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Some approximation properties of new Kantorovich type q-analogue of Balázs–Szabados operators

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Abstract
In this paper, we define a new Kantorovich type q-analogue of the Balázs–Szabados operators, we give some local approximation properties of these operators and prove a Voronovskaja type theorem.

Keywords: q-calculus; Bernstein operators; q-Balázs–Szabados operators

1 Introduction

The well-known Bernstein polynomials belonging to a function $f(x)$ defined on the interval $[0,1]$ are given as follows:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} \quad (n = 1,2,\ldots).$$

If $f(x)$ is continuous on $[0,1]$, the polynomials $B_n(f,x)$ converge uniformly to $f(x)$. These polynomials have an important role in approximation theory and also in other fields of mathematics (see [14], [2], [25], [8], and [13]).

In [4] Balázs defined and studied approximation properties of Bernstein type rational functions

$$R_n(f;x) = \frac{1}{(1+a_n x)^n}\sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)\binom{n}{k}(a_n x)^k \quad (n = 1,2,\ldots), \tag{1}$$

where f is a real- and single-valued function defined on $[0,\infty)$, a_n and b_n are real numbers which are suitably chosen and do not depend on x. In [5] Balázs and Szabados together improved the estimate in [4] by choosing suitable a_n and b_n under some restrictions for $f(x)$.

On the other hand, in the last three decades q-calculus has gained a significant role in the approximation of functions by positive linear operators. Firstly, we give some notations and definitions of q-calculus. For any nonnegative integer n, the q-integer of the number

n is defined as

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} & \text{if } q \neq 1, \\ n & \text{if } q = 1, \end{cases} \quad \text{where } q \text{ is a positive real number.}$$

The q-factorial is defined by $[n]_q! = [1]_q[2]_q...[n]_q$ and $[0]_q! = 1$. For integers $0 \leq k \leq n$, the q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[k]_q![n-k]_q!} & \text{if } q \neq 1, \\ \binom{n}{k} & \text{if } q = 1. \end{cases}$$

The definite q-integral is defined by

$$\int_0^b f(t)\,d_qt = b(1-q)\sum_{j=0}^{\infty} f\left(bq^j\right)q^j, \quad 0 < q < 1, b > 0,$$

and

$$\int_a^b f(t)\,d_qt = \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt, \quad 0 < a < b.$$

For example, the q-integral of the function $f(t) = t^n$ on the interval $[0,1]$ is

$$\int_0^1 t^n\,d_qt = \frac{1}{[n+1]}.$$

One may read [2], [3], and [12] for more information on q-calculus.

Bernstein polynomials based on the q-integers were firstly introduced by Lupas [15] in 1987, and another generalization of Bernstein polynomials based on the q-integers was introduced by Phillips [23] in 1996. The q-Bernstein polynomials quickly gained the popularity, and then many operators based on the q-integers were introduced and examined by some other authors.

Different q-analogues of Balázs–Szabados operators have recently been studied by Doğru [7] and Ozkan ([22] and [21]). Approximation properties of the q-Balázs–Szabados complex operators are studied by Mahmudov in [16] and by İspir and Özkan in [11]. The Balázs–Szabados operator based on the q-integers defined by Mahmudov in [16] is as follows:

$$R_{n,q}(f,x) = \frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} f\left(\frac{[k]_q}{b_n}\right)\begin{bmatrix} n \\ k \end{bmatrix}_q (a_nx)^k \prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_qa_nx\right), \tag{2}$$

where $q > 0$, f is a real-valued function defined on $[0,\infty)$, $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^{\beta}$, $0 < \beta \leq \frac{2}{3}$, $n \in \mathbb{N}$, and $x \neq -\frac{1}{a_n}$.

In [16] the following equalities for the q-Balázs–Szabados operators $R_{n,q}(f,x)$ are given:

$$R_{n,q}(1,x) = 1, R_{n,q}(t,x) = \frac{x}{(1+a_nx)},$$

$$R_{n,q}\left(t^2,x\right)=\frac{x}{b_n(1+a_nx)^2}+\frac{x^2}{(1+a_nx)^2},$$
$$R_{n,q}\left((t-x),x\right)=\frac{-a_nx^2}{1+a_nx},\qquad R_{n,q}\left((t-x)^2,x\right)=\frac{x+a_n^2b_nx^4}{b_n(1+a_nx)^2}.$$

On the other hand, q-Balázs–Szabados–Kantorovich operator (q-BSK operator) defined by Ozkan in [22] is as follows:

$$\widetilde{R}_n(f;q,x)=\frac{b_n}{\prod_{s=0}^{n-1}(1+q^sa_nx)}\sum_{j=0}^{n}q^{j(j-1)/2}\begin{bmatrix}n\\j\end{bmatrix}_q(a_nx)^j\int_{\frac{q[j]_q}{b_n}}^{\frac{q[j+1]_q}{b_n}}f(t)\,d_qt, \tag{3}$$

where f is a nondecreasing and continuous function on $[0,\infty)$, $a_n=[n]_q^{\beta-1}$ and $b_n=[n]_q^{\beta}$ for all $n\in\mathbb{N}$, $q\in(0,1)$, and $0<\beta\leq\frac{2}{3}$. Since f is nondecreasing and from the definition of q-integral, q-BSK operator is a positive operator.

The operators defined by (2) are summation type operators, which are not capable of approximating integrable functions. On the other hand, to guarantee the positivity of the q-BSK operators defined by (3), f must be a nondecreasing function. The main motivation of this paper is to construct a new Kantorovich type q-analogue of the Balázs–Szabados operators that approximates also the integrable functions on the interval $[0,\infty)$ and maintain the positivity without nondecreasing restriction on f, obtain the local approximation properties, and establish a Voronovskaja type theorem for these new operators. We define the operators as follows:

$$R_{n,q}^{*}(f,x)=\sum_{k=0}^{n}r_{n,k}(q,x)\int_0^1 f\left(\frac{[k]_q+q^kt}{b_n}\right)d_qt,$$

where $r_{n,k}(q,x)=\frac{1}{(1+a_nx)^n}\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_q(a_nx)^k\prod_{s=0}^{n-k-1}(1+(1-q)[s]_qa_nx)$, $q\in(0,1)$, $a_n=[n]_q^{\beta-1}$, $b_n=[n]_q^{\beta}$, $0<\beta\leq\frac{2}{3}$, $n\in\mathbb{N}$, $x\geq0$, and f is a real-valued continuous function defined on $[0,\infty)$. The operators $R_{n,q}^{*}(f,x)$ are positive and linear operators. Since the lower limit of the q-integral is 0, f does not need to be a nondecreasing function, the operators are positive for all real-valued continuous functions defined on $[0,\infty)$.

