Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On some Schwarz type inequalities

Miodrag Mateljević[1] and Adel Khalfallah[2*]

*Correspondence:
khelifa@kfupm.edu.sa
[2]Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, Dhahran, 31261, Kingdom of Saudi Arabia
Full list of author information is available at the end of the article

Abstract
First, we establish some Schwarz type inequalities for mappings with bounded Laplacian, then we obtain boundary versions of the Schwarz lemma.

MSC: 30C80; 31A05

Keywords: Schwarz's lemma; Poisson's equation; Harmonic functions; Subharmonic functions

1 Introduction and preliminaries

Motivated by the role of the Schwarz lemma in complex analysis and numerous fundamental results, see for instance [16, 19] and references therein, in 2016, the first author [1](a) has posted on ResearchGate the project "Schwarz lemma, the Carathéodory and Kobayashi Metrics and Applications in Complex Analysis".[a] Various discussions regarding the subject can also be found in the Q&A section on ResearchGate under the question "What are the most recent versions of the Schwarz lemma?" [1](b).[b] In this project and in [16], cf. also [13], we developed the method related to holomorphic mappings with strip codomain (we refer to this method as the approach via the Schwarz–Pick lemma for holomorphic maps from the unit disc into a strip). It is worth mentioning that the Schwarz lemma has been generalized in various directions; see [2, 4, 7, 8, 13, 14, 18, 21] and the references therein.

Recently Wang and Zhu [20] and Chen and Kalaj [5] have studied boundary Schwarz lemma for solutions of Poisson's equation. They improved Heinz's theorem [10] and Theorem A below. We found that Theorem A is a forgotten result of Hethcote [11], published in 1977.

Note that previously Burgeth [3] improved the above result of Heinz and Theorem A for real-valued functions (it is easy to extend his result for complex-valued functions; see below) by removing the assumption $f(0) = 0$ but it is overlooked in the literature. Recently, Mateljević and Sveltik [18] proved a Schwarz lemma for real harmonic functions with values in $(-1, 1)$ using a completely different approach than Burgeth [3] and showed that the inequalities obtained are sharp.

In this paper, we further develop the method initiated in [18]. More precisely, we show that, if $\mathbb{U}$ denotes the open unit disc and $f : \mathbb{U} \to (-1, 1)$, $f \in C^2(\mathbb{U})$ and is continuous on $\overline{\mathbb{U}}$, and $|\Delta f| \le c$ on $\mathbb{U}$ for some $c > 0$, then the mapping $u = f \pm \frac{c}{4}(1 - |z|^2)$ is subharmonic or superharmonic and we estimate the harmonic function $P[u^*]$; see Theorem 2. Next, we extend the previous result to complex-valued functions; see Corollary 1. As an application, we provide an elementary proof of a theorem of Chen and Kalaj [5] giving an estimate of the solutions of the Poisson equations. Finally, we establish Schwarz lemmas at the boundary for solutions of $|\Delta f| \leq c$. Our results are generalizations of Theorem 1.1 [20] and Theorem 2 [5].

The proofs are mainly based on two ingredients, the first of which is a sharp Schwarz lemma for real harmonic functions with values in $(-1,1)$, see Theorem B, and the second is the principle of harmonic majoration, which is a consequence of the maximum principle for subharmonic functions.

1.1 Notations and background

In this paper $\mathbb{T}$ denotes the unit circle.

Recall that a real-valued function u, defined in an open subset D of the complex plane $\mathbb{C}$, is harmonic if it satisfies Laplace's equation $\Delta u = 0$ on D.

A real-valued function $u \in \mathcal{C}^2(D)$ is called subharmonic if $\Delta u(z) \geq 0$ for all $z \in D$.

