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Strongly (η, ω)-convex functions with nonnegative modulus

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Abstract

We introduce a new class of functions called strongly (η, ω)-convex functions. This class of functions generalizes some recently introduced notions of convexity, namely, the η-convex functions and strongly η-convex functions. We also establish inequalities of the Hermite–Hadamard–Fejér's type, which generalize results of Delavar and Dragomir (Math. Inequal. Appl. 20(1):203–216, 2017) and Awan et al. (Filomat 31(18):5783–5790, 2017). In addition, we obtain some new results for this class of functions. Finally, we apply our results to the k-Riemann–Liouville fractional integral operators to obtain more results in this direction.

MSC: 26A51; 26D15

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1 Introduction

The field of mathematical inequalities, derived from different families of convexity, has been a booming area in recent times. The literature is replete with plethora of such results. The theory of inequalities, especially integral inequalities, has found its place in many areas of mathematical sciences. For instance, it is generally known that there are functions whose integrals cannot be computed analytically, but estimates of such integrals would suffice. Hence an inequality is desired in this case. With the help of convexity the Jensen, Jensen–Steffensen, Slater, Favard, Berwald, Fejér, Hermite–Hadamard inequalities, and their generalizations have all been established. In this work, we concern ourselves with the Fejér and Hermite–Hadamard inequalities.

We start our discussion by collating the following foundational definition and results.

Definition 1 ([3]) A function $F : J \to \mathbb{R}$ is said to be η-convex with respect to $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if the following inequality holds:

$$F\big(\tau x + (1 - \tau)y\big) \le F(y) + \tau\eta\big(F(x), F(y)\big)$$

for all $x, y \in J$ and $\tau \in [0, 1]$.

We recapture the classical definition of convexity if the bifunction $\eta(x,y) = x - y$. Recently, Delavar and Dragomir [1] obtained the following theorems for the class of η-convex functions.

Theorem 2 ([1]) *Let $F : [\alpha, \beta] \to \mathbb{R}$ with $\alpha < \beta$. Suppose that the function F satisfies the following conditions*:

(a) *F is η-convex and η bounded above on $F([\alpha,\beta]) \times F([\alpha,\beta])$;*

(b) *$F \in L_\infty([\alpha,\beta])$.*

Then we have the following inequalities:

$$\begin{aligned} &F\left(\frac{\alpha+\beta}{2}\right) - \frac{K_\eta}{2} \\ &\quad\le \frac{1}{\beta-\alpha}\int_\alpha^\beta F(u)\,du \\ &\quad\le \frac{\eta(F(\alpha),F(\beta)) + \eta(F(\beta),F(\alpha))}{4} + \frac{F(\alpha)+F(\beta)}{2} \\ &\quad\le \frac{F(\alpha)+F(\beta)}{2} + \frac{K_\eta}{2}, \end{aligned} \tag{1}$$

where K_η is an upper bound of η.

Theorem 3 ([1]) *Let $F : [\alpha,\beta] \to \mathbb{R}$ and $G : (\alpha,\beta) \to [0,\infty)$ with $\alpha < \beta$. Suppose that the functions F and G satisfy the following conditions*:

(a). *F is η-convex;*

(b). *$F \in L_\infty([\alpha,\beta])$;*

(c). *$G \in L_1((\alpha,\beta))$;*

(d). *$G(\alpha+\beta-u) = G(u)$ for all $u \in (\alpha,\beta)$.*

Then we have the following inequality:

$$\begin{aligned} &\int_\alpha^\beta F(u)G(u)\,du \\ &\quad\le \left(\frac{\eta(F(\alpha),F(\beta)) + \eta(F(\beta),F(\alpha))}{2(\beta-\alpha)}\right)\int_\alpha^\beta (\beta-u)G(u)\,du \\ &\qquad + \left(\frac{F(\alpha)+F(\beta)}{2}\right)\int_\alpha^\beta G(u)\,du. \end{aligned} \tag{2}$$

Definition 1 was further generalized by Awan et al.

Definition 4 ([2]) A function $F : J \subset \mathbb{R} \to \mathbb{R}$ is said to be strongly η-convex with respect to $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and modulus $\mu \ge 0$ if

$$F\bigl(\tau x + (1-\tau)y\bigr) \le F(y) + \tau\eta\bigl(F(x),F(y)\bigr) - \mu\tau(1-\tau)(x-y)^2$$

for all $x, y \in J$ and $\tau \in [0,1]$.

Example 5 The function $F(x) = x^2$ is strongly η-convex with respect to the function $\eta(x,y) = 2x + y$ and modulus $\mu = 1$.

For related and recent results associated with the η-convex functions, we refer the interested reader to the papers [4–10] and the references therein.

The authors in [2] proved the following result.

Theorem 6 ([2]) *Let $F:[\alpha,\beta]\to\mathbb{R}$ with $\alpha<\beta$. Suppose that the function F satisfies the following conditions*:

(a) *F is strongly η-convex with respect to modulus $\mu\geq 0$ and η bounded above on $F([\alpha,\beta])\times F([\alpha,\beta])$;*

(b) *$F\in L_\infty([\alpha,\beta])$.*

Then we have the following inequalities:

$$\begin{aligned} & F\left(\frac{\alpha+\beta}{2}\right)-\frac{K_\eta}{2}+\frac{\mu}{12}(\beta-\alpha)^2 \\ & \quad\leq \frac{1}{\beta-\alpha}\int_\alpha^\beta F(u)\,du \\ & \quad\leq \frac{\eta(F(\alpha),F(\beta))+\eta(F(\beta),F(\alpha))}{4}+\frac{F(\alpha)+F(\beta)}{2}-\frac{\mu}{6}(\beta-\alpha)^2 \\ & \quad\leq \frac{F(\alpha)+F(\beta)}{2}+\frac{K_\eta}{2}-\frac{\mu}{6}(\beta-\alpha)^2, \end{aligned} \tag{3}$$

where K_η is an upper bound of η.

Stimulated by the above-mentioned work, we aim to achieve the following goals:

1. to introduce a new class of functions in Sect. 2, which generalizes preexisting notions of convexity;
2. to extend Theorems 3 and 6 to this new class of functions (see Sect. 3) and then apply the results obtained thereafter to the k-Riemann–Liouville fractional integrals;
3. finally, to establish many new integral inequalities in this direction.

2 A new class of convexity

We now introduce a new definition as a generalization of Definition 4.

Definition 7 A function $F:J\subset\mathbb{R}\to\mathbb{R}$ is said to be strongly (η,ω)-convex with respect to $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $\omega:[0,1]\to[0,\infty)$, and modulus $\mu\geq 0$ if

$$F\bigl(\tau x+(1-\tau)y\bigr)\leq F(y)+\omega(\tau)\eta\bigl(F(x),F(y)\bigr)-\mu\tau(1-\tau)(x-y)^2 \tag{4}$$

for all $x,y\in J$ and $\tau\in[0,1]$.

