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Certain Grüss-type inequalities via tempered fractional integrals concerning another function

Gauhar Rahman[1], Kottakkaran Sooppy Nisar[2], Saima Rashid[3] and Thabet Abdeljawad[4,5,6*]

*Correspondence:
tabdeljawad@psu.edu.sa
[4]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Kingdom of Saudi Arabia
[5]Department of Medical Research, China Medical University, Taichung, Taiwan
Full list of author information is available at the end of the article

Abstract

We study a generalized left sided tempered fractional (GTF)-integral concerning another function Ψ in the kernel. Then we investigate several kinds of inequalities such as Grüss-type and certain other related inequalities by utilizing the GTF-integral. Additionally, we present various special cases of the main result. By utilizing the connection between GTF-integral and Riemann–Liouville integral concerning another function Ψ in the kernel, certain distinct particular cases of the main result are also presented. Furthermore, certain other inequalities can be formed by applying various kinds of conditions on the function Ψ.

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1 Introduction

The field of fractional calculus deals with the integrals and differentiation of an arbitrary non-integer order. In the last three centuries, this field has been considered as the most power tool in describing the anomalous kinetics and its wide applications in diverse domains. Numerous mathematical, statistical, engineering, physical, chemical, and biological phenomena can be modeled by utilizing ordinary differential equations involving fractional derivatives. Many a mathematician and physicist has contributed to the development of the theories of fractional calculus. The interesting reader is referred to [1–3] and the references therein. In practical applications, a numerous types of fractional integrals and derivatives operators, such as the Riemann–Liouville, Caputo, Riesz, Hilfer, Hadamard, Erdélyi–Kolber, Saigo, and Marichev–Saigo–Maeda operators, were extensively studied by various researchers. We refer the reader to [2–4].

Later, mathematicians introduced the notion of fractional conformable integrals and derivatives which are cited therein. Khalil et al. [5] introduced fractional conformable derivatives operators with some shortcoming. Abdeljawad [6] investigated the properties of the fractional conformable derivative operators. Jarad et al. [7] defined generalized fractional conformable integral and derivative operators. Anderson and Unless [8] presented the idea of a conformable derivative by employing local proportional derivatives. In [9], Abdeljawad and Baleanu gave certain monotonicity results for fractional difference operators with discrete exponential kernels. In [10], Abdeljawad and Baleanu have defined a fractional derivative operator with exponential kernel and the discrete version. In [11], Atangana and Baleanu defined a new fractional derivative operator with the non-local and non-singular kernel. A fractional derivative without a singular kernel can be found in the work of Caputo and Fabrizio [12]. Certain properties of fractional derivatives without a singular kernel can be found in the work of Losada and Nieto [13]. In [14–16], the authors studied stability analysis and a numerical scheme for fractional Klein–Gordon equations, existence results in Banach space for a nonlinear impulsive system and results for mild solutions of fractional coupled hybrid boundary value problems.

A variety of such types of distinguished operators led researchers to establish new ideas and fractional integral inequalities by utilizing these new operators. In [17, 18], Hasib et al. established various inequalities by using AB-fractional and Saigo fractional integral operators. Recently Alzabut et al. and Rahman et al. [19, 20] studied generalized proportional derivatives and integral operators and established a certain Gronwall inequality and the Minkowski inequalities involving the said operators. Rahman et al. [21, 22] presented fractional integral inequalities for a family of positive continuous and decreasing functions and inequalities for convex functions by employing proportional Hadamard fractional integrals. Recently, researchers presented several various remarkable inequalities with properties and applications for the fractional conformable integrals and proportional integrals. The interested reader may consult [23–33].

2 Preliminaries

In this section, we consider some well-known definitions and mathematical preliminaries.

Definition 2.1 ([34]) Suppose that the functions $\mathcal{U},\mathcal{V}:[a_1,b_1]\to\mathbb{R}$ are positive with $\mathcal{A}\leq\mathcal{U}(\vartheta)\leq\mathcal{B}$ and $\mathcal{C}\leq\mathcal{V}(\vartheta)\leq\mathcal{D}$, for all $\vartheta\in[a_1,b_1]$, then the following inequality holds:

$$\begin{aligned}&\left|\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\mathcal{U}(\vartheta)\mathcal{V}(\vartheta)\,d\vartheta-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\mathcal{U}(\vartheta)\,d\vartheta\,\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\mathcal{V}(\vartheta)\,d\vartheta\right|\\&\quad\leq\frac{1}{4}(\mathcal{B}-\mathcal{A})(\mathcal{D}-\mathcal{C}),\end{aligned}\tag{2.1}$$

where the constants $\mathcal{B}$, $\mathcal{A}$, $\mathcal{C}$, $\mathcal{D}\in\mathbb{R}$ and $\frac{1}{4}$ is the sharp value of inequality (2.1).