Let P be the Poisson kernel, i.e., the function

$$P\big(z, e^{i\theta}\big) = \frac{1-|z|^2}{|z-e^{i\theta}|^2},$$

and let G be the Green function on the unit disc, i.e., the function

$$G(z,w) = \frac{1}{2\pi} \log\left|\frac{1-z\overline{w}}{z-w}\right|, \quad z, w \in \mathbb{U}, z \neq w.$$

Let $\phi \in L^1(\mathbb{T})$ be an integrable function on the unit circle. Then the function $P[\phi]$ given by

$$P[\phi](z) = \frac{1}{2\pi}\int_0^{2\pi} P\big(z, e^{i\theta}\big)\phi\big(e^{i\theta}\big)\,d\theta$$

is harmonic in $\mathbb{U}$ and has a radial limit that agrees with ϕ almost everywhere on $\mathbb{T}$.

For $g \in \mathcal{C}(\overline{\mathbb{U}})$, let

$$G[g](z) = \int_{\mathbb{U}} G(z,w)g(w)\,dm(w),$$

$|z| < 1$ and let $dm(w)$ denote the Lebesgue measure in $\mathbb{U}$.

If we consider the function

$$u(z) := P[\phi](z) - G[g](z),$$

then u satisfies the Poisson equation

$$\begin{cases} \Delta u = g & \text{on the disc } \mathbb{U}, \\ \lim_{r\to 1^-} u(re^{i\theta}) = \phi(e^{i\theta}) & \text{a.e. on the circle.} \end{cases}$$

One can easily see that the previous equation has a *non-unique* solution. Indeed, the Poisson kernel $P(z) = \frac{1-|z|^2}{|1-z|^2}$ is a harmonic function on the unit disc and $\lim_{r\to 1^-} P(re^{i\theta}) = 0$ a.e., but $P \neq 0$.

It is well known that, if ϕ is continuous on the unit circle, then the harmonic function $P[\phi]$ extends continuously on $\mathbb{T}$ and equals ϕ on $\mathbb{T}$; see Hörmander [12].

The following is a consequence of the maximum principle for subharmonic functions.

Theorem (Harmonic majoration) *Let u be a subharmonic function in $\mathcal{C}^2(\mathbb{U}) \cap \mathcal{C}(\overline{\mathbb{U}})$. Then*

$$u \le P[u|_{\mathbb{T}}] \quad \text{on } \mathbb{U}.$$

2 The Schwarz lemma for harmonic functions

In [10], Heinz proved that, if f is a harmonic mapping f from the unit disc into itself such that $f(0) = 0$, then

$$\left|f(z)\right| \le \frac{4}{\pi} \arctan |z|.$$

Moreover, this inequality is sharp for each point $z \in \mathbb{U}$.

This inequality for functions from the unit disk to unit ball of $\mathbb{C}^n$ are discussed in [9] to establish Landau's theorem for p-harmonic mappings in several variables.

Later, in 1977, Hethcote [11] improved the above result of Heinz by removing the assumption $f(0) = 0$ and showed the following.

Theorem A ([11]) *If f is a harmonic mapping from the unit disc into itself, then*

$$\left|f(z) - \frac{1-|z|^2}{1+|z|^2} f(0)\right| \le \frac{4}{\pi} \arctan |z|$$

holds for all $z \in \mathbb{U}$.

We remark that the estimate of Theorem A cannot be sharp for all values z in the unit disc.

Recently, Mateljević and Sveltik [18] proved a Schwarz lemma for real harmonic functions with values in $(-1, 1)$ using a completely different approach from Burgeth [3].

Theorem B ([18]) *Let $u : \mathbb{U} \to (-1, 1)$ be a harmonic function such that $u(0) = b$. Then*

$$m_b\big(|z|\big) \le u(z) \le M_b\big(|z|\big) \quad \text{for all } z \in \mathbb{U}.$$

Moreover, this inequality is sharp for each $z \in \mathbb{U}$, where $M_b(r) := \frac{4}{\pi} \arctan \frac{a+r}{1+ar}$, $m_b(r) := \frac{4}{\pi} \arctan \frac{a-r}{1-ar}$, and $a = \tan \frac{b\pi}{4}$.