Evidently, by taking $\omega(\tau)=\tau$ we get Definition 4. Substituting $\tau=0$ into (4), we obtain

$$\omega(0)\eta\bigl(F(x),F(y)\bigr)\geq 0.$$

For $\tau=1$, we get

$$\omega(1)\eta\bigl(F(x),F(y)\bigr)\geq F(x)-F(y).$$

If, in addition, we set $x = y$ in (4), then we obtain

$$\omega(\tau)\eta\big(F(x),F(x)\big) \geq 0.$$

We now present an example of a strongly (η,ω)-convex function.

Example 8 Let $F(x) = x^2$. The function F is strongly (η,ω)-convex with respect to the bifunction $\eta(x,y) = 2x + y$, $\omega(\tau) = \tau$, and modulus $\mu = 1$. To see this, let $\tau \in [0,1]$. Then

$$\begin{aligned}
&F(y) + \omega(\tau)\eta\big(F(x),F(y)\big) - \mu\tau(1-\tau)(x-y)^2 \\
&\quad = y^2 + \tau\big(2x^2 + y^2\big) - \tau(1-\tau)(y-x)^2 \\
&\quad = \tau^2x^2 + 2xy\tau(1-\tau) + (1-\tau)^2y^2 + \tau\big(x^2 + 2y^2\big) \\
&\quad \geq \tau^2x^2 + 2xy\tau(1-\tau) + (1-\tau)^2y^2 \\
&\quad = F\big(\tau x + (1-\tau)y\big).
\end{aligned}$$

We wrap up this section by showing, by means of the next example, that the class of strongly (η,ω)-convex functions is wider than the class of strongly η-convex functions.

Example 9 The function $F(x) = \sqrt{x}$ defined on $[0,1]$ is strongly (η,ω)-convex with respect to $\eta(p,q) = \sqrt{|p^2 - q^2|}$ $(p,q \in \mathbb{R})$, $\omega(\tau) = \sqrt{\tau}$ $(\tau \in [0,1])$, and $\mu = 0$. To prove this claim, let $x,y,\tau \in [0,1]$. Then

$$\begin{aligned}
&F(y) + \omega(\tau)\eta\big(F(x),F(y)\big) - \mu\tau(1-\tau)(x-y)^2 \\
&\quad = \sqrt{y} + \sqrt{\tau}\sqrt{|x-y|} \\
&\quad \geq \sqrt{y + \tau(x-y)} \\
&\quad = F\big(\tau x + (1-\tau)y\big).
\end{aligned}$$

Next, we argue that there are no $\eta : [0,1] \times [0,1] \to \mathbb{R}$ and $\mu \geq 0$ for which F is strongly η-convex. We prove this by contradiction. Suppose there are $\eta : [0,1] \times [0,1] \to \mathbb{R}$ and $\mu \geq 0$ such that F is strongly η-convex. Then for all $x,y \in [0,1]$,

$$\sqrt{\tau x + (1-\tau)y} \leq \sqrt{y} + \tau\eta(\sqrt{x},\sqrt{y}) - \mu\tau(1-\tau)(x-y)^2, \quad \tau \in [0,1].$$

Let $x > 0$ and $y = 0$. We get

$$\sqrt{\tau x} \leq \tau\eta(\sqrt{x},0) - \mu\tau(1-\tau)x^2, \quad \tau \in [0,1].$$

This implies

$$\sqrt{x} \leq \sqrt{\tau}\eta(\sqrt{x},0) - \mu\sqrt{\tau}(1-\tau)x^2, \quad \tau \in (0,1].$$

Taking limit as $\tau \to 0^+$, we obtain $x = 0$, contradicting the fact that $x > 0$. Therefore our claim is justified.

3 Main results

We break this section into three subsections. We start by presenting Hermite–Hadamard–Fejér-type results and give an application to the k-Riemann–Liouville fractional integral. Thereafter, we conclude by establishing three more theorems for the class of (η,ω)-convex functions.

3.1 Inequalities of the Hermite–Hadamard–Fejér type

Theorem 10 *Let $F:[\alpha,\beta]\to\mathbb{R}$ and $G:(\alpha,\beta)\to[0,\infty)$ with $\alpha<\beta$. Suppose that the functions F and G satisfy the following conditions*:

(a) *F is strongly (η,ω)-convex with modulus $\mu\geq 0$, η bounded above on $F([\alpha,\beta])\times F([\alpha,\beta])$, and $\omega\in L_\infty([0,1])$;*
(b) *$F\in L_\infty([\alpha,\beta])$;*
(c) *$G\in L_1((\alpha,\beta))$;*
(d) *$G(\alpha+\beta-u)=G(u)$ for all $u\in(\alpha,\beta)$.*

Then we have the following inequalities:

$$\begin{aligned}
&\left(F\left(\frac{\alpha+\beta}{2}\right)-\omega\left(\frac{1}{2}\right)K_\eta\right)\int_\alpha^\beta G(u)\,du+\frac{\mu}{4}\int_\alpha^\beta(2u-\alpha-\beta)^2G(u)\,du\\
&\quad\leq\int_\alpha^\beta F(u)G(u)\,du\\
&\quad\leq\left(\frac{\eta(F(\alpha),F(\beta))+\eta(F(\beta),F(\alpha))}{2}\right)\int_\alpha^\beta\omega\left(\frac{\beta-u}{\beta-\alpha}\right)G(u)\,du\\
&\qquad+\left(\frac{F(\alpha)+F(\beta)}{2}\right)\int_\alpha^\beta G(u)\,du-\mu\int_\alpha^\beta(u-\alpha)(\beta-u)G(u)\,du\\
&\quad\leq\left(\frac{F(\alpha)+F(\beta)}{2}\right)\int_\alpha^\beta G(u)\,du+K_\eta\int_\alpha^\beta\omega\left(\frac{\beta-u}{\beta-\alpha}\right)G(u)\,du\\
&\qquad-\mu\int_\alpha^\beta(u-\alpha)(\beta-u)G(u)\,du,
\end{aligned}\tag{5}$$

where K_η is an upper bound of η.