Definition 2.2 ([35, 36]) The function $\mathcal{U}(\varrho)$ will be in the space $L_{p,r}[0,\infty[$ if

$$L_{p,r}[0,\infty[=\left\{\mathcal{U}:\|\mathcal{U}\|_{L_{p,r}[0,\infty[}=\left(\int_r^s\left|\mathcal{U}(\varrho)\right|^p\varrho^r\,d\varrho\right)^{\frac{1}{p}}<\infty,1\leq p<\infty,r\geq 0\right\}.\tag{2.2}$$

If we consider $r=0$, then (2.2) gives

$$L_{p}[0,\infty[=\left\{\mathcal{U}:\|\mathcal{U}\|_{L_{p}[0,\infty[}=\left(\int_r^s\left|\mathcal{U}(\varrho)\right|^p\,d\varrho\right)^{\frac{1}{p}}<\infty,1\leq p<\infty\right\}.$$

Definition 2.3 ([37]) Suppose that we have the function $\mathcal{U}\in L_1[0,\infty[$ and assume that the function Ψ is positive, monotone and an increasing on $[0,\infty[$ and let Ψ' be continuous on

$[0,\infty[$ with $\Psi(0)=0$. Then the Lebesgue real-valued measurable function $\mathcal{U}$ defined on $[0,\infty[$ is said to be in the space $\chi_{\Psi}^{p}(0,\infty)$, $(1\leq p<\infty)$ for which

$$\|\mathcal{U}\|_{\chi_{\Psi}^{p}}=\left(\int_{r}^{s}\left|\mathcal{U}(\varrho)\right|^{p}\Psi'(\varrho)\,d\varrho\right)^{\frac{1}{p}}<\infty,\quad 1\leq p<\infty.$$

When $p=\infty$, then

$$\|\mathcal{U}\|_{\chi_{\Psi}^{\infty}}=\textbf{ess}\sup_{0\leq\varrho<\infty}\left[\Psi'(\varrho)\mathcal{U}(\varrho)\right].$$

Note that the space $\chi_{\Psi}^{p}(0,\infty)$ coincides with the space $L_{p}[0,\infty[$ if $\Psi(\varrho)=\varrho$ for $1\leq p<\infty$ and similarly with the space $L_{p,r}[1,\infty[$ if $\Psi(\varrho)=\ln\varrho$ for $1\leq p<\infty$.

The tempered fractional integral was first studied by Buschman [38], but Li et al. [39] and Meerschaert et al. [40] have described the associated tempered fractional calculus more explicitly. Fernandez and Ustaoğlu [41] investigated several analytic properties of tempered fractional integrals.

Definition 2.4 ([39, 40]) Suppose that $[a,b]$ is a real interval and $\kappa,\xi\in\mathbb{C}$ with $\Re(\kappa)>0$ and $\Re(\xi)\geq 0$, then the left sided tempered fractional integral is defined by

$$\left({}_{a}\mathcal{J}^{\kappa,\xi}\mathcal{U}\right)(\varrho)=\frac{1}{\Gamma(\kappa)}\int_{a}^{\varrho}e^{-\xi(\varrho-t)}(\varrho-t)^{\kappa-1}\mathcal{U}(t)\,dt,\quad a<\varrho.\tag{2.3}$$

Remark 2.1 Setting $\xi=0$ in (2.3) yields the following Riemann–Liouville fractional integral:

$$\left({}_{a}\mathcal{J}^{\kappa}\mathcal{U}\right)(\varrho)=\frac{1}{\Gamma(\kappa)}\int_{a}^{\varrho}(\varrho-t)^{\kappa-1}\mathcal{U}(t)\,dt,\quad \varrho>a.\tag{2.4}$$

The tempered fractional integral (2.3) satisfies the following semigroup property:

$${}_{a}\mathcal{J}^{\kappa,\xi}\left({}_{a}\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho)\right)={}_{a}\mathcal{J}^{\kappa+\lambda,\xi}\mathcal{U}(\varrho).$$

In [42], Fahad et al. defined the following general form of the generalized tempered fractional integral concerning another function.

Definition 2.5 Let $\mathcal{U}$ be an integrable function in the space $\chi_{\Psi}^{p}(0,\infty)$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and let Ψ' be continuous on $[0,\infty[$ with $\Psi(0)=0$. Then the left (GTF)-integral of a function $\mathcal{U}$ concerning another function Ψ in the kernel is defined by

$$\left({}_{a}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}\right)(\rho)=\frac{1}{\Gamma(\kappa)}\int_{a}^{\rho}e^{-\xi(\Psi(\rho)-\Psi(\varrho))}\left(\Psi(\rho)-\Psi(\varrho)\right)^{\kappa-1}\Psi'(\varrho)\mathcal{U}(\varrho)\,d\varrho,\quad a<\rho,\tag{2.5}$$

where $\xi>0$, $\kappa\in\mathbb{C}$ with $\Re(\kappa)>0$ and $\Gamma(\cdot)$ is the well-known gamma function.