Clearly Theorem B improves Theorem A for real harmonic functions, as one can check the following elementary proposition.

Proposition 2.1 *Let b be in $(-1, 1)$ and $r \in [0, 1)$. Then*

(1) $M_b(r) \le \frac{1-r^2}{1+r^2} b + \frac{4}{\pi} \arctan r =: A_b(r)$ *and* $m_b(r) \ge \frac{1-r^2}{1+r^2} b - \frac{4}{\pi} \arctan r$.

(2) *The mapping $b \mapsto M_b(r)$ is increasing on $(-1, 1)$.*

Using a standard rotation, we can extend Theorem B for *complex* harmonic functions from the unit disc into itself.

Theorem 1 *Let $f : \mathbb{U} \to \mathbb{U}$ be a harmonic function from the unit disc into itself. Then*

$$\left|f(z)\right| \leq M_{|f(0)|}\left(|z|\right)$$

holds for all $z \in \mathbb{U}$.

Proof Fix z_0 in the unit disc and choose unimodular λ such that $\lambda f(z_0) = |f(z_0)|$.

Define $u(z) = \Re(\lambda f(z))$.

Hence, using Theorem B, we get

$$\left|f(z_0)\right| = u(z_0) \leq M_{u(0)}\left(|z_0|\right) \leq M_{|f(0)|}\left(|z_0|\right),$$

as the mapping $b \mapsto M_b(|z_0|)$ is increasing. □

3 Schwarz lemma for mappings with bounded Laplacian

The following theorem is our main result of this section.

Theorem 2 *Let f be a $C^2(\mathbb{U})$ real-valued function, continuous on $\overline{\mathbb{U}}$ and $f^* = f|_{\mathbb{T}}$. Let $b = P[f^*](0)$, $c \in \mathbb{R}$ and K be a positive number such that $K \geq \|P[f^*]\|_\infty$.*

(i) *If f satisfies $\Delta f \geq -c$, then*

$$f(z) \leq KM_{b/K}\left(|z|\right) + \frac{c}{4}\left(1 - |z|^2\right)$$

holds for all $z \in \mathbb{U}$.

(ii) *If f satisfies $\Delta f \leq c$, then*

$$f(z) \geq Km_{b/K}\left(|z|\right) - \frac{c}{4}\left(1 - |z|^2\right)$$

holds for all $z \in \mathbb{U}$.

Proof (i) Define $f^0(z) = f(z) + \frac{c}{4}(|z|^2 - 1)$, and set $P[f^*](0) = b$. Then f^0 is subharmonic and $f^0 \leq P[f^*]$. As $\frac{1}{K}P[f^*]$ is a real harmonic function with codomain $(-1, 1)$, by Theorem B, we obtain $P[f^*](z) \leq KM_{b/K}(|z|)$. Thus

$$f(z) \leq KM_{b/K}\left(|z|\right) + \frac{c}{4}\left(1 - |z|^2\right), \quad \text{for all } z \in \mathbb{U}.$$

(ii) If f satisfies $\Delta f \leq c$, then define $f_0(z) = f(z) - \frac{c}{4}(|z|^2 - 1)$, and set $P[f^*](0) = b$. In a similar way, we show that the inequality

$$f(z) \geq Km_{b/K}\left(|z|\right) - \frac{c}{4}\left(1 - |z|^2\right)$$

holds for all $z \in \mathbb{U}$. □

For complex-valued functions with bounded Laplacian from the unit disc into itself, we prove the following.

Corollary 1 *Suppose that* $f : \mathbb{U} \to \mathbb{U}$, $f \in C^2(\mathbb{U})$ *and continuous on* $\overline{\mathbb{U}}$, *and* $|\Delta f| \le c$ *on* $\mathbb{U}$ *for some* $c > 0$. *Then*

$$\left|f(z)\right| \le M_b\left(|z|\right) + \frac{c}{4}\left(1 - |z|^2\right)$$

holds for all $z \in \mathbb{U}$, *where* $b = |P[f^*](0)|$.