Proof For all $\tau\in[0,1]$, we have

$$F\left(\frac{\alpha+\beta}{2}\right)=F\left[\frac{1}{2}\left(\frac{\alpha+\beta+\tau(\beta-\alpha)}{2}\right)+\left(1-\frac{1}{2}\right)\left(\frac{\alpha+\beta-\tau(\beta-\alpha)}{2}\right)\right].$$

Since F is strongly (η,ω)-convex, we obtain

$$\begin{aligned}
F\left(\frac{\alpha+\beta}{2}\right)&\leq\omega\left(\frac{1}{2}\right)\eta\left(F\left(\frac{\alpha+\beta+\tau(\beta-\alpha)}{2}\right),F\left(\frac{\alpha+\beta-\tau(\beta-\alpha)}{2}\right)\right)\\
&\quad+F\left(\frac{\alpha+\beta-\tau(\beta-\alpha)}{2}\right)-\frac{\mu\tau^2}{4}(\beta-\alpha)^2
\end{aligned}\tag{6}$$

for all $\tau\in[0,1]$. Since K_η is an upper bound of η, we get

$$F\left(\frac{\alpha+\beta}{2}\right)\leq\omega\left(\frac{1}{2}\right)K_\eta+F\left(\frac{\alpha+\beta-\tau(\beta-\alpha)}{2}\right)-\frac{\mu\tau^2}{4}(\beta-\alpha)^2,\quad\tau\in[0,1],$$

that is,

$$F\left(\frac{\alpha+\beta}{2}\right)-\omega\left(\frac{1}{2}\right)K_\eta \le F\left(\frac{\alpha+\beta-\tau(\beta-\alpha)}{2}\right)-\frac{\mu\tau^2}{4}(\beta-\alpha)^2 \tag{7}$$

for all $\tau\in[0,1]$. Similarly, we can also write

$$F\left(\frac{\alpha+\beta}{2}\right)=F\left[\frac{1}{2}\left(\frac{\alpha+\beta-\tau(\beta-\alpha)}{2}\right)+\left(1-\frac{1}{2}\right)\left(\frac{\alpha+\beta+\tau(\beta-\alpha)}{2}\right)\right].$$

From this inequality we get

$$F\left(\frac{\alpha+\beta}{2}\right)-\omega\left(\frac{1}{2}\right)K_\eta \le F\left(\frac{\alpha+\beta+\tau(\beta-\alpha)}{2}\right)-\frac{\mu\tau^2}{4}(\beta-\alpha)^2 \tag{8}$$

for all $\tau\in[0,1]$. Adding (7) and (8), we obtain the following inequality for $\tau\in[0,1]$:

$$\begin{aligned}F\left(\frac{\alpha+\beta}{2}\right)-\omega\left(\frac{1}{2}\right)K_\eta &\le \frac{1}{2}F\left(\frac{\alpha+\beta-\tau(\beta-\alpha)}{2}\right)+\frac{1}{2}F\left(\frac{\alpha+\beta+\tau(\beta-\alpha)}{2}\right)\\ &\quad-\frac{\mu\tau^2}{4}(\beta-\alpha)^2.\end{aligned}\tag{9}$$

Multiplying (9) by $G(\frac{\alpha+\beta+\tau(\beta-\alpha)}{2})$, integrating over $(0,1)$ with respect to the variable τ, and using item (d) and a change of variable, we get

$$\begin{aligned}&\left(F\left(\frac{\alpha+\beta}{2}\right)-\omega\left(\frac{1}{2}\right)K_\eta\right)\frac{2}{\beta-\alpha}\int_{\frac{\alpha+\beta}{2}}^{\beta}G(u)\,du\\ &\quad\le\frac{1}{\beta-\alpha}\int_{\alpha}^{\frac{\alpha+\beta}{2}}F(u)G(u)\,du+\frac{1}{\beta-\alpha}\int_{\frac{\alpha+\beta}{2}}^{\beta}F(u)G(u)\,du\\ &\qquad-\frac{\mu}{2}\frac{1}{\beta-\alpha}\int_{\frac{\alpha+\beta}{2}}^{\beta}(2u-\alpha-\beta)^2G(u)\,du.\end{aligned}$$

This implies that

$$\begin{aligned}&2\left(F\left(\frac{\alpha+\beta}{2}\right)-\omega\left(\frac{1}{2}\right)K_\eta\right)\int_{\frac{\alpha+\beta}{2}}^{\beta}G(u)\,du\\ &\quad\le\int_{\alpha}^{\beta}F(u)G(u)\,du-\frac{\mu}{2}\int_{\frac{\alpha+\beta}{2}}^{\beta}(2u-\alpha-\beta)^2G(u)\,du.\end{aligned}\tag{10}$$

Multiplying again (9) by $G(\frac{\alpha+\beta-\tau(\beta-\alpha)}{2})$ and proceeding as before, we get

$$\begin{aligned}&2\left(F\left(\frac{\alpha+\beta}{2}\right)-\omega\left(\frac{1}{2}\right)K_\eta\right)\int_{\alpha}^{\frac{\alpha+\beta}{2}}G(u)\,du\\ &\quad\le\int_{\alpha}^{\beta}F(u)G(u)\,du-\frac{\mu}{2}\int_{\alpha}^{\frac{\alpha+\beta}{2}}(2u-\alpha-\beta)^2G(u)\,du.\end{aligned}\tag{11}$$

Adding (10) and (11) gives

$$\left(F\left(\frac{\alpha+\beta}{2}\right)-\omega\left(\frac{1}{2}\right)K_\eta\right)\int_\alpha^\beta G(u)\,du$$
$$\leq \int_\alpha^\beta F(u)G(u)\,du-\frac{\mu}{4}\int_\alpha^\beta(2u-\alpha-\beta)^2G(u)\,du, \tag{12}$$

which gives the first inequality.

Next, we prove the second inequality. For this, let u be any element in $[\alpha,\beta]$. Then u can be expressed as

$$u=\tau\alpha+(1-\tau)\beta \quad \text{with } \tau=\frac{\beta-u}{\beta-\alpha}.$$

Using the strong (η,ω)-convexity of F, we obtain

$$F(u)\leq F(\beta)+\omega\left(\frac{\beta-u}{\beta-\alpha}\right)\eta\big(F(\alpha),F(\beta)\big)-\mu(u-\alpha)(\beta-u).$$

Multiplying this inequality by $G(u)$ and integrating over (α,β) with respect to the variable u, we get

$$\int_\alpha^\beta F(u)G(u)\,du\leq F(\beta)\int_\alpha^\beta G(u)\,du+\eta\big(F(\alpha),F(\beta)\big)\int_\alpha^\beta\omega\left(\frac{\beta-u}{\beta-\alpha}\right)G(u)\,du$$
$$-\mu\int_\alpha^\beta(u-\alpha)(\beta-u)G(u)\,du. \tag{13}$$

Similarly, we can also write

$$u=\tau\beta+(1-\tau)\alpha \quad \text{with } \tau=\frac{u-\alpha}{\beta-\alpha}.$$