The paper is organized as follows. In Sect. 2 we introduce a new Kantorovich type q-analogue of the Balázs–Szabados operators, we give a recurrence formula and evaluate the moments of these operators. In Sect. 3 we study local approximation properties and prove a Voronovkaja type theorem.

2 Operators and estimation of moments

Definition 1 Let $0<q<1$. For $f:[0,\infty)\to\mathbb{R}$, a new Kantorovich type q-analogue of the Balázs–Szabados operator is defined as follows:

$$R_{n,q}^{*}(f,x)=\sum_{k=0}^{n}r_{n,k}(q,x)\int_0^1 f\left(\frac{[k]_q+q^kt}{b_n}\right)d_qt,$$

where $r_{n,k}(q,x)=\frac{1}{(1+a_nx)^n}\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_q(a_nx)^k\prod_{s=0}^{n-k-1}(1+(1-q)[s]_qa_nx)$, $a_n=[n]_q^{\beta-1}$, $b_n=[n]_q^{\beta}$, $0<\beta\leq\frac{2}{3}$, $n\in\mathbb{N}$, $x\geq0$.

In the case $q = 1$, these polynomials reduce to

$$R_n^*(f,x) = \sum_{k=0}^{n} r_{n,k}(x) \int_0^1 f\left(\frac{k+t}{b_n}\right) dt,$$

where $r_{n,k}(x) = \frac{1}{(1+a_n x)^n}\binom{n}{k}(a_n x)^k$, $a_n = n^{\beta-1}$, $b_n = n^{\beta}$, $0 < \beta \le \frac{2}{3}$, $n \in \mathbb{N}$, $x \ge 0$, and in this case they coincide with the q-BSK operators defined by (3). It can be easily seen that $R_n^*(t^m,x) = \widetilde{R}_n(t^m;x)$ for $m = 0,1,2$.

Lemma 2 *For all $n \in \mathbb{N}$, $x \in [0,\infty)$, $m \in \mathbb{Z}^+ \cup \{0\}$, and $0 < q < 1$, we have*

$$R_{n,q}^*(t^m,x) = \sum_{j=0}^{m} \binom{m}{j} \frac{1}{b_n^{m-j}[m-j+1]_q} \sum_{i=0}^{m-j} \binom{m-j}{i} (a_n)^i (q^n-1)^i R_{n,q}(t^{i+j},x),$$

where $R_{n,q}(f,x)$ is the q-Balázs–Szabados operator defined in [16].

Proof By direct calculation, the recurrence formula is obtained as follows:

$$R_{n,q}^*(t^m,x) = \sum_{k=0}^{n} r_{n,k}(q,x) \int_0^1 \left(\frac{[k]_q + q^k t}{b_n}\right)^m d_q t.$$

Using the binomial formula for $([k]_q + q^k t)^m$ and evaluating the q-integral, we get

$$\begin{aligned}
R_{n,q}^*(t^m,x) &= \sum_{k=0}^{n} r_{n,k}(q,x) \sum_{j=0}^{m} \binom{m}{j} \frac{q^{(m-j)k}[k]_q^j}{b_n^m[m-j+1]_q} \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{[m-j+1]_q} \sum_{k=0}^{n} q^{(m-j)k} \frac{[k]_q^j}{(b_n)^m} r_{n,k}(q,x) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{(b_n)^{m-j}[m-j+1]_q} \\
&\quad \times \sum_{k=0}^{n} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^k-1)^i \frac{[k]_q^j}{b_n^j} r_{n,k}(q,x) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{(b_n)^{m-j}[m-j+1]_q} \\
&\quad \times \sum_{i=0}^{m-j} \binom{m-j}{i} \frac{(q^n-1)^i}{[n]_q^i} \sum_{k=0}^{n} \frac{[k]_q^{i+j}}{(b_n)^j} r_{n,k}(q,x) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{(b_n)^{m-j}[m-j+1]_q} \\
&\quad \times \sum_{i=0}^{m-j} \binom{m-j}{i} (a_n)^i (q^n-1)^i \sum_{k=0}^{n} \frac{[k]_q^{i+j}}{(b_n)^{i+j}} r_{n,k}(q,x).
\end{aligned}$$

Since the last summation is $R_{n,q}(t^{i+j},x)$, we get

$$R^*_{n,q}(t^m,x) = \sum_{j=0}^{m} \binom{m}{j} \frac{1}{b_n^{m-j}[m-j+1]_q} \times \sum_{i=0}^{m-j} \binom{m-j}{i} (a_n)^i (q^n-1)^i R_{n,q}(t^{i+j},x). \qquad \square$$

In the following lemma, we calculate $R_{n,q}(f,x)$ for the monomials $f(t)=t^m$ for $m=3,4$, and we give an estimation for $R_{n,q}((t-x)^4,x)$ which later will be used to estimate the fourth-order central moment of $R^*_{n,q}(f,x)$ that is needed for the Voronoskaja type theorem.