Proof Fix z_0 in the unit disc and choose λ such that $\lambda f(z_0) = |f(z_0)|$. Define $u(z) = \Re(\lambda f(z))$ (we say that u is a real-valued harmonic associated to complex-valued harmonic f at z_0). We have $\Delta u = \Re(\lambda \Delta f)$. As u is a real function with codomain $(-1, 1)$ satisfying $|\Delta u| \le c$, by Theorem 2, we get

$$\left|u(z)\right| \le M_{b_1}\left(|z|\right) + \frac{c}{4}\left(1 - |z|^2\right), \quad \text{where } b_1 = P\left[u^*\right](0).$$

We have $b_1 = P[u^*](0) = \Re(\lambda P[f^*](0)) \le |P[f^*](0)|$. Hence

$$\left|f(z_0)\right| \le M_b\left(|z_0|\right) + \frac{c}{4}\left(1 - |z_0|^2\right),$$

where $b = |P[f^*](0)|$, as the mapping $b \mapsto M_b(|z_0|)$ is increasing. □

Under the conditions of the previous theorem and using Proposition 2.1 we obtain

$$\left|f(z) - \frac{1 - |z|^2}{1 + |z|^2} P\left[f^*\right](0)\right| \le \frac{4}{\pi} \arctan |z| + \frac{c}{4}\left(1 - |z|^2\right). \tag{3.1}$$

3.1 Applications

For a given continuous function $g : G \to \mathbb{C}$, Chen and Kalaj [5] established some Schwarz type Lemmas for mappings f in G satisfying the Poisson equation $\Delta f = g$, where G is a subset of the complex plane $\mathbb{C}$. Then they applied these results to obtain a Landau type theorem, which is a partial answer to the open problem in [6].

We provide a different and an elementary proof of Theorem C, giving a Schwarz type lemma for mappings satisfying Poisson's equations.

Theorem C ([5]) *Let* $g \in C(\overline{\mathbb{U}})$ *and* $\phi \in C(\mathbb{T})$. *If a complex-valued function* f *satisfies* $\Delta f = g$ *in* $\mathbb{U}$ *and* $f = \phi$ *in* $\mathbb{T}$, *then for* $z \in \mathbb{U}$

$$\left|f(z) - P[\phi](0)\frac{1 - |z|^2}{1 + |z|^2}\right| \le \frac{4}{\pi}\left\|P[\phi]\right\|_\infty \arctan |z| + \frac{1}{4}\|g\|_\infty\left(1 - |z|^2\right), \tag{3.2}$$

where $\|P[\phi]\|_\infty = \sup_{z\in\mathbb{U}} |P[\phi](z)|$ *and* $\|g\|_\infty = \sup_{z\in\mathbb{U}} |g(z)|$.

Now we show that Theorem 2 implies Theorem C.

We will consider first the case when f is a *real*-valued $C^2(\mathbb{U})$ function, continuous on $\overline{\mathbb{U}}$, satisfying $\Delta f = g$ and $f^* = \phi$. Let $K := \|P[\phi]\|_\infty$. By Theorem 2, we have

$$m_{b/K}\left(|z|\right)K - \|g\|_\infty \frac{(1 - |z|^2)}{4} \le f(z) \le M_{b/K}\left(|z|\right)K + \|g\|_\infty \frac{(1 - |z|^2)}{4},$$

where $b = P[\phi](0)$. Using Proposition 2.1(1), we get

$$M_{b/K}\big(|z|\big)K \le \frac{1-|z|^2}{1+|z|^2}b + \frac{4K}{\pi}\arctan|z|$$

and

$$m_{b/K}\big(|z|\big)K \ge \frac{1-|z|^2}{1+|z|^2}b - \frac{4K}{\pi}\arctan|z|.$$

Hence, the following inequality:

$$\left|f(z) - \frac{1-|z|^2}{1+|z|^2}P[\phi](0)\right| \le \frac{4}{\pi}\big\|P[\phi]\big\|_\infty \arctan|z| + \frac{1}{4}\|g\|_\infty\big(1-|z|^2\big)$$

holds for all $z \in \mathbb{U}$.