Applying again the strong (η,ω)-convexity of F gives

$$F(u)\leq F(\alpha)+\omega\left(\frac{u-\alpha}{\beta-\alpha}\right)\eta\big(F(\beta),F(\alpha)\big)-\mu(u-\alpha)(\beta-u).$$

Multiplying this inequality by $G(u)$, proceeding as outlined before, and noting that $G(\alpha+u(\beta-\alpha))=G(\beta-u(\beta-\alpha))$, we get

$$\int_\alpha^\beta F(u)G(u)\,du\leq F(\alpha)\int_\alpha^\beta G(u)\,du+\eta\big(F(\beta),F(\alpha)\big)\int_\alpha^\beta\omega\left(\frac{\beta-u}{\beta-\alpha}\right)G(u)\,du$$
$$-\mu\int_\alpha^\beta(u-\alpha)(\beta-u)G(u)\,du, \tag{14}$$

where we have used the fact that

$$\int_\alpha^\beta\omega\left(\frac{u-\alpha}{\beta-\alpha}\right)G(u)\,du=\int_\alpha^\beta\omega\left(\frac{\beta-u}{\beta-\alpha}\right)G(u)\,du.$$

Now adding (13) and (14) gives

$$\int_{\alpha}^{\beta} F(u)G(u)\,du \le \left(\frac{\eta(F(\alpha),F(\beta))+\eta(F(\beta),F(\alpha))}{2}\right)\int_{\alpha}^{\beta}\omega\left(\frac{\beta-u}{\beta-\alpha}\right)G(u)\,du + \left(\frac{F(\alpha)+F(\beta)}{2}\right)\int_{\alpha}^{\beta}G(u)\,du - \mu\int_{\alpha}^{\beta}(u-\alpha)(\beta-u)G(u)\,du. \quad (15)$$

The last inequality follows by using the upper bound K_η in (15):

$$\begin{aligned}
&\left(\frac{\eta(F(\alpha),F(\beta))+\eta(F(\beta),F(\alpha))}{2}\right)\int_{\alpha}^{\beta}\omega\left(\frac{\beta-u}{\beta-\alpha}\right)G(u)\,du \\
&\quad + \left(\frac{F(\alpha)+F(\beta)}{2}\right)\int_{\alpha}^{\beta}G(u)\,du - \mu\int_{\alpha}^{\beta}(u-\alpha)(\beta-u)G(u)\,du \\
&\le \left(\frac{F(\alpha)+F(\beta)}{2}\right)\int_{\alpha}^{\beta}G(u)\,du + K_\eta\int_{\alpha}^{\beta}\omega\left(\frac{\beta-u}{\beta-\alpha}\right)G(u)\,du \\
&\quad - \mu\int_{\alpha}^{\beta}(u-\alpha)(\beta-u)G(u)\,du.
\end{aligned}$$

This completes the proof. □

Corollary 11 *Let $F:[\alpha,\beta]\to\mathbb{R}$ and $G:(\alpha,\beta)\to[0,\infty)$ with $\alpha<\beta$. Suppose that the functions F and G satisfy the following conditions*:

(a) *F is strongly η-convex with modulus $\mu\ge 0$ and η bounded above on $F([\alpha,\beta])\times F([\alpha,\beta])$*;

(b) *$F\in L_\infty([\alpha,\beta])$*;

(c) *$G\in L_1((\alpha,\beta))$*;

(d) *$G(\alpha+\beta-u)=G(u)$ for all $u\in(\alpha,\beta)$.*

Then we have the following inequalities:

$$\begin{aligned}
&\left(F\left(\frac{\alpha+\beta}{2}\right)-\frac{K_\eta}{2}\right)\int_{\alpha}^{\beta}G(u)\,du + \frac{\mu}{4}\int_{\alpha}^{\beta}(2u-\alpha-\beta)^2G(u)\,du \\
&\le \int_{\alpha}^{\beta}F(u)G(u)\,du \\
&\le \left(\frac{\eta(F(\alpha),F(\beta))+\eta(F(\beta),F(\alpha))}{2(\beta-\alpha)}\right)\int_{\alpha}^{\beta}(\beta-u)G(u)\,du \\
&\quad + \left(\frac{F(\alpha)+F(\beta)}{2}\right)\int_{\alpha}^{\beta}G(u)\,du - \mu\int_{\alpha}^{\beta}(u-\alpha)(\beta-u)G(u)\,du \\
&\le \left(\frac{F(\alpha)+F(\beta)}{2}\right)\int_{\alpha}^{\beta}G(u)\,du + \frac{K_\eta}{\beta-\alpha}\int_{\alpha}^{\beta}(\beta-u)G(u)\,du \\
&\quad - \mu\int_{\alpha}^{\beta}(u-\alpha)(\beta-u)G(u)\,du,
\end{aligned}$$

where K_η is an upper bound of η.

Proof The desired result follows by setting $\omega(\tau)=\tau$ in Theorem 10. □

Remark 12 If we take $\mu = 0$ in Corollary 11, then we recapture Theorem 3 due to Delavar and Dragomir. By taking $G(\tau) = 1$ for all $\tau \in (\alpha, \beta)$ in Corollary 11 we recover Theorem 6 due to Awan et al. Also, Corollary 11 reduces to Theorem 2 by taking $G \equiv 1$ and $\mu = 0$.

Corollary 13 *Let $F : [\alpha, \beta] \to \mathbb{R}$ and $H : (\alpha, \beta) \to [0, \infty)$ with $\alpha < \beta$. Suppose that the functions F and H satisfy the following conditions*:

(a) *F is strongly (η, ω)-convex with modulus $\mu \geq 0$, η bounded above on $F([\alpha, \beta]) \times F([\alpha, \beta])$, and $\omega \in L_\infty([0,1])$*;
(b) *$F \in L_\infty([\alpha, \beta])$*;
(c) *$H \in L_1((\alpha, \beta))$*.