Lemma 3 *For $q\in(0,1)$, for all $n\in\mathbb{N}$, we have the following*:

$$R_{n,q}(t^3,x) = \frac{q^3[n-1]_q[n-2]_q}{a_n b_n^2}\left(\frac{a_n x}{1+a_n x}\right)^3 + \frac{q[n-1]_q(2+q)}{a_n b_n^2}\left(\frac{a_n x}{1+a_n x}\right)^2 + \frac{1}{a_n b_n^2}\frac{a_n x}{1+a_n x},$$

$$R_{n,q}(t^4,x) = \frac{q^6[n]_q[n-1]_q[n-2]_q[n-3]_q}{b_n^4}\left(\frac{a_n x}{1+a_n x}\right)^4 + \frac{(q^5+2q^4+3q^3)[n]_q[n-1]_q[n-2]_q}{b_n^4}\left(\frac{a_n x}{1+a_n x}\right)^3 + \frac{(q^3+3q^2+3q)[n]_q[n-1]_q}{b_n^4}\left(\frac{a_n x}{1+a_n x}\right)^2 + \frac{[n]_q}{b_n^4}\left(\frac{a_n x}{1+a_n x}\right),$$

$$R_{n,q}\bigl((t-x)^4,x\bigr) \le \frac{1}{b_n^2}C(q,a) \quad \textit{for } x\in[0,a],$$

where $C(q,a)$ is a positive constant which depends on q and a.

Proof

$$R_{n,q}(t^3,x) = \frac{1}{(1+a_n x)^n}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q^3}{b_n^3}(a_n x)^k \prod_{s=0}^{n-k-1}\bigl(1+(1-q)[s]_q a_n x\bigr),$$

since $[k]_q\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = [n]_q\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]_q$ and $b_n=[n]_q a_n$, we have

$$R_{n,q}(t^3,x) = \frac{1}{a_n^3}\frac{1}{(1+a_n x)^n}\sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{[k]_q^2}{[n]_q^2}(a_n x)^k \prod_{s=0}^{n-k-1}\bigl(1+(1-q)[s]_q a_n x\bigr).$$

Now by using the facts that

$$[k]_q = q[k-1]_q+1 \quad \text{and} \quad [k-1]_q\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = [n-1]_q\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q,$$

we get

$$R_{n,q}(t^3,x) = \frac{1}{a_n^3}\frac{1}{(1+a_n x)^n}$$

$$\begin{aligned}
&\times \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{(q[k-1]_q+1)^2}{[n]_q^2} (a_n x)^k \prod_{s=0}^{n-k-1} \bigl(1+(1-q)[s]_q a_n x\bigr) \\
&= \frac{q^3[n-1]_q[n-2]_q}{a_n b_n^2} \left(\frac{a_n x}{1+a_n x}\right)^3 \sum_{k=0}^{n-3} r_{n-3,k}(q,x) \\
&\quad + \frac{q[n-1]_q(2+q)}{a_n b_n^2} \left(\frac{a_n x}{1+a_n x}\right)^2 \sum_{k=0}^{n-2} r_{n-2,k}(q,x) + \frac{1}{a_n b_n^2} \frac{a_n x}{1+a_n x} \sum_{k=0}^{n-1} r_{n-1,k}(q,x) \\
&= \frac{q^3[n-1]_q[n-2]_q}{a_n b_n^2} \left(\frac{a_n x}{1+a_n x}\right)^3 + \frac{q[n-1]_q(2+q)}{a_n b_n^2} \left(\frac{a_n x}{1+a_n x}\right)^2 \\
&\quad + \frac{1}{a_n b_n^2} \frac{a_n x}{1+a_n x}
\end{aligned}$$

$R_{n,q}(t^4,x)$ is calculated in a similar way.

For the estimation of $R_{n,q}((t-x)^4,x)$, we write the formula explicitly:

$$\begin{aligned}
&R_{n,q}\bigl((t-x)^4,x\bigr) \\
&\quad = R_{n,q}\bigl((t^4,x\bigr) - 4xR_{n,q}\bigl((t^3,x\bigr) + 6x^2 R_{n,q}\bigl((t^2,x\bigr) - 4x^3 R_{n,q}((t,x) + x^4 \\
&\quad = \left[q^6 \frac{[n-1]_q[n-2]_q[n-3]_q}{[n]_q^3(1+a_n x)^4} - 4q^3 \frac{[n-1]_q[n-2]_q}{[n]_q^2(1+a_n x)^3} + \frac{6}{(1+a_n x)^2} - \frac{4}{1+a_n x} + 1\right] x^4 \\
&\qquad + \left[\bigl(q^5+2q^4+3q^3\bigr) \frac{[n-1]_q[n-2]_q}{a_n[n]_q^3(1+a_n x)^3} - \frac{4q(2+q)[n-1]_q}{a_n[n]_q(1+a_n x)^2} + \frac{6}{a_n[n]_q(1+a_n x)^2}\right] x^3 \\
&\qquad + \left[\bigl(q^3+3q^2+3q\bigr) \frac{[n-1]_q}{a_n^2[n]_q^3(1+a_n x)^2} - \frac{4}{a_n^2[n]_q^2(1+a_n x)}\right] x^2 \\
&\qquad + \frac{1}{a_n^3[n]_q^3(1+a_n x)} x.
\end{aligned}$$

by using the facts that $q[n-1]_q = [n]_q - 1$, $q^2[n-2]_q = [n]_q - [2]_q$, and $q^3[n-3]_q = [n]_q - [3]_q$, we get

$$\begin{aligned}
R_{n,q}\bigl((t-x)^4,x\bigr) &= \frac{S_1}{a_n^3[n]_q^3(1+a_n x)} x + \frac{S_2}{a_n^2[n]_q^3(1+a_n x)^2} x^2 \\
&\quad + \frac{S_3}{a_n[n]_q^3(1+a_n x)^3} x^3 \frac{S_4}{[n]_q^3(1+a_n x)^4} x^4,
\end{aligned}$$

where

$$\begin{aligned}
&S_1 = 1, \qquad S_2 = \bigl(q^2+3q+3\bigr)\bigl([n]_q-1\bigr) - 4[n]_q(1+a_n x), \\
&S_3 = \bigl(q^2+2q+3\bigr)\bigl([n]_q-1\bigr)\bigl([n]_q-[2]_q\bigr) \\
&\qquad - 4\bigl(1+[2]_q\bigr)[n]_q\bigl([n]_q-1\bigr)(1+a_n x) + 6[n]_q^2(1+a_n x), \quad \text{and} \\
&S_4 = \bigl([n]_q-1\bigr)\bigl([n]_q-[2]_q\bigr)\bigl([n]_q-[3]_q\bigr) - 4[n]_q\bigl([n]_q-1\bigr)\bigl([n]_q-[2]_q\bigr)(1+a_n x) \\
&\qquad + 6[n]_q^3(1+a_n x)^2 - 4[n]_q^3(1+a_n x)^3 + [n]_q^3(1+a_n x)^4.
\end{aligned}$$

