikeness of $\mathcal{P}_{2 n-1}$ is $r^{*}\left(\mathcal{P}_{2 n-1}\right)=r_{2}^{*}$, where $r_{2}^{*}$ denotes the smallest positive root of the equation

$$
\sum_{k=1}^{n-1} \frac{2 r^{2}}{z_{k}^{2}-r^{2}}-1=0 .
$$

Example 7 For $n=2$ we have

$$
\mathcal{P}_{3}(z):=\frac{P_{3}(z)}{P_{3}^{\prime}(0)}=\frac{\frac{1}{2}\left(5 z^{3}-3 z\right)}{-\frac{3}{2}}=-\frac{1}{3}\left(5 z^{3}-3 z\right),
$$

Figure 3 The image of $\mathbb{U}\left(\frac{1}{\sqrt{5}}\right)$ under $\mathcal{P}_{3}(z)=-\frac{1}{3}\left(5 z^{3}-3 z\right)$

where the roots of $P_{3}$ are $z_{0}=0$ and $z_{1}= \pm \sqrt{15} / 5$. From Corollary 4 it follows that the radius of starlikeness of $\mathcal{P}_{3}$ is $r^{*}\left(\mathcal{P}_{3}\right)=1 / \sqrt{5} \simeq 0.4472135954 \ldots$, where $1 / \sqrt{5}$ denotes the smallest positive root of the equation

$$
\frac{2 r^{2}}{z_{1}^{2}-r^{2}}-1=-\frac{3\left(5 r^{2}-1\right)}{5 r^{2}-3}=0 .
$$

According to the above result, as Fig. 3, the domain $\mathcal{P}_{3}(\mathbb{U}(1 / \sqrt{5}))$ is starlike.

Remark 2 All the figures inserted in this article have been obtained using MAPLE ${ }^{\text {mw }}$ software.

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## Authors' contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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