Then we have the following inequalities:

$$
\begin{aligned}
&\left(F\left(\frac{\alpha+\beta}{2}\right) - \omega\left(\frac{1}{2}\right)K_\eta\right)\int_\alpha^\beta H(u)\,du \\
&\quad + \frac{\mu}{8}\int_\alpha^\beta (2u-\alpha-\beta)^2\bigl(H(u)+H(\alpha+\beta-u)\bigr)\,du \\
&\quad \leq \frac{1}{2}\int_\alpha^\beta F(u)\bigl(H(u)+H(\alpha+\beta-u)\bigr)\,du \\
&\quad \leq \left(\frac{\eta(F(\alpha),F(\beta))+\eta(F(\beta),F(\alpha))}{4}\right)\int_\alpha^\beta \left[\omega\left(\frac{\beta-u}{\beta-\alpha}\right)+\omega\left(\frac{u-\alpha}{\beta-\alpha}\right)\right]H(u)\,du \\
&\quad + \left(\frac{F(\alpha)+F(\beta)}{2}\right)\int_\alpha^\beta H(u)\,du - \frac{\mu}{2}\int_\alpha^\beta (u-\alpha)(\beta-u)\bigl(H(u)+H(\alpha+\beta-u)\bigr)\,du \\
&\quad \leq \left(\frac{F(\alpha)+F(\beta)}{2}\right)\int_\alpha^\beta H(u)\,du + \frac{K_\eta}{2}\int_\alpha^\beta \left[\omega\left(\frac{\beta-u}{\beta-\alpha}\right)+\omega\left(\frac{u-\alpha}{\beta-\alpha}\right)\right]H(u)\,du \\
&\quad - \frac{\mu}{2}\int_\alpha^\beta (u-\alpha)(\beta-u)\bigl(H(u)+H(\alpha+\beta-u)\bigr)\,du,
\end{aligned}
$$

where K_η is an upper bound of η.

Proof Let $G : (\alpha, \beta) \to \mathbb{R}$ be the function defined by

$$G(u) = H(u) + H(\alpha+\beta-u), \quad u \in (\alpha, \beta).$$

Since $H \in L_1((\alpha, \beta))$, it follows also that $G \in L_1((\alpha, \beta))$. Also, by the definition of the function G we have that for $u \in (\alpha, \beta)$,

$$G(\alpha+\beta-u) = H(\alpha+\beta-u) + H(u) = G(u).$$

Hence, items (c) and (d) of Theorem 10 are satisfied. Therefore, applying Theorem 10 to the function G, we get the desired inequalities. □

3.2 Application to the k-Riemann–Liouville fractional operators

We start by recalling the definition of the k-Riemann–Liouville fractional integrals: the left- and right-sided k-Riemann–Liouville fractional integral operators ${}_k\mathcal{J}^{\epsilon}_{\alpha^+}$ and ${}_k\mathcal{J}^{\epsilon}_{\beta^-}$ of order $\epsilon > 0$ for a real-valued continuous function $F(x)$ are defined as

$$ {}_k\mathcal{J}^{\epsilon}_{\alpha^+}F(x) = \frac{1}{k\Gamma_k(\epsilon)}\int_\alpha^x (x-t)^{\frac{\epsilon}{k}-1}F(t)\,dt, \quad x > \alpha, \tag{16}$$

and

$$
{}_k\mathcal{J}^{\epsilon}_{\beta^-}F(x) = \frac{1}{k\Gamma_k(\epsilon)} \int_x^{\beta} (t-x)^{\frac{\epsilon}{k}-1} F(t)\,dt, \quad x < \beta, \tag{17}
$$

where $k > 0$, and Γ_k is the k-gamma function given by

$$
\Gamma_k(x) := \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}}\,dt, \quad \mathrm{Re}(x) > 0,
$$

with the properties $\Gamma_k(x+k) = x\Gamma_k(x)$ and $\Gamma_k(k) = 1$.

In what follows, we will need the following functions $\mathcal{U}, \mathcal{V}, \mathcal{W} : [\alpha, \beta] \to \mathbb{R}$ defined by

$$
\begin{aligned}
&\mathcal{U}(x) = (2x - \alpha - \beta)^2, \\
&\mathcal{V}(x) = (x - \alpha)(\beta - x),
\end{aligned}
$$

and

$$
\mathcal{W}(x) = \omega\left(\frac{\beta - x}{\beta - \alpha}\right) + \omega\left(\frac{x - \alpha}{\beta - \alpha}\right).
$$

Applying Corollary 13, we get the following result.

Corollary 14 *Let $F : [\alpha, \beta] \to \mathbb{R}$ with $\alpha < \beta$. Suppose that the function F satisfies the following conditions*:

(a) *F is strongly (η, ω)-convex with modulus $\mu \geq 0$, η bounded above on $F([\alpha, \beta]) \times F([\alpha, \beta])$, and $\omega \in L_{\infty}([0,1])$;*

(b) *$F \in L_{\infty}([\alpha, \beta])$.*

Then we have the following inequalities:

$$
\begin{aligned}
&F\left(\frac{\alpha+\beta}{2}\right) - \omega\left(\frac{1}{2}\right)K_{\eta} \\
&\quad + \frac{\mu}{8}\frac{\Gamma_k(\epsilon+k)}{(\beta-\alpha)^{\frac{\epsilon}{k}}}\left[{}_k\mathcal{J}^{\epsilon}_{\alpha^+}\mathcal{U}(\beta) + {}_k\mathcal{J}^{\epsilon}_{\beta^-}\mathcal{U}(\alpha)\right] \\
&\leq \frac{\Gamma_k(\epsilon+k)}{2(\beta-\alpha)^{\frac{\epsilon}{k}}}\left[{}_k\mathcal{J}^{\epsilon}_{\alpha^+}F(\beta) + {}_k\mathcal{J}^{\epsilon}_{\beta^-}F(\alpha)\right] \\
&\leq \frac{\Gamma_k(\epsilon+k)}{(\beta-\alpha)^{\frac{\epsilon}{k}}}\left(\frac{\eta(F(\alpha),F(\beta)) + \eta(F(\beta),F(\alpha))}{4}\right){}_k\mathcal{J}^{\epsilon}_{\alpha^+}\mathcal{W}(\beta) \\
&\quad + \frac{F(\alpha)+F(\beta)}{2} - \frac{\mu}{2}\frac{\Gamma_k(\epsilon+k)}{(\beta-\alpha)^{\frac{\epsilon}{k}}}\left[{}_k\mathcal{J}^{\epsilon}_{\alpha^+}\mathcal{V}(\beta) + {}_k\mathcal{J}^{\epsilon}_{\beta^-}\mathcal{V}(\alpha)\right] \\
&\leq \frac{K_{\eta}}{2}\frac{\Gamma_k(\epsilon+k)}{(\beta-\alpha)^{\frac{\epsilon}{k}}}{}_k\mathcal{J}^{\epsilon}_{\alpha^+}\mathcal{W}(\beta) \\
&\quad + \frac{F(\alpha)+F(\beta)}{2} - \frac{\mu}{2}\frac{\Gamma_k(\epsilon+k)}{(\beta-\alpha)^{\frac{\epsilon}{k}}}\left[{}_k\mathcal{J}^{\epsilon}_{\alpha^+}\mathcal{V}(\beta) + {}_k\mathcal{J}^{\epsilon}_{\beta^-}\mathcal{V}(\alpha)\right],
\end{aligned} \tag{18}
$$

where K_{η} is an upper bound of η.