Now, if we take into account the powers of $[n]_q$ in S_1, S_2, S_3, S_4 and the facts that $\frac{1}{1+a_nx} \leq 1$ and $\frac{a_nx}{1+a_nx} \leq 1$, we see that, for $x \in [0,a]$, $R_{n,q}((t-x)^4,x) \leq \frac{1}{b_n^2}C(q,a)$. □

In the following lemma, we give a formula for the *mth*-order central moments of the q-Balázs–Szabados operators $R_{n,q}(f,x)$ in terms of the well-known q-Bernstein polynomials.

Lemma 4 *For all $n \in \mathbb{N}$, $x \in [0,\infty)$, we have*

$$R_{n,q}\bigl((t-x)^m,x\bigr) = \frac{1}{a_n^m}\sum_{j=0}^{m}\binom{m}{j}(-a_nx)^{m-j}B_{n,q}\left(t^j,\frac{a_nx}{1+a_nx}\right),$$

where $B_{n,q}(f,x) = \sum_{k=0}^{n} f(\frac{[k]_q}{[n]_q})\left[{n \atop k}\right]_q(x)^k\prod_{s=0}^{n-k-1}(1-q^sx)$ are the q-Bernstein polynomials.

Proof By writing $R_{n,q}((t-x)^m,x)$ explicitly, we have

$$\begin{aligned} R_{n,q}\bigl((t-x)^m,x\bigr) &= \frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{[k]_q}{b_n}-x\right)^m (a_nx)^k \\ &\quad\times\prod_{s=0}^{n-k-1}\bigl(1+(1-q)[s]_qa_nx\bigr). \end{aligned}$$

Now, since $b_n = [n]_qa_n$,

$$\begin{aligned} R_{n,q}\bigl((t-x)^m,x\bigr) &= \frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{a_n^m}\left(\frac{[k]_q}{[n]_q}-a_nx\right)^m (a_nx)^k \\ &\quad\times\prod_{s=0}^{n-k-1}\bigl(1+(1-q)[s]_qa_nx\bigr) \\ &= \frac{1}{a_n^m}\sum_{j=0}^{m}\binom{m}{j}(-a_nx)^{m-j}\sum_{k=0}^{n}\left(\frac{[k]_q}{[n]_q}\right)^j\begin{bmatrix} n \\ k \end{bmatrix}_q\left(\frac{a_nx}{1+a_nx}\right)^k\left(\frac{1}{1+a_nx}\right)^{n-k} \\ &\quad\times\prod_{s=0}^{n-k-1}\bigl(1+(1-q)[s]_qa_nx\bigr) \\ &= \frac{1}{a_n^m}\sum_{j=0}^{m}\binom{m}{j}(-a_nx)^{m-j} \\ &\quad\times\sum_{k=0}^{n}\left(\frac{[k]_q}{[n]_q}\right)^j\begin{bmatrix} n \\ k \end{bmatrix}_q\left(\frac{a_nx}{1+a_nx}\right)^k\prod_{s=0}^{n-k-1}\left(1-q^s\frac{a_nx}{1+a_nx}\right) \\ &= \frac{1}{a_n^m}\sum_{j=0}^{m}\binom{m}{j}(-a_nx)^{m-j}B_{n,q}\left(t^j,\frac{a_nx}{1+a_nx}\right), \end{aligned}$$

which gives us the desired formula. □

Similar results for the new Kantorovich type q-analogue of the Balázs–Szabados operators $R_{n,q}^*(f,x)$ as in Lemma 3 are given in the next lemma.

Lemma 5 *For all $n \in \mathbb{N}$, $x \in [0,\infty)$, and $0 < q < 1$, we have the following equalities*:

$$\begin{aligned}
R_{n,q}^{*}(1,x) &= 1,\\
R_{n,q}^{*}(t,x) &= \frac{2q}{[2]_q}\frac{x}{1+a_nx} + \frac{1}{[2]_q b_n},\\
R_{n,q}^{*}(t^2,x) &= \frac{q[n-1]_q}{[n]_q}\frac{4q^3+q^2+q}{[2]_q[3]_q}\left(\frac{x}{1+a_nx}\right)^2 + \frac{4q^3+5q^2+3q}{b_n[2]_q[3]_q}\frac{x}{1+a_nx}\\
&\quad + \frac{1}{[3]_q b_n^2}.
\end{aligned}$$

Proof The proof is done by using the recurrence formula given in Lemma 2. $R_{n,q}^{*}(1,x)$ is obvious.

$$\begin{aligned}
R_{n,q}^{*}(t,x) &= \frac{1}{[2]_q b_n}\left(R_{n,q}(1,x) + a_n\left(q^n-1\right)R_{n,q}(t,x)\right) + R_{n,q}(t,x)\\
&= \frac{1}{[2]_q b_n} + \frac{2q}{[2]_q}R_{n,q}(t,x).
\end{aligned}$$

