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Hermite–Hadamard inequality for fractional integrals of Caputo–Fabrizio type and related inequalities

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Abstract

In this article, firstly, Hermite–Hadamard's inequality is generalized via a fractional integral operator associated with the Caputo–Fabrizio fractional derivative. Then a new kernel is obtained and a new theorem valid for convex functions is proved for fractional order integrals. Also, some applications of our main findings are given.

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Keywords: Caputo–Fabrizio Fractional integral; Convexity

1 Introduction

Fractional calculus has been appealing to many researchers over the last decades [1–5]. Some researchers have found that different fractional derivatives with different singular or nonsingular kernels need to be identified by real-world problems in different fields of engineering and science [6–12]. These different fractional operators are also used in integral inequalities [13–21]. Thus, fractional calculus plays an important role in the development of inequality theory. One of the best-known inequalities, the Hermite–Hadamard inequality, which is generalized by means of several fractional integral operators, is given now.

Theorem 1 (See [22]) *Let $f : I \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds*:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

In the field of fractional analysis, many researchers have focused on defining new operators and modeling and implementing of the problems based on their features. The features that make the operators different from each other include singularity and locality, while kernel expression of the operator is presented with functions such as the power law, the exponential function or a Mittag-Leffler function. The distinctive feature of the Caputo–Fabrizio operator is that it has a non-singular kernel. With the help of the Caputo–Fabrizio operator, new studies have been made on many modeling problems and real-world problems. This is so because the definition of Caputo–Fabrizio is very effective in better describing heterogeneousness and systems with different scales with memory effects. The main basic feature of the Caputo–Fabrizio definition can be explained as a real power turned into the integer by the Laplace transformation, thus the exact solution can be easily found for various problems. Now, we will proceed by some necessary definitions and preliminary results which are used and referred throughout this paper.

Definition 1 (See [1, 23, 24]) Let $f \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$, then the definition of the left fractional derivative in the sense of Caputo and Fabrizio becomes

$$\left({}_{a}^{\mathrm{CFC}}D^{\alpha}f\right)(t) = \frac{B(\alpha)}{1-\alpha}\int_{a}^{t} f'(x)e^{\frac{-\alpha(t-x)^{\alpha}}{1-\alpha}}\,dx$$

and the associated fractional integral is

$$\left({}_{a}^{\mathrm{CF}}I^{\alpha}f\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\int_{a}^{t} f(x)\,dx,$$

where $B(\alpha) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$. For the right fractional derivative we have

$$\left({}^{\mathrm{CFC}}D_{b}^{\alpha}f\right)(t) = \frac{-B(\alpha)}{1-\alpha}\int_{t}^{b} f'(x)e^{\frac{-\alpha(x-t)^{\alpha}}{1-\alpha}}\,dx$$

and the associated fractional integral is

$$\left({}^{\mathrm{CF}}I_{b}^{\alpha}f\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\int_{t}^{b} f(x)\,dx.$$

Fractional derivative and integral operators have recently been used to generalize existing kernels. The kernel which we will generalize with the help of a Caputo–Fabrizio fractional integral operator is proven by Dragomir and Agarwal.

Lemma 1 (See [25]) *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I$ with $a < b$. If $f' \in L[a,b]$ then the following equality holds*:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_{a}^{b} f(x)\,dx = \frac{b-a}{2}\int_{0}^{1}(1-2t)f'\bigl(ta+(1-t)b\bigr)\,dt.$$

In the following section, we will prove a theorem which is a variant of the Hermite–Hadamard inequality.

2 A generalization of Hermite–Hadamard inequality via the Caputo–Fabrizio fractional operator

Theorem 2 *Let a function $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be convex on $[a,b]$ and $f \in L_1[a,b]$. If $\alpha \in [0,1]$, then the following double inequality holds*:

$$f\left(\frac{a+b}{2}\right) \le \frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\alpha}f\right)(k) + \left({}^{\mathrm{CF}}I_{b}^{\alpha}f\right)(k) - \frac{2(1-\alpha)}{B(\alpha)}f(k)\right] \le \frac{f(a)+f(b)}{2}, \tag{1}$$

where $k \in [a,b]$ and $B(\alpha) > 0$ is a normalization function.