If f is a complex-valued function, we may consider $u = \Re(\lambda f)$, where λ is a complex number of modulus 1. Indeed, we have

$$u(z) - \frac{1-|z|^2}{1+|z|^2}P\big[u^*\big](0) = \Re\left(\lambda\left(f(z) - \frac{1-|z|^2}{1+|z|^2}P[\phi](0)\right)\right),$$

where $u^* = \Re(\lambda\phi)$ on $\mathbb{T}$. Now, one can choose λ such that

$$\left|f(z) - \frac{1-|z|^2}{1+|z|^2}P[\phi](0)\right| = u(z) - \frac{1-|z|^2}{1+|z|^2}P\big[u^*\big](0).$$

4 Boundary Schwarz lemmas

We establish Schwarz lemmas at the boundary for solutions of $|\Delta f| \le c$. Our results are generalizations of Theorem 1.1 [20] and Theorem 2 [5].

Theorem 3 *Suppose* $f \in C^2(\mathbb{U})$, *continuous on* $\overline{\mathbb{U}}$ *with codomain* $(-1,1)$, *such that* $\Delta f \ge -c$. *If* f *is differentiable at* $z = 1$ *with* $f(1) = 1$, *then the following inequality holds*:

$$f_x(1) \ge \frac{2}{\pi}\tan\frac{\pi}{4}(1-b) - \frac{c}{2},$$

where

$$b = P\big[f^*\big](0).$$

Before giving the proof, one can easily show that

$$M_b'(r) = \frac{4}{\pi}\left[\frac{1-a^2}{(a^2+1)r^2 + 4ar + a^2 + 1}\right].$$

Hence

$$M_b'(1) = \frac{2}{\pi}\left[\frac{1-a}{1+a}\right] = \frac{2}{\pi}\tan\frac{\pi}{4}(1-b),$$

as $a = \tan\frac{b\pi}{4}$.

Proof Since f is differentiable at $z = 1$, we know that

$$f(z) = 1 + f_z(1)(z-1) + f_{\bar{z}}(1)(\bar{z}-1) + o\big(|z-1|\big).$$

That is,

$$f_x(1) = \lim_{r\to 1^-} \frac{f(r)-1}{r-1}.$$

On the other hand, Theorem 2(i) leads to

$$1 - f(r) \geq 1 - M_b(r) - \frac{c}{4}\big(1-r^2\big).$$

Dividing by $(1-r)$ and letting $r \to 1^-$, we get

$$f_x(1) \geq M_b'(1) - \frac{c}{2}. \tag{4.1}$$

Thus

$$f_x(1) \geq \frac{2}{\pi} \tan\frac{\pi}{4}(1-b) - \frac{c}{2}.$$

□

Corollary 2 *Suppose $f \in C^2(\mathbb{U})$, with codomain $(-1,1)$, is continuous on $\overline{\mathbb{U}}$ and is differentiable at $z = 1$ with $f(1) = 1$.*

(i) *If $\Delta f \geq -c$, then*

$$f_x(1) \geq \frac{2}{\pi} - b - \frac{c}{2}.$$

(ii) *If $|\Delta f| \leq c$ and $f(0) = 0$, then $|b| \leq \frac{c}{4}$ and*

$$f_x(1) \geq \frac{2}{\pi} - \frac{3}{4}c,$$

where $b = P[f^*](0)$.