Now, by using the equality for $R_{n,q}(t,x)$ which is given in [16], we get

$$R_{n,q}^{*}(t,x) = \frac{1}{[2]_q b_n} + \frac{2q}{[2]_q}\frac{x}{1+a_nx}.$$

In a similar way,

$$\begin{aligned}
&R_{n,q}^{*}\left(t^2,x\right)\\
&\quad= \frac{1}{[3]_q b_n^2}\left(R_{n,q}(1,x) + 2a_n\left(q^n-1\right)R_{n,q}(t,x) + a_n^2\left(q^n-1\right)^2 R_{n,q}\left(t^2,x\right)\right)\\
&\qquad + \frac{2}{[2]_q b_n}\left(R_{n,q}(t,x) + a_n\left(q^n-1\right)R_{n,q}\left(t^2,x\right)\right) + R_{n,q}\left(t^2,x\right)\\
&\quad= \frac{1}{[3]_q b_n^2} + \frac{4q^2+2q}{[2]_q[3]_q b_n}\frac{x}{1+a_nx}\\
&\qquad + \frac{4q^3+q^2+q}{[2]_q[3]_q}\left(\frac{x}{b_n(1+a_nx)} + \frac{q[n-1]_q}{[n]_q}\left(\frac{x}{1+a_nx}\right)^2\right)\\
&\quad= \frac{1}{[3]_q b_n^2} + \frac{4q^3+5q^2+3q}{b_n[2]_q[3]_q}\frac{x}{1+a_nx}\\
&\qquad + \frac{q[n-1]_q}{[n]_q}\frac{4q^3+q^2+q}{[2]_q[3]_q}\left(\frac{x}{1+a_nx}\right)^2.
\end{aligned}$$

□

Lemma 6 *For all $n \in \mathbb{N}$ and $0 < q < 1$, we have the following estimations*:

$$\begin{aligned}
\left(R_{n,q}^{*}(t-x,x)\right)^2 &\leq \frac{2}{b_n}\left\{\frac{(1-q^n)^2}{b_n}\left(\frac{1}{1+q} + \frac{a_nx}{1-q}\right)^2 + \frac{1}{b_n}\right\}, \quad x \in [0,\infty),\\
R_{n,q}^{*}\left((t-x)^2,x\right) &\leq \frac{2}{b_n}\left(\frac{1}{[3]_q b_n} + \frac{x(1+a_n^2 b_n x^3)}{(1+a_nx)^2}\right), \quad x \in [0,\infty),
\end{aligned}$$

$$R^*_{n,q}\big((t-x)^4,x\big) \le \frac{1}{b_n^2} C_1(q,a) \quad \textit{for } x \in [0,a],$$

where $C_1(q,a)$ *is a positive constant which depends on* q *and* a.

Proof First, we estimate $(R^*_{n,q}(t-x,x))^2$. For $x \in [0,\infty)$,

$$\begin{aligned}
\big(R^*_{n,q}(t-x,x)\big)^2 &= \left(\frac{2q}{[2]_q}\frac{x}{1+a_nx} - x + \frac{1}{[2]_q b_n}\right)^2 \\
&\le 2\left(\frac{(1-q)x}{[2]_q(1+a_nx)} + \frac{a_nx^2}{1+a_nx}\right)^2 + 2\left(\frac{1}{[2]_q b_n}\right)^2 \\
&= \frac{2}{b_n^2}\left(\frac{1-q^n}{1+q}\frac{a_nx}{1+a_nx} + \frac{1-q^n}{1-q}(a_nx)\frac{a_nx}{1+a_nx}\right)^2 + 2\left(\frac{1}{[2]_q b_n}\right)^2 \\
&\le \frac{2}{b_n^2}\left(\frac{1-q^n}{1+q} + \frac{1-q^n}{1-q}(a_nx)\right)^2 + 2\left(\frac{1}{[2]_q b_n}\right)^2 \\
&\le \frac{2}{b_n^2}\left\{\big(1-q^n\big)^2\left(\frac{1}{1+q} + \frac{a_nx}{1-q}\right)^2 + 1\right\} \\
&= \frac{2}{b_n}\left\{\frac{(1-q^n)^2}{b_n}\left(\frac{1}{1+q} + \frac{a_nx}{1-q}\right)^2 + \frac{1}{b_n}\right\}.
\end{aligned}$$

For the estimation of $R^*_{n,q}((t-x)^2,x)$, we use $R_{n,q}((t-x)^2,x)$ which is calculated in [16]. For $x \in [0,\infty)$,

$$\begin{aligned}
R^*_{n,q}\big((t-x)^2,x\big) &= \sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 \left(\frac{[k]_q + q^k t}{b_n} - x\right)^2 d_qt \\
&= \sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 \left(\frac{q^k t}{b_n} + \frac{[k]_q}{b_n} - x\right)^2 d_qt \\
&\le 2\sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 \left(\frac{q^k t}{b_n}\right)^2 d_qt + 2\sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 \left(\frac{[k]_q}{b_n} - x\right)^2 d_qt \\
&= 2\sum_{k=0}^{n} r_{n,k}(q,x)\frac{q^{2k}}{[3]_q b_n^2} + 2R_{n,q}\big((t-x)^2,x\big) \\
&\le \frac{2}{[3]_q b_n^2} + 2\left(\frac{x + a_n^2 b_n x^4}{b_n(1+a_nx)^2}\right) \\
&= \frac{2}{b_n}\left(\frac{1}{[3]_q b_n} + \frac{x(1+a_n^2 b_n x^3)}{(1+a_nx)^2}\right).
\end{aligned}$$

Now, for $x \in [0,a]$, we use similar calculations for the estimation of $R^*_{n,q}((t-x)^4,x)$:

$$\begin{aligned}
R^*_{n,q}\big((t-x)^4,x\big) &= \sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 \left(\frac{[k]_q + q^k t}{b_n} - x\right)^4 d_qt \\
&= \sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 \left(\frac{q^k t}{b_n} + \frac{[k]_q}{b_n} - x\right)^4 d_qt
\end{aligned}$$

$$
\begin{aligned}
&\le 4\sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 \left(\frac{q^k t}{b_n}\right)^4 d_q t + 4\sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 \left(\frac{[k]_q}{b_n} - x\right)^4 d_q t \\
&= 4\sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 \left(\frac{q^k t}{b_n}\right)^4 d_q t + 4R_{n,q}\big((t-x)^4, x\big).
\end{aligned}
$$