Proof Let

$$H(x)=\frac{(\beta-x)^{\frac{\epsilon}{k}-1}}{k\Gamma_k(\epsilon)},\quad x\in(\alpha,\beta),$$

where $\epsilon, k>0$. The function H clearly satisfies the conditions of Corollary 13 since

$$\begin{aligned}\int_\alpha^\beta H(x)\,dx&=\frac{1}{k\Gamma_k(\epsilon)}\int_\alpha^\beta(\beta-x)^{\frac{\epsilon}{k}-1}\,dx\\&=\frac{(\beta-\alpha)^{\frac{\epsilon}{k}}}{\Gamma_k(\epsilon+k)}<\infty.\end{aligned}$$

We obtain the intended inequalities by applying Corollary 13 to the function H and the following identities:

$$\begin{aligned}&\int_\alpha^\beta(2u-\alpha-\beta)^2\big(H(u)+H(\alpha+\beta-u)\big)\,du={}_k\mathcal{J}^\epsilon_{\alpha^+}\mathcal{U}(\beta)+{}_k\mathcal{J}^\epsilon_{\beta^-}\mathcal{U}(\alpha),\\&\int_\alpha^\beta F(u)\big(H(u)+H(\alpha+\beta-u)\big)\,du={}_k\mathcal{J}^\epsilon_{\alpha^+}F(\beta)+{}_k\mathcal{J}^\epsilon_{\beta^-}F(\alpha),\\&\int_\alpha^\beta\left[\omega\left(\frac{\beta-u}{\beta-\alpha}\right)+\omega\left(\frac{u-\alpha}{\beta-\alpha}\right)\right]H(u)\,du={}_k\mathcal{J}^\epsilon_{\alpha^+}\mathcal{W}(\beta),\end{aligned}$$

and

$$\int_\alpha^\beta(u-\alpha)(\beta-u)\big(H(u)+H(\alpha+\beta-u)\big)\,du={}_k\mathcal{J}^\epsilon_{\alpha^+}\mathcal{V}(\beta)+{}_k\mathcal{J}^\epsilon_{\beta^-}\mathcal{V}(\alpha).\qquad\square$$

Corollary 15 *Let $F:[\alpha,\beta]\to\mathbb{R}$ with $\alpha<\beta$. Suppose that the function F satisfies the following conditions*:

(a) *F is strongly η-convex with modulus $\mu\geq 0$ and η bounded above on $F([\alpha,\beta])\times F([\alpha,\beta])$;*

(b) *$F\in L_\infty([\alpha,\beta])$.*

Then we have the following inequalities:

$$\begin{aligned}&F\left(\frac{\alpha+\beta}{2}\right)-\frac{K_\eta}{2}\\&\quad+\frac{\mu}{8}\frac{\Gamma_k(\epsilon+k)}{(\beta-\alpha)^{\frac{\epsilon}{k}}}\left[{}_k\mathcal{J}^\epsilon_{\alpha^+}\mathcal{U}(\beta)+{}_k\mathcal{J}^\epsilon_{\beta^-}\mathcal{U}(\alpha)\right]\\&\leq\frac{\Gamma_k(\epsilon+k)}{2(\beta-\alpha)^{\frac{\epsilon}{k}}}\left[{}_k\mathcal{J}^\epsilon_{\alpha^+}F(\beta)+{}_k\mathcal{J}^\epsilon_{\beta^-}F(\alpha)\right]\\&\leq\frac{\eta(F(\alpha),F(\beta))+\eta(F(\beta),F(\alpha))}{4}\\&\quad+\frac{F(\alpha)+F(\beta)}{2}-\frac{\mu}{2}\frac{\Gamma_k(\epsilon+k)}{(\beta-\alpha)^{\frac{\epsilon}{k}}}\left[{}_k\mathcal{J}^\epsilon_{\alpha^+}\mathcal{V}(\beta)+{}_k\mathcal{J}^\epsilon_{\beta^-}\mathcal{V}(\alpha)\right]\\&\leq\frac{K_\eta}{2}+\frac{F(\alpha)+F(\beta)}{2}\\&\quad-\frac{\mu}{2}\frac{\Gamma_k(\epsilon+k)}{(\beta-\alpha)^{\frac{\epsilon}{k}}}\left[{}_k\mathcal{J}^\epsilon_{\alpha^+}\mathcal{V}(\beta)+{}_k\mathcal{J}^\epsilon_{\beta^-}\mathcal{V}(\alpha)\right],\end{aligned}\tag{19}$$

where K_η is an upper bound of η.

Proof The proof follows by setting $\omega(\tau) = \tau$, $\tau \in [0,1]$, in Corollary 14. For this, we notice that

$$\mathcal{W}(x) = 1, \quad x \in [\alpha,\beta],$$

and thus

$$_k\mathcal{J}^{\epsilon}_{\alpha^+}\mathcal{W}(\beta) = \frac{(\beta-\alpha)^{\frac{\epsilon}{k}}}{\Gamma_k(\epsilon+k)}. \qquad \square$$

By substituting $\mu = 0$ in the corollary, we obtain the following result for the η-convex functions.

Corollary 16 *Let $F : [\alpha,\beta] \to \mathbb{R}$ with $\alpha < \beta$. Suppose that the function F satisfies the following conditions:*
(a) *F is η-convex and η bounded above on $F([\alpha,\beta]) \times F([\alpha,\beta])$;*
(b) *$F \in L_\infty([\alpha,\beta])$.*
Then we have the following inequalities:

$$\begin{aligned} &F\left(\frac{\alpha+\beta}{2}\right) - \frac{K_\eta}{2} \\ &\quad\le \frac{\Gamma_k(\epsilon+k)}{2(\beta-\alpha)^{\frac{\epsilon}{k}}}\left[{}_k\mathcal{J}^{\epsilon}_{\alpha^+}F(\beta) + {}_k\mathcal{J}^{\epsilon}_{\beta^-}F(\alpha)\right] \\ &\quad\le \frac{\eta(F(\alpha),F(\beta)) + \eta(F(\beta),F(\alpha))}{4} + \frac{F(\alpha)+F(\beta)}{2} \\ &\quad\le \frac{K_\eta}{2} + \frac{F(\alpha)+F(\beta)}{2}, \end{aligned} \tag{20}$$

where K_η is an upper bound of η.

3.3 More integral inequalities

We now proceed to obtain more results associated with this new class of functions. For this, we will need the following lemma.