Proof Since f is a convex function on $[a,b]$ we can write

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) &\le \frac{2}{b-a}\int_a^b f(x)\,dx \\ &= \frac{2}{b-a}\left(\int_a^k f(x)\,dx + \int_k^b f(x)\,dx\right). \end{aligned} \tag{2}$$

By multiplying both sides of (2) with $\frac{\alpha(b-a)}{2B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}f(k)$ we have

$$\begin{aligned} &\frac{2(1-\alpha)}{B(\alpha)}f(k) + \frac{\alpha(b-a)}{B(\alpha)}f\left(\frac{a+b}{2}\right) \\ &\le \frac{2(1-\alpha)}{B(\alpha)}f(k) + \frac{\alpha}{B(\alpha)}\left(\int_a^k f(x)\,dx + \int_k^b f(x)\,dx\right) \\ &= \left(\frac{1-\alpha}{B(\alpha)}f(k) + \frac{\alpha}{B(\alpha)}\int_a^k f(x)\,dx\right) + \left(\frac{1-\alpha}{B(\alpha)}f(k) + \frac{\alpha}{B(\alpha)}\int_k^b f(x)\,dx\right) \\ &= \left({}_{a}^{\mathrm{CF}}I^{\alpha}f\right)(k) + \left({}^{\mathrm{CF}}I_{b}^{\alpha}f\right)(k). \end{aligned} \tag{3}$$

So, the proof of the first inequality in (1) is completed by reorganizing the last inequality. For the proof of the second inequality in (1), if we use the right hand side of Hadamard inequality, we can write

$$\frac{2}{b-a}\int_a^b f(x)\,dx \le f(a)+f(b). \tag{4}$$

By making the same operations with (2) in (4), we have

$$\left({}_{a}^{\mathrm{CF}}I^{\alpha}f\right)(k) + \left({}^{\mathrm{CF}}I_{b}^{\alpha}f\right)(k) \le \frac{2(1-\alpha)}{B(\alpha)}f(k) + \frac{\alpha(b-a)}{2B(\alpha)}\big(f(a)+f(b)\big). \tag{5}$$

By reorganizing (5), the proof of the second inequality in (1) is completed. □

Theorem 3 *Let $f,g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. If $fg \in L([a,b])$, then we have the following inequality*:

$$\frac{2B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\alpha}fg\right)(k) + \left({}^{\mathrm{CF}}I_{b}^{\alpha}fg\right)(k) - \frac{2(1-\alpha)}{B(\alpha)}f(k)g(k)\right] \le \frac{2}{3}M(a,b) + \frac{1}{3}N(a,b),$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b),$$

$$N(a,b)=f(a)g(b)+f(b)g(a),$$

and $k\in[a,b]$, $B(\alpha)>0$ *is a normalization function.*

Proof Since f and g are convex functions on $[a,b]$, we have

$$f\big(ta+(1-t)b\big)\le tf(a)+(1-t)f(b),\quad \forall t\in[0,1], a,b\in I,$$

and

$$g\big(ta+(1-t)b\big)\le tg(a)+(1-t)g(b),\quad \forall t\in[0,1], a,b\in I.$$

Multiplying above inequalities both sides, we have

$$\begin{aligned}&f\big(ta+(1-t)b\big)g\big(ta+(1-t)b\big)\\&\quad\le t^2f(a)g(a)+(1-t)^2f(b)g(b)+t(1-t)\big[f(a)g(b)+f(b)g(a)\big].\end{aligned}\tag{6}$$

Integrating (6) with respect to t over $[0,1]$, and making the change of variable, we obtain

$$\frac{2}{b-a}\int_a^b f(x)g(x)\,dx\le\frac{2}{3}\big[f(a)g(a)+f(b)g(b)\big]+\frac{1}{3}\big[f(a)g(b)+f(b)g(a)\big],$$

which implies

$$\frac{2}{b-a}\left[\int_a^t f(x)g(x)\,dx+\int_t^b f(x)g(x)\,dx\right]\le\frac{2}{3}M(a,b)+\frac{1}{3}N(a,b).$$

By multiplying both sides with $\frac{\alpha(b-a)}{2B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}f(k)g(k)$ we have

$$\begin{aligned}&\frac{\alpha}{B(\alpha)}\left[\int_a^k f(x)g(x)\,dx+\int_k^b f(x)g(x)\,dx\right]+\frac{2(1-\alpha)}{B(\alpha)}f(k)g(k)\\&\quad\le\frac{\alpha(b-a)}{2B(\alpha)}\left[\frac{2}{3}M(a,b)+\frac{1}{3}N(a,b)\right]+\frac{2(1-\alpha)}{B(\alpha)}f(k)g(k).\end{aligned}$$