Proof (i) Using the inequality $M_b \leq A_b$ from Proposition 2.1 and $M_b(1) = A_b(1) = 1$, we get

$$M_b'(1) \geq A_b'(1) = \frac{2}{\pi} - b.$$

(ii) The estimate $|b| \leq \frac{c}{4}$ follows directly from Theorem 2 using the assumption $f(0) = 0$. □

Remark 4 One can also prove directly that $M_b'(1) \geq A_b'(1)$, that is,

$$\frac{2}{\pi}\tan\frac{\pi}{4}(1-b) \geq \frac{2}{\pi} - b \quad \text{for } b \in [0,1). \tag{4.2}$$

Using the convexity of the tangent function, we get

$$\tan x \geq 2\left(x - \frac{\pi}{4}\right) + 1 \quad \text{for } x \in [0, \pi/2).$$

For $b \in [0,1)$, let us substitute x by $\frac{\pi}{4}(1-b)$, we obtain

$$\frac{2}{\pi}\tan\frac{\pi}{4}(1-b) \geq \frac{2}{\pi} - b.$$

The following theorem is a generalization of Theorem 2 in [5] where the authors proved a Schwarz lemma on the boundary for a function f satisfying $\Delta f = g$ and under the assumption $f(0) = 0$.

Theorem 5 *Suppose that $f \in \mathcal{C}^2(\mathbb{U}) \cap \mathcal{C}(\overline{\mathbb{U}})$ is a function of $\mathbb{U}$ into $\mathbb{U}$ satisfying $|\Delta f| \leq c$, where $0 \leq c < \frac{4}{\pi}\tan\frac{\pi}{4}(1-b)$. If, for some $\xi \in \mathbb{T}$, $\lim_{r\to 1^-}|f(r\xi)| = 1$, then*

$$\liminf_{r\to 1^-}\frac{|f(\xi)-f(r\xi)|}{1-r} \geq \frac{2}{\pi}\tan\frac{\pi}{4}(1-b) - \frac{c}{2},$$

where $b = |P[f^*](0)|$.

If, in addition, we assume that $f(0) = 0$, then

$$|b| \leq \frac{c}{4}$$

and

$$\liminf_{r\to 1^-}\frac{|f(\xi)-f(r\xi)|}{1-r} \geq \frac{2}{\pi} - \frac{3}{4}c.$$

Proof Using Corollary 1, we have

$$\bigl|f(\xi)-f(r\xi)\bigr| \geq 1 - \bigl|f(r\xi)\bigr| \geq 1 - M_b(r) - \frac{c}{4}\bigl(1-r^2\bigr).$$

Thus

$$\liminf_{r\to 1^-}\frac{|f(\xi)-f(r\xi)|}{1-r} \geq \lim_{r\to 1^-}\frac{1-M_b(r)-\frac{c}{4}(1-r^2)}{1-r} = M_b'(1) - \frac{c}{2}.$$

The conclusion follows as $M_b'(1) = \frac{2}{\pi}\tan\frac{\pi}{4}(1-b)$.

If in addition, we assume that $f(0) = 0$, using the inequality (3.1), we obtain $|b| < \frac{c}{4}$. Hence

$$\liminf_{r\to 1^-}\frac{|f(\xi)-f(r\xi)|}{1-r} \geq \frac{2}{\pi}\tan\frac{\pi}{4}(1-b) - \frac{c}{2} \geq \frac{2}{\pi} - b - \frac{c}{2} \geq \frac{2}{\pi} - \frac{3}{4}c.$$

The second estimate follows from the inequality (4.2). □

Acknowledgements

This article was supported by the Deanship of Scientific Research (DSR) at King Fahd University of Petroleum and Minerals (KFUPM).

Funding

The first author is partially supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia, Grant No. 174 032. The second author would like to acknowledge the support provided by the Deanship of Scientific Research (DSR) at King Fahd University of Petroleum and Minerals (KFUPM).

Availability of data and materials

Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript read and approved the final manuscript.

Author details
[1]Faculty of mathematics, University of Belgrade, Belgrade, Republic of Serbia. [2]Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, Dhahran, 31261, Kingdom of Saudi Arabia.

Endnotes
[a] Motivated by Krantz' paper [15].
[b] The subject has been presented at Belgrade analysis seminar [17].

Publisher's Note

Received: 29 January 2020 Accepted: 5 June 2020 Published online: 12 June 2020

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