Lemma 17 ([11]) *Let $F : J \subset \mathbb{R} \to \mathbb{R}$, and let $\alpha,\beta \in J$ with $\alpha < \beta$. Suppose F satisfies the following conditions:*
(a) *F is differentiable in the interior of J denoted by J°;*
(b) *$F' \in L_1([\alpha,\beta])$.*
Then, for any $\lambda \in \mathbb{R}$, we have the identity

$$\lambda F(\alpha) + (1-\lambda)F(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(u)\,du = (\beta-\alpha)\int_0^1 (\tau-\lambda)F'\big((1-\tau)\alpha+\tau\beta\big)\,d\tau.$$

Theorem 18 *Assume that a function F satisfies the conditions of Lemma 17. If, in addition, $|F'|$ is strongly (η,ω)-convex on $[\alpha,\beta]$ with modulus $\mu \ge 0$ and $\omega \in L_\infty([0,1])$, then for any*

$\lambda \in \mathbb{R}$, *we have the following inequalities*:

$$\left|\lambda F(\alpha) + (1-\lambda)F(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(u)\,du\right| \leq \begin{cases} h(\lambda - \frac{1}{2})|F'(\alpha)| + h\eta(|F'(\beta)|, |F'(\alpha)|)\mathcal{W}_1(\lambda) + \frac{\mu h^3(1-2\lambda)}{12}, & \lambda \geq 1, \\ h(\lambda^2 - \lambda + \frac{1}{2})|F'(\alpha)| + h\eta(|F'(\beta)|, |F'(\alpha)|)\mathcal{W}_2(\lambda) \\ \quad + \frac{\mu h^3(2\lambda^4 - 4\lambda^3 + 2\lambda - 1)}{12}, & 0 < \lambda < 1, \\ h(\frac{1}{2} - \lambda)|F'(\alpha)| + h\eta(|F'(\beta)|, |F'(\alpha)|)\mathcal{W}_3(\lambda) + \frac{\mu h^3(2\lambda-1)}{12}, & \lambda \leq 0, \end{cases} \tag{21}$$

where $h = \beta - \alpha$,

$$\mathcal{W}_1(\lambda) = \int_0^1 (\lambda - \tau)\omega(\tau)\,d\tau,$$
$$\mathcal{W}_2(\lambda) = \int_0^\lambda (\lambda - \tau)\omega(\tau)\,d\tau + \int_\lambda^1 (\tau - \lambda)\omega(\tau)\,d\tau,$$

and

$$\mathcal{W}_3(\lambda) = \int_0^1 (\tau - \lambda)\omega(\tau)\,d\tau.$$

Proof We start by observing that

$$\int_0^1 |\tau - \lambda|\,d\tau = \begin{cases} \lambda - \frac{1}{2}, & \lambda \geq 1, \\ \lambda^2 - \lambda + \frac{1}{2}, & 0 < \lambda < 1, \\ \frac{1}{2} - \lambda, & \lambda \leq 0, \end{cases} \tag{22}$$

and

$$\int_0^1 |\tau - \lambda|\tau(1-\tau)\,d\tau = \begin{cases} \frac{2\lambda - 1}{12}, & \lambda \geq 1, \\ \frac{4\lambda^3 - 2\lambda^4 - 2\lambda + 1}{12}, & 0 < \lambda < 1, \\ \frac{1-2\lambda}{12}, & \lambda \leq 0. \end{cases} \tag{23}$$

Now using Lemma 17 and the strong (η, ω)-convexity of $|F'|$, we get

$$\begin{aligned} &\left|\lambda F(\alpha) + (1-\lambda)F(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(u)\,du\right| \\ &\quad = \left|(\beta - \alpha)\int_0^1 (\tau - \lambda)F'\big((1-\tau)\alpha + \tau\beta\big)\,d\tau\right| \\ &\quad \leq (\beta - \alpha)\int_0^1 |\tau - \lambda|\big|F'\big((1-\tau)\alpha + \tau\beta\big)\big|\,d\tau \\ &\quad \leq (\beta - \alpha)\int_0^1 |\tau - \lambda|\big[\big|F'(\alpha)\big| + \omega(\tau)\eta\big(\big|F'(\beta)\big|, \big|F'(\alpha)\big|\big) - \mu\tau(1-\tau)(\beta-\alpha)^2\big]\,d\tau \end{aligned}$$

$$= (\beta-\alpha)\left|F'(\alpha)\right| \int_0^1 |\tau-\lambda|\,d\tau + (\beta-\alpha)\eta\left(\left|F'(\beta)\right|,\left|F'(\alpha)\right|\right)\int_0^1 |\tau-\lambda|\omega(\tau)\,d\tau$$
$$- \mu(\beta-\alpha)^3 \int_0^1 |\tau-\lambda|\tau(1-\tau)\,d\tau.$$

Hence the desired result is obtained by using (22) and (23). □

Theorem 19 *Assume that a function F satisfies the conditions of Lemma 17. If, in addition, $|F'|^q$ $(q>1)$ is strongly (η,ω)-convex on $[\alpha,\beta]$ with modulus $\mu \ge 0$ and $\omega \in L_\infty([0,1])$, then for any $\lambda \in \mathbb{R}$, we have the following inequalities*:

$$\left|\lambda F(\alpha) + (1-\lambda)F(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(u)\,du\right| \le \begin{cases} (\beta-\alpha)(\frac{\lambda^{p+1}-(\lambda-1)^{p+1}}{p+1})^{\frac{1}{p}}\mathcal{A}_q(\eta,\omega;\mu), & \lambda \ge 1, \\ (\beta-\alpha)(\frac{\lambda^{p+1}-(1-\lambda)^{p+1}}{p+1})^{\frac{1}{p}}\mathcal{A}_q(\eta,\omega;\mu), & 0<\lambda<1, \\ (\beta-\alpha)(\frac{(1-\lambda)^{p+1}-(-\lambda)^{p+1}}{p+1})^{\frac{1}{p}}\mathcal{A}_q(\eta,\omega;\mu), & \lambda \le 0, \end{cases} \tag{24}$$

where $\frac{1}{p}+\frac{1}{q}=1$, and

$$\mathcal{A}_q(\eta,\omega;\mu) = \left(\left|F'(\alpha)\right|^q + \eta\left(\left|F'(\beta)\right|^q,\left|F'(\alpha)\right|^q\right)\int_0^1 \omega(\tau)\,d\tau - \frac{\mu(\beta-\alpha)^2}{6}\right)^{\frac{1}{q}}.$$

Proof From identity (22) we get

$$\int_0^1 |\tau-\lambda|^p\,d\tau = \begin{cases} \frac{\lambda^{p+1}-(\lambda-1)^{p+1}}{p+1}, & \lambda \ge 1, \\ \frac{\lambda^{p+1}-(1-\lambda)^{p+1}}{p+1}, & 0<\lambda<1, \\ \frac{(1-\lambda)^{p+1}-(-\lambda)^{p+1}}{p+1}, & \lambda \le 0, \end{cases} \tag{25}$$

where $p>1$.