By evaluating the q-integral and using Lemma 3, we get

$$
\begin{aligned}
R^*_{n,q}\big((t-x)^4, x\big) &\le \frac{4}{[5]_q b_n^4} + 4\frac{1}{b_n^2} C(q,a) \\
&\le \frac{4}{b_n^2} + \frac{4}{b_n^2} C(q,a) \\
&\le \frac{1}{b_n^2} C_1(q,a). \qquad \square
\end{aligned}
$$

Lemma 7 *Assume that* $0 < q_n < 1$, $q_n \to 1$, $q_n^n \to \mu$ *as* $n \to \infty$ *and* $0 < \beta < \frac{1}{2}$. *Then we have the following limits*:

$$
\begin{aligned}
&\text{(i)} \quad \lim_{n\to\infty} b_{n,q_n} R^*_{n,q_n}(t-x, x) = \frac{1}{2}, \\
&\text{(ii)} \quad \lim_{n\to\infty} b_{n,q_n} R^*_{n,q_n}\big((t-x)^2, x\big) = x,
\end{aligned}
$$

where $b_{n,q_n} = [n]_{q_n}^{\beta}$

Proof To prove this lemma, we use the formulas of $R^*_{n,q_n}(t,x)$ and $R^*_{n,q_n}(t^2,x)$ given in Lemma 5. The first statement is trivial

$$
\begin{aligned}
&\lim_{n\to\infty} b_{n,q_n} R^*_{n,q_n}(t-x, x) \\
&\quad= \lim_{n\to\infty} b_{n,q_n}\big(R^*_{n,q_n}(t,x) - x\big) \\
&\quad= \lim_{n\to\infty} b_{n,q_n}\left(\frac{q_n - 1}{[2]_{q_n}}\frac{x}{1 + a_{n,q_n}x} - \frac{a_{n,q_n}x^2}{1 + a_{n,q_n}x} + \frac{1}{[2]_{q_n} b_{n,q_n}}\right) \\
&\quad= \frac{1}{2}.
\end{aligned}
$$

For the second statement, we write

$$
\begin{aligned}
&\lim_{n\to\infty} b_{n,q_n} R^*_{n,q_n}\big((t-x)^2, x\big) \\
&\quad= \lim_{n\to\infty} b_{n,q_n}\big\{R^*_{n,q_n}\big(t^2, x\big) - x^2 - 2xR^*_{n,q_n}(t-x, x)\big\} \\
&\quad= \lim_{n\to\infty} b_{n,q_n}\left\{\frac{1}{[3]_{q_n} b_{n,q_n}^2} + \frac{4q_n^3 + 5q_n^2 + 3q_n}{b_{n,q_n}[2]_{q_n}[3]_{q_n}}\frac{x}{1 + a_{n,q_n}x}\right. \\
&\qquad\left. + \frac{q_n[n-1]_{q_n}}{[n]_{q_n}}\frac{4q_n^3 + q_n^2 + q_n}{[2]_{q_n}[3]_{q_n}}\left(\frac{x}{1 + a_{n,q_n}x}\right)^2 - x^2 - 2xR^*_{n,q_n}(t-x, x)\right\} \\
&\quad= \lim_{n\to\infty} b_{n,q_n}\left\{\frac{1}{[3]_{q_n} b_{n,q_n}^2} + \frac{4q_n^3 + 5q_n^2 + 3q_n}{b_{n,q_n}[2]_{q_n}[3]_{q_n}}\frac{x}{1 + a_{n,q_n}x}\right.
\end{aligned}
$$

$$+\frac{q_n[n-1]_{q_n}}{[n]_{q_n}}\frac{4q_n^3+q_n^2+q_n}{[2]_{q_n}[3]_{q_n}}\left(\frac{x}{1+a_{n,q_n}x}\right)^2-x^2-2xR^*_{n,q_n}(t-x,x)\biggr\}.$$

Now, by substituting the following limits into the last equality

$$\begin{aligned}&\lim_{n\to\infty}\frac{q_n[n-1]_{q_n}}{[n]_{q_n}}=1,\qquad \lim_{n\to\infty}\frac{x^2}{(1+a_{n,q_n}x)^2}=x^2,\\&\lim_{n\to\infty}b_{n,q_n}\frac{4q_n^3+5q_n^2+3q_n}{b_{n,q_n}[2]_{q_n}[3]_{q_n}}\frac{x}{1+a_{n,q_n}x}=2x,\\&\lim_{n\to\infty}b_{n,q_n}\frac{1}{[3]_{q_n}b_{n,q_n}^2}=0\quad\text{and}\quad\lim_{n\to\infty}b_{n,q_n}R^*_{n,q_n}(t-x,x)=\frac{1}{2},\end{aligned}$$

we get

$$\begin{aligned}&\lim_{n\to\infty}b_{n,q_n}R^*_{n,q_n}\bigl((t-x)^2,x\bigr)\\&\quad=\lim_{n\to\infty}b_{n,q_n}\frac{4q_n^3+q_n^2+q_n}{[2]_{q_n}[3]_{q_n}}x^2-\lim_{n\to\infty}b_{n,q_n}x^2+x\\&\quad=\lim_{n\to\infty}b_{n,q_n}\left(\frac{4q_n^3+q_n^2+q_n}{[2]_{q_n}[3]_{q_n}}-1\right)x^2+x\\&\quad=\lim_{n\to\infty}a_{n,q_n}\bigl(1-q_n^n\bigr)\left(\frac{3q_n^3-q_n^2-q_n-1}{1+q_n-q_n^3-q_n^4}\right)x^2+x\\&\quad=x,\end{aligned}$$

which proves the lemma. □

3 Local approximation

In this section we establish local approximation theorem for the new Kantorovich type q-analogue of the Balázs–Szabados operators. Let $C_B[0,\infty)$ be the space of all real-valued continuous bounded functions f on $[0,\infty)$, endowed with the norm $\|f\|=\sup_{x\in[0,\infty)}|f(x)|$. We consider the Peetre's K-functional:

$$K_2(f,\delta):=\inf\bigl\{\|f-g\|+\delta\bigl\|g''\bigr\|:g\in C_B^2[0,\infty)\bigr\},\quad \delta\geq 0,$$

where

$$C_B^2[0,\infty):=\bigl\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\bigr\}.$$