Thus

$$\big({}_a^{\mathrm{CF}}I^{\alpha}fg\big)(k)+\big({}^{\mathrm{CF}}I_b^{\alpha}fg\big)(k)\le\frac{\alpha(b-a)}{2B(\alpha)}\left[\frac{2}{3}M(a,b)+\frac{1}{3}N(a,b)\right]+\frac{2(1-\alpha)}{B(\alpha)}f(k)g(k),$$

and with suitable rearrangements, the proof is completed. □

Theorem 4 *Let* $f,g:I\subseteq\mathbb{R}\to\mathbb{R}$ *be a convex function. If* $fg\in L([a,b])$, *the set of integrable functions, then*

$$\begin{aligned}&2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\big[\big({}_a^{\mathrm{CF}}I^{\alpha}fg\big)(k)+\big({}^{\mathrm{CF}}I_b^{\alpha}fg\big)\big]k+\frac{1-\alpha}{\alpha(b-a)}f(k)g(k)\\&\quad\le\frac{2}{3}M(a,b)+\frac{4}{3}N(a,b),\end{aligned}\tag{7}$$

where $M(a,b)$ and $N(a,b)$ are given in Theorem 3 and $k \in [a,b]$, $B(\alpha) > 0$ is a normalization function.

Proof Since f and g are convex functions on $[a,b]$, for $t = \frac{1}{2}$, we have

$$f\left(\frac{a+b}{2}\right) \le \frac{f((1-t)a+tb)+f(ta+(1-t)b)}{2}, \quad \forall a,b \in I, t \in [0,1],$$

and

$$g\left(\frac{a+b}{2}\right) \le \frac{g((1-t)a+tb)+g(ta+(1-t)b)}{2}, \quad \forall a,b \in I, t \in [0,1].$$

Multiplying the above inequalities at both sides, we have

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&\quad \le \frac{1}{4}\bigl[f\bigl((1-t)a+tb\bigr)g\bigl((1-t)a+tb\bigr)+f\bigl(ta+(1-t)b\bigr)g\bigl(ta+(1-t)b\bigr) \\
&\qquad +f\bigl((1-t)a+tb\bigr)g\bigl(ta+(1-t)b\bigr)+f\bigl(ta+(1-t)b\bigr)g\bigl((1-t)a+tb\bigr)\bigr] \\
&\quad \le \frac{1}{4}\bigl[f\bigl((1-t)a+tb\bigr)g\bigl((1-t)a+tb\bigr)+f\bigl(ta+(1-t)b\bigr)g\bigl(ta+(1-t)b\bigr) \\
&\qquad + 2\bigl\{t(1-t)\bigl[f(a)g(a)+f(b)g(b)\bigr]+(1-t)^2f(a)g(b)+t^2f(b)g(a)\bigr\}\bigr].
\end{aligned}$$

Integrating the above inequality with respect to t over $[0,1]$ and making the change of variable, one obtains

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&\quad \le \frac{1}{2(b-a)}\int_a^b f(x)g(x)\,dx+\frac{1}{3}\bigl[f(a)g(a)+f(b)g(b)\bigr]+\frac{2}{3}\bigl[f(a)g(b)+f(b)g(a)\bigr].
\end{aligned}$$

Thus

$$4f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{2}{(b-a)}\int_a^b f(x)g(x)\,dx+\frac{4}{3}M(a,b)+\frac{8}{3}N(a,b).$$

By multiplying both sides with $\frac{\alpha(b-a)}{2B(\alpha)}$ and subtracting $\frac{2(1-\alpha)}{B(\alpha)}f(k)g(k)$ we have

$$\begin{aligned}
&\frac{2\alpha(b-a)}{B(\alpha)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{\alpha}{B(\alpha)}\left[\int_a^k f(x)g(x)\,dx+\int_k^b f(x)g(x)\,dx\right] \\
&\qquad -\frac{2(1-\alpha)}{B(\alpha)}f(k)g(k) \\
&\quad \le \frac{\alpha(b-a)}{2B(\alpha)}\left[\frac{4}{3}M(a,b)+\frac{8}{3}N(a,b)\right]-\frac{2(1-\alpha)}{B(\alpha)}f(k)g(k),
\end{aligned}$$

and one arrives at

$$\begin{aligned}&\frac{2\alpha(b-a)}{B(\alpha)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{\alpha}{B(\alpha)}\left[\left({}_{a}^{\mathrm{CF}}I^{\alpha}fg\right)(k)+\left({}^{\mathrm{CF}}I_{b}^{\alpha}fg\right)\right](k)\\&\quad\le\frac{\alpha(b-a)}{2B(\alpha)}\left[\frac{4}{3}M(a,b)+\frac{8}{3}N(a,b)\right]-\frac{2(1-\alpha)}{B(\alpha)}f(k)g(k).\end{aligned}$$