Using again Lemma 17, the strong (η,ω)-convexity of $|F'|^q$, and the Hölder inequality, we obtain

$$\left|\lambda F(\alpha) + (1-\lambda)F(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(u)\,du\right|$$
$$= \left|(\beta-\alpha)\int_0^1 (\tau-\lambda)F'\big((1-\tau)\alpha+\tau\beta\big)\,d\tau\right|$$
$$\le (\beta-\alpha)\int_0^1 |\tau-\lambda|\left|F'\big((1-\tau)\alpha+\tau\beta\big)\right|\,d\tau$$
$$\le (\beta-\alpha)\left(\int_0^1 |\tau-\lambda|^p\,d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \left|F'\big((1-\tau)\alpha+\tau\beta\big)\right|^q\,d\tau\right)^{\frac{1}{q}}$$
$$\le (\beta-\alpha)\left(\int_0^1 |\tau-\lambda|^p\,d\tau\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 \left[\left|F'(\alpha)\right|^q + \omega(\tau)\eta\left(\left|F'(\beta)\right|^q,\left|F'(\alpha)\right|^q\right) - \mu\tau(1-\tau)(\beta-\alpha)^2\right]\,d\tau\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&= (\beta-\alpha)\biggl(\int_0^1 |\tau-\lambda|^p\,d\tau\biggr)^{\frac{1}{p}} \\
&\quad\times \biggl(\bigl|F'(\alpha)\bigr|^q + \eta\bigl(\bigl|F'(\beta)\bigr|^q, \bigl|F'(\alpha)\bigr|^q\bigr)\int_0^1 \omega(\tau)\,d\tau - \frac{\mu(\beta-\alpha)^2}{6}\biggr)^{\frac{1}{q}}.
\end{aligned}
$$

The desired result is obtained by employing identity (25). □

Theorem 20 *Assume that a function F satisfies the conditions of Lemma 17. If, in addition, $|F'|^q$ ($q \geq 1$) is strongly (η,ω)-convex on $[\alpha,\beta]$ with modulus $\mu \geq 0$ and $\omega \in L_\infty([0,1])$, then for any $\lambda \in \mathbb{R}$, we have the following inequalities:*

$$
\begin{aligned}
&\biggl|\lambda F(\alpha) + (1-\lambda)F(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(u)\,du\biggr| \\
&\quad\leq \begin{cases}
h(\lambda-\frac{1}{2})^{1-\frac{1}{q}}[|F'(\alpha)|^q(\lambda-\frac{1}{2}) + \eta(|F'(\beta)|^q,|F'(\alpha)|^q)\mathcal{W}_1(\lambda) \\
\qquad + \frac{\mu h^2(1-2\lambda)}{12}]^{\frac{1}{q}}, & \lambda \geq 1, \\
h(\lambda^2-\lambda+\frac{1}{2})^{1-\frac{1}{q}}[|F'(\alpha)|(\lambda^2-\lambda+\frac{1}{2}) + \eta(|F'(\beta)|^q,|F'(\alpha)|^q)\mathcal{W}_2(\lambda) \\
\qquad + \frac{\mu h^2(2\lambda^4-4\lambda^3+2\lambda-1)}{12}]^{\frac{1}{q}}, & 0 < \lambda < 1, \\
h(\frac{1}{2}-\lambda)^{1-\frac{1}{q}}[|F'(\alpha)|(\frac{1}{2}-\lambda) + \eta(|F'(\beta)|,|F'(\alpha)|)\mathcal{W}_3(\lambda) + \frac{\mu h^2(2\lambda-1)}{12}]^{\frac{1}{q}}, & \lambda \leq 0,
\end{cases}
\end{aligned}
$$

where h, $\mathcal{W}_1(\lambda)$, $\mathcal{W}_2(\lambda)$, and $\mathcal{W}_3(\lambda)$ are defined in Theorem 18.

Proof Applying Lemma 17, the strong (η,ω)-convexity of $|F'|^q$, and the Hölder inequality, we get

$$
\begin{aligned}
&\biggl|\lambda F(\alpha) + (1-\lambda)F(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(u)\,du\biggr| \\
&\quad= \biggl|(\beta-\alpha)\int_0^1 (\tau-\lambda)F'\bigl((1-\tau)\alpha+\tau\beta\bigr)\,d\tau\biggr| \\
&\quad\leq (\beta-\alpha)\int_0^1 |\tau-\lambda|\bigl|F'\bigl((1-\tau)\alpha+\tau\beta\bigr)\bigr|\,d\tau \\
&\quad\leq (\beta-\alpha)\biggl(\int_0^1 |\tau-\lambda|\,d\tau\biggr)^{1-\frac{1}{q}}\biggl(\int_0^1 |\tau-\lambda|\bigl|F'\bigl((1-\tau)\alpha+\tau\beta\bigr)\bigr|^q\,d\tau\biggr)^{\frac{1}{q}} \\
&\quad\leq (\beta-\alpha)\biggl(\int_0^1 |\tau-\lambda|\,d\tau\biggr)^{1-\frac{1}{q}} \\
&\qquad\times \biggl(\int_0^1 |\tau-\lambda|\bigl[\bigl|F'(\alpha)\bigr|^q + \omega(\tau)\eta\bigl(\bigl|F'(\beta)\bigr|^q,\bigl|F'(\alpha)\bigr|^q\bigr) - \mu\tau(1-\tau)(\beta-\alpha)^2\bigr]\,d\tau\biggr)^{\frac{1}{q}} \\
&\quad= (\beta-\alpha)\biggl(\int_0^1 |\tau-\lambda|\,d\tau\biggr)^{1-\frac{1}{q}} \\
&\qquad\times \biggl(\bigl|F'(\alpha)\bigr|^q\int_0^1 |\tau-\lambda|\,d\tau + \eta\bigl(\bigl|F'(\beta)\bigr|^q,\bigl|F'(\alpha)\bigr|^q\bigr)\int_0^1 |\tau-\lambda|\omega(\tau)\,d\tau \\
&\qquad - \mu(\beta-\alpha)^2\int_0^1 |\tau-\lambda|\tau(1-\tau)\,d\tau\biggr)^{\frac{1}{q}}.
\end{aligned}
$$

The intended result is reached by employing identities (22) and (23). □

4 Conclusion

We introduced the notion of (η,ω)-convexity. We established inequalities of the Hermite–Hadamard–Fejér type and many novel results for the class of (η,ω)-convex functions. Applications are also provided by employing Corollary 13 to the k-Riemann–Liouville fractional integral operators. We anticipate that this new class of functions will inspire further investigation in this direction. Some further work in this direction can be found in [12–29].

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