Then, from the known result in [6], there exists an absolute constant $C_0>0$ such that

$$K_2(f,\delta)\leq C_0\omega_2(f,\sqrt{\delta}),\tag{4}$$

where

$$\omega_2(f,\sqrt{\delta}):=\sup_{0<h\leq\sqrt{\delta}}\sup_{x\pm h\in[0,\infty)}\bigl|f(x-h)-2f(x)+f(x+h)\bigr|$$

is the second modulus of smoothness of $f \in C_B[0,\infty)$. Also, we let

$$\omega(f,\delta) = \sup_{0<h\le\delta}\sup_{x\in[0,\infty)}\left|f(x+h)-f(x)\right|.$$

The first main result of the local approximation for our operators $R^*_{n,q}(f,x)$ is stated in the following theorem.

Theorem 8 *There exists an absolute constant* $C>0$ *such that*

$$\left|R^*_{n,q}(f,x)-f(x)\right| \le C\omega_2\left(f,\sqrt{\delta_n(x)}\right)+\omega\left(f,\left|\theta\frac{x}{1+a_nx}+\eta_n\right|\right),$$

where $f \in C_B[0,\infty)$, $\theta = \frac{2q}{[2]_q}$, $\eta_n = \frac{1}{[2]_q b_n}$, $0 \le x < \infty$, $0<q<1$, *and* $\delta_n(x) = \frac{2}{b_n}\{(\frac{1}{[3]_q b_n}+\frac{x(1+a_n^2b_nx^3)}{(1+a_nx)^2}) + (\frac{(1-q^n)^2}{b_n}(\frac{1}{1+q}+\frac{a_nx}{1-q})^2+\frac{1}{b_n})\}$.

Proof Let

$$\widetilde{R}^*_{n,q}(f,x) = R^*_{n,q}(f,x)+f(x)-f\left(\theta\frac{x}{1+a_nx}+\eta_n\right),$$

where $f \in C_B[0,\infty]$, $\theta = \frac{2q}{[2]_q}$, $\eta_n = \frac{1}{[2]_q b_n}$. By using Taylor's formula, we have

$$g(t) = g(x)+g'(x)(t-x)+\int_x^t (t-s)g''(s)\,ds, \quad g \in C_B^2[0,\infty),$$

then we have

$$\begin{aligned}\widetilde{R}^*_{n,q}(g,x) &= g(x)+R^*_{n,q}\left(\int_x^t (t-s)g''(s)\,ds,x\right)\\ &\quad -\int_x^{\theta\frac{x}{1+a_nx}+\eta_n}\left(\theta\frac{x}{1+a_nx}+\eta_n-s\right)g''(s)\,ds.\end{aligned}$$

Hence

$$\begin{aligned}&\left|\widetilde{R}^*_{n,q}(g,x)-g(x)\right|\\ &\quad\le \widetilde{R}^*_{n,q}\left(\left|\int_x^t |t-s|\left|g''(s)\right|ds\right|,x\right)\\ &\qquad+\left|\int_x^{\theta\frac{x}{1+a_nx}+\eta_n}\left|\theta\frac{x}{1+a_nx}+\eta_n-s\right|\left|g''(s)\right|ds\right| \qquad (5)\\ &\quad\le \left\|g''\right\|R^*_{n,q}\left((t-x)^2,x\right)+\left\|g''\right\|\left(\theta\frac{x}{1+a_nx}+\eta_n-x\right)^2\\ &\quad\le \left\|g''\right\|\frac{2}{b_n}\left(\frac{1}{[3]_qb_n}+\frac{x(1+a_n^2b_nx^3)}{(1+a_nx)^2}\right)\\ &\qquad+\left\|g''\right\|\frac{2}{b_n}\left\{\frac{(1-q^n)^2}{b_n}\left(\frac{1}{1+q}+\frac{a_nx}{1-q}\right)^2+\frac{1}{b_n}\right\} \qquad (6)\\ &\quad=\left\|g''\right\|\delta_n(x). \qquad (7)\end{aligned}$$

Using (7) and the uniform boundedness of $\widetilde{R}^*_{n,q}$, we get

$$\begin{aligned}\left|R^*_{n,q}(f,x)-f(x)\right| &\le \left|\widetilde{R}^*_{n,q}(f-g,x)\right| + \left|\widetilde{R}^*_{n,q}(g,x)-g(x)\right| \\ &\quad + \left|f(x)-g(x)\right| + \left|f\left(\theta\frac{x}{1+a_nx}+\eta_n\right)-f(x)\right| \\ &\le 4\|f-g\| + \left\|g''\right\|\delta_n(x) + \omega\left(f,\left|\theta\frac{x}{1+a_nx}+\eta_n-x\right|\right).\end{aligned}$$

If we take the infimum on the right-hand side over all $g \in C^2_B[0,\infty)$, we obtain

$$\left|R^*_{n,q}(f,x)-f(x)\right| \le 4K_2\bigl(f;\delta_n(x)\bigr) + \omega\left(f,\left|\theta\frac{x}{1+a_nx}+\eta_n-x\right|\right),$$

which together with (4) gives the proof of the theorem. □

Corollary 9 *Let $a>0$, $q_n \in (0,1)$, $q_n \to 1$ as $n\to\infty$. Then, for each $f \in C[0,\infty)$, the sequence of operators $R^*_{n,q_n}(f,x)$ converges to f uniformly on $[0,a]$.*

In the following theorem we give a Voronovskaja type result for the new Kantorovich type q-analogue of the Balázs–Szabados operators.

Theorem 10 *Assume that $q_n \in (0,1)$, $q_n \to 1$, and $q_n^n \to \mu$ as $n\to\infty$, and let $0<\beta<\frac{1}{2}$. For any $f \in C^2_B[0,\infty)$, the following equality holds*:

$$\lim_{n\to\infty} b_{n,qn}\bigl(R^*_{n,q_n}(f,x)-f(x)\bigr) = \frac{1}{2}f'(x) + \frac{1}{2}xf''(x)$$

uniformly on $[0,a]$.