Multiplying both sides of the above inequality by $\frac{2B(\alpha)}{\alpha(b-a)}$, we get the required inequality (7). □

3 Some new results related with Caputo–Fabrizio fractional operator

In this section, firstly, we will generalize a lemma, then we will put forward a theorem with the help of the lemma.

Lemma 2 *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I$ with $a<b$. If $f'\in L_1[a,b]$ and $\alpha\in[0,1]$, the following equality holds*:

$$\begin{aligned}&\frac{b-a}{2}\int_0^1(1-2t)f'\bigl(ta+(1-t)b\bigr)\,dt-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\\&\quad=\frac{f(a)+f(b)}{2}-\frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\alpha}f\right)(k)+\left({}^{\mathrm{CF}}I_{b}^{\alpha}f\right)(k)\right],\end{aligned}$$

where $k\in[a,b]$ and $B(\alpha)>0$ is a normalization function.

Proof It is easy to see that

$$\begin{aligned}&\int_0^1(1-2t)f'\bigl(ta+(1-t)b\bigr)\,dt\\&\quad=\frac{f(a)+f(b)}{b-a}-\frac{2}{(b-a)^2}\left(\int_a^k f(x)\,dx+\int_k^b f(x)\,dx\right).\end{aligned}$$

By multiplying both sides with $\frac{\alpha(b-a)^2}{2B(\alpha)}$ and subtracting $\frac{2(1-\alpha)}{B(\alpha)}f(k)$ we have

$$\begin{aligned}&\frac{\alpha(b-a)^2}{2B(\alpha)}\int_0^1(1-2t)f'\bigl(ta+(1-t)b\bigr)\,dt-\frac{2(1-\alpha)}{B(\alpha)}f(k)\\&\quad=\frac{\alpha(b-a)(f(a)+f(b))}{2B(\alpha)}+\frac{2(1-\alpha)}{B(\alpha)}f(k)-\frac{\alpha}{B(\alpha)}\left(\int_a^k f(x)\,dx+\int_k^b f(x)\,dx\right)\\&\quad=\frac{\alpha(b-a)(f(a)+f(b))}{2B(\alpha)}-\left(\frac{(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha}{B(\alpha)}\int_a^k f(x)\,dx\right)\\&\qquad-\left(\frac{(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha}{B(\alpha)}\int_k^b f(x)\,dx\right)\\&\quad=\frac{\alpha(b-a)(f(a)+f(b))}{2B(\alpha)}-\left[\left({}_{a}^{\mathrm{CF}}I^{\alpha}f\right)(k)+\left({}^{\mathrm{CF}}I_{b}^{\alpha}f\right)(k)\right].\end{aligned}$$

Thus, the proof is completed. □

Theorem 5 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable positive mapping on I° and $|f'|$ be convex on $[a,b]$ where $a, b \in I$ with $a < b$. If $f' \in L_1[a,b]$ and $\alpha \in [0,1]$, the following inequality holds*:

$$\left|\frac{f(a)+f(b)}{2} + \frac{2(1-\alpha)}{\alpha(b-a)}f(k) - \frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\alpha}f\right)(k) + \left({}^{\mathrm{CF}}I_{b}^{\alpha}f\right)(k)\right]\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8},$$

where $k \in [a,b]$ and $B(\alpha) > 0$ is a normalization function.