Proof Let $f \in C^2_B[0,\infty)$ and $x \in [0,\infty)$ be fixed. By using Taylor's formula, we write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{8}$$

where the function $r(t,x)$ is the Peano form of the remainder, $r(t,x) \in C_B[0,\infty)$ and $\lim_{t\to x} r(t,x)=0$. Applying R^*_{n,q_n} to (8), we obtain

$$\begin{aligned}&b_{n,q_n}\bigl(R^*_{n,q_n}(f,x)-f(x)\bigr) \\ &\quad = f'(x)b_{n,q_n}R^*_{n,q_n}(t-x,x) + \frac{1}{2}f''(x)b_{n,q_n}R^*_{n,q_n}\bigl((t-x)^2,x\bigr) \\ &\qquad + b_{n,q_n}R^*_{n,q_n}\bigl(r(t,x)(t-x)^2,x\bigr).\end{aligned}$$

By using the Cauchy–Schwarz inequality, we get

$$R^*_{n,q_n}\bigl(r(t,x)(t-x)^2,x\bigr) \le \sqrt{R^*_{n,q_n}\bigl(r^2(t,x),x\bigr)}\sqrt{R^*_{n,q_n}\bigl((t-x)^4,x\bigr)}. \tag{9}$$

We observe that $r^2(x,x)=0$ and $r^2(.,x) \in C_B[0,\infty)$. Now from Corollary 9 it follows that

$$\lim_{n\to\infty} R^*_{n,q_n}\bigl(r^2(t,x),x\bigr) = r^2(x,x) = 0 \tag{10}$$

uniformly with respect to $x \in [0,a]$. Finally, from (9), (10), and Lemma 7, we get immediately

$$\lim_{n\to\infty} b_{n,q_n} R^*_{n,q_n}\bigl(r(t,x)(t-x)^2,x\bigr) = 0,$$

which completes the proof. □

Theorem 11 *Let $\alpha \in (0,1]$ and A be any subset of the interval $[0,\infty)$. Then, if $f \in C_B[0,\infty)$ is locally* $\mathrm{Lip}(\alpha)$, *i.e., the condition*

$$\bigl|f(y)-f(x)\bigr| \le L|y-x|^\alpha, \quad y \in A \text{ and } x \in [0,\infty) \tag{11}$$

holds, then, for each $x \in [0,\infty)$, we have

$$\bigl|R^*_{n,q}(f,x)-f(x)\bigr| \le L\bigl\{\lambda_n^{\frac{\alpha}{2}}(x) + 2\bigl(d(x,A)\bigr)^\alpha\bigr\},$$

where L is a constant depending on α and f; and $d(x,A)$ is the distance between x and A defined as

$$d(x,A) = \inf\bigl\{|t-x| : t \in A\bigr\}.$$

Proof Let $\bar{A}$ be the closure of A in $[0,\infty)$. Then there exists a point $x_0 \in \bar{A}$ such that $|x - x_0| = d(x,A)$. By the triangle inequality

$$\bigl|f(t)-f(x)\bigr| \le \bigl|f(t)-f(x_0)\bigr| + \bigl|f(x)-f(x_0)\bigr|$$

and by (11), we get

$$\begin{aligned}
\bigl|R^*_{n,q}(f,x)-f(x)\bigr| &\le R^*_{n,q}\bigl(\bigl|f(t)-f(x_0)\bigr|,x\bigr) + R^*_{n,q}\bigl(\bigl|f(x)-f(x_0)\bigr|,x\bigr) \\
&\le L\bigl\{R^*_{n,q}\bigl(|t-x_0|^\alpha,x\bigr) + |x-x_0|^\alpha\bigr\} \\
&\le L\bigl\{R^*_{n,q}\bigl(|t-x|^\alpha + |x-x_0|^\alpha,x\bigr) + |x-x_0|^\alpha\bigr\} \\
&\le L\bigl\{R^*_{n,q}\bigl(|t-x|^\alpha,x\bigr) + 2|x-x_0|^\alpha\bigr\}.
\end{aligned}$$

Now, by using the Hölder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we get

$$\begin{aligned}
\bigl|R^*_{n,q}(f,x)-f(x)\bigr| &\le L\bigl\{\bigl[R^*_{n,q}\bigl(|t-x|^{\alpha p},x\bigr)\bigr]^{\frac{1}{p}}\bigl[R^*_{n,q}\bigl(1^q,x\bigr)\bigr]^{\frac{1}{q}} + 2\bigl(d(x,A)\bigr)^\alpha\bigr\} \\
&= M\bigl\{\bigl[R^*_{n,q}\bigl(|t-x|^2,x\bigr)\bigr]^{\frac{\alpha}{2}} + 2\bigl(d(x,A)\bigr)^\alpha\bigr\} \\
&\le M\left\{\left[\frac{2}{b_n}\left(\frac{1}{[3]_q b_n} + \frac{x(1+a_n^2 b_n x^3)}{(1+a_n x)^2}\right)\right]^{\frac{\alpha}{2}} + 2\bigl(d(x,A)\bigr)^\alpha\right\} \\
&= M\bigl\{\lambda_n(x)^{\frac{\alpha}{2}} + 2\bigl(d(x,A)\bigr)^\alpha\bigr\},
\end{aligned}$$

and the proof is completed. □

4 Conclusion

By using the notion of q-integers, we introduced a new Kantorovich type q-analogue of the Balázs–Szabados operators. The new operators have some advantages compared with other studies: they are positive for all real-valued continuous functions on the interval $[0, \infty)$ and they are capable of approximating integrable functions. In the case $q = 1$, the operators coincide with the ones defined in [22]. We established the moments of the operators with the help of the recurrence formula. We studied the local approximation properties of these new operators in terms of modulus of continuity and proved a Voronovskaja type theorem.

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