Proof By using Lemma 2, the properties of the absolute value and the convexity of $|f'|$ we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} + \frac{2(1-\alpha)}{\alpha(b-a)}f(k) - \frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\alpha}f\right)(k) + \left({}^{\mathrm{CF}}I_{b}^{\alpha}f\right)(k)\right]\right| \\
&\quad\le \frac{b-a}{2}\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right|\,dt \\
&\quad\le \frac{b-a}{2}\int_0^1 |1-2t|\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right)dt \\
&\quad= \frac{b-a}{2}\left(\int_0^{\frac{1}{2}} (1-2t)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right)dt\right. \\
&\qquad\left. + \int_{\frac{1}{2}}^1 (2t-1)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right)dt\right) \\
&\quad= \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.
\end{aligned}$$

So the proof is completed. □

Theorem 6 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable positive mapping on I° and $|f'|^q$ be convex on $[a,b]$ where $p > 1$, $p^{-1} + q^{-1} = 1$, $a, b \in I$ with $a < b$. If $f' \in L_1[a,b]$ and $\alpha \in [0,1]$, the following inequality holds*:

$$\left|\frac{f(a)+f(b)}{2} + \frac{2(1-\alpha)}{\alpha(b-a)}f(k) - \frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\alpha}f\right)(k) + \left({}^{\mathrm{CF}}I_{b}^{\alpha}f\right)(k)\right]\right| \le \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

where $k \in [a,b]$ and $B(\alpha) > 0$ is a normalization function.

Proof By a similar argument to the proof of the previous theorem, but now using Lemma 2, the Hölder inequality and convexity of $|f'|^q$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} + \frac{2(1-\alpha)}{\alpha(b-a)}f(k) - \frac{B(\alpha)}{\alpha(b-a)}\left[\left({}_{a}^{\mathrm{CF}}I^{\alpha}f\right)(k) + \left({}^{\mathrm{CF}}I_{b}^{\alpha}f\right)(k)\right]\right| \\
&\quad\le \frac{b-a}{2}\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right|\,dt
\end{aligned}$$

$$\le \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(ta+(1-t)b\right)\right|^q\,dt\right)^{\frac{1}{q}}$$
$$= \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

So the proof is completed. □

4 Application to special means

It is very important to give an application in terms of efficiency and usefulness of the results obtained. At the same time, the accuracy of the findings will be confirmed by the application to special means for real numbers a_1, a_2 such that $a_1 \neq a_2$:

(1) The arithmetic mean

$$\mathcal{A} = \mathcal{A}(a_1,a_2) = \frac{a_1+a_2}{2}, \quad a_1,a_2 \in \mathbb{R}.$$

(2) The generalized logarithmic mean

$$\mathcal{L} = \mathcal{L}_r^r(a_1,a_2) = \frac{a_2^{\mathfrak{r}+1}-a_1^{\mathfrak{r}+1}}{(r+1)(a_2-a_1)}, \quad \mathfrak{r} \in \mathbb{R} \setminus \{-1,0\backslash\}, a_1,a_2 \in \mathbb{R}, a_1 \neq a_2.$$

Now, using the results in Sect. 3, we have some applications to the special means of real numbers.

Proposition 1 *Let $a_1,a_2 \in \mathbb{R}^+$, $a_1 < a_2$, then*

$$\left|\mathcal{A}\left(a_1^2,a_2^2\right) - \mathcal{L}_2^2(a_1,a_2)\right| \le \frac{(a_1-a_2)}{4}\left[|a_1|+|a_2|\right].$$

Proof In Theorem 5, if we set $f(z) = z^2$ with $\alpha = 1$ and $B(\alpha) = B(1) = 1$, then we obtain the result immediately. □

Proposition 2 *Let $a_1,a_2 \in \mathbb{R}^+$, $a_1 < a_2$, then*

$$\left|\mathcal{A}\left(e^{a_1},e^{a_2}\right) - \mathcal{L}\left(e^{a_1},e^{a_2}\right)\right| \le \frac{(a_2-a_1)}{8}\left(e^{a_1}+e^{a_2}\right).$$

Proof In Theorem 5, if we set $f(z) = e^z$ with $\alpha = 1$ and $B(\alpha) = B(1) = 1$, then we obtain the result immediately. □

Proposition 3 *Let $a_1,a_2 \in \mathbb{R}^+$, $a_1 < a_2$, then*

$$\left|\mathcal{A}\left(a_1^n,a_2^n\right) - \mathcal{L}_n^n(a_1,a_2)\right| \le \frac{n(a_1-a_2)}{8}\left(\left|a_1^{n-1}\right|+\left|a_2^{n-1}\right|\right).$$

Proof In Theorem 5, if we set $f(z) = z^n$ where n is an even number with $\alpha = 1$ and $B(\alpha) = B(1) = 1$, then we obtain the result immediately. □

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Authors' contributions
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