Remark 2.2 The left GTF-integral (2.5) will reduce the following fractional integrals:

i. setting $\Psi(\rho)=\rho$, then the left tempered fractional integral (2.3) will be obtained,
ii. setting $\xi=0$, then the left generalized RL-fractional integral operator defined by [1] will be obtained,
iii. setting $\Psi(\rho)=\ln\rho$, the left Hadamard GTF-integral defined by [42] will be obtained,

$$\begin{aligned}&\left({}_a^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}\right)(\rho)\\&\quad=\frac{1}{\Gamma(\kappa)}\int_a^{\rho}\exp\bigl[-\xi(\ln\rho-\ln\varrho)\bigr](\ln\rho-\ln\varrho)^{\kappa-1}\frac{\mathcal{U}(\varrho)}{\varrho}\,d\varrho,\quad a<\rho,\end{aligned}\tag{2.6}$$

iv. setting $\Psi(\rho)=\frac{\rho^{\kappa}}{\kappa}$, $\kappa>0$ and $\xi=0$, the left Katugampola [36] fractional integral operator will be obtained,
v. setting $\Psi(\rho)=\rho$ and $\xi=0$, the left Riemann–Liouville fractional integral (2.4) will be obtained,
vi. setting $\Psi(\rho)=\frac{\rho^{\alpha+s}}{\alpha+s}$ and $\xi=0$ (where $\alpha\in(0,1]$, $s\in\mathbb{R}$ and $\mu+s\neq0$), then the generalized fractional conformable integral given in [43] will be obtained.

In this manuscript, we will consider the following one sided GTF-integral.

Definition 2.6 Let $\mathcal{U}$ be an integrable function in the space $\chi_{\Psi}^{p}(0,\infty)$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and let Ψ' is continuous on $[0,\infty[$ with $\Psi(0)=0$. Then the one sided (GTF) integral of a function $\mathcal{U}$ concerning another function Ψ in the kernel is defined by

$$\left({}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}\right)(\rho)=\frac{1}{\Gamma(\kappa)}\int_0^{\rho}e^{-\xi(\Psi(\rho)-\Psi(\varrho))}\bigl(\Psi(\rho)-\Psi(\varrho)\bigr)^{\kappa-1}\Psi'(\varrho)\mathcal{U}(\varrho)\,d\varrho.\tag{2.7}$$

One can easily derive the following results.

Theorem 2.1 *Let* $\mathcal{U}:[0,\rho]\subseteq[0,\infty[\to\mathbb{R}$ *be an GTF-integral operator concerning another function* Ψ, *then we have*

$${}^{\Psi}\mathcal{J}^{\kappa,\xi}\left({}^{\Psi}\mathcal{J}^{\mu,\xi}\mathcal{U}\right)(\rho)=\left({}^{\Psi}\mathcal{J}^{\kappa+\mu,\xi}\mathcal{U}\right)(\rho),$$

where $\kappa,\mu>0$.

Theorem 2.2 *The GTF-integral operator* ${}^{\Psi}\mathcal{J}^{\kappa,\xi}:L_1[0,\rho]\to L_1[0,\rho]$ *will satisfy the following linearity property*:

$${}^{\Psi}\mathcal{J}^{\kappa,\xi}(\alpha\mathcal{U}_1+\beta\mathcal{U}_2)=\alpha\,{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}_1+\beta\,{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}_2$$

where $\mathcal{U}_1,\mathcal{U}_2\in L_1[0,\rho]$ *and* $\alpha,\beta\in\mathbb{R}$.

The main goal of this manuscript is to establish certain inequalities such as Grüss-type and several other inequalities by utilizing the GTF-integral (2.1). Also, certain special and particular cases of the main result are presented.

3 Main results

We present generalizations of certain inequalities by utilizing the GTF-integral operator (2.7) containing another function Ψ in its kernel in this section.

Theorem 3.1 *Suppose that we have the function $\mathcal{U} \in \chi_{\Psi}^{p}(0,\infty)$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0) = 0$. Moreover, let ϕ_1 and ϕ_2 be two integrable functions defined on $[0,\infty[$ such that*

$$\phi_1(\varrho) \leq \mathcal{U}(\varrho) \leq \phi_2(\varrho), \quad \varrho \in [0,\infty[. \tag{3.1}$$

Then, for $\varrho > 0$, $\kappa,\lambda > 0$, we have

$$\begin{aligned} &{}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\phi_1(\varrho) \\ &\quad \geq {}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\phi_1(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho). \end{aligned} \tag{3.2}$$

Proof Applying (3.1) for all $\rho \geq 0$ and $\zeta \geq 0$, we have

$$\big(\phi_2(\rho) - \mathcal{U}(\rho)\big)\big(\mathcal{U}(\zeta) - \phi_1(\zeta)\big) \geq 0.$$

It follows that

$$\phi_2(\rho)\mathcal{U}(\zeta) + \phi_1(\zeta)\mathcal{U}(\rho) \geq \phi_2(\rho)\phi_1(\zeta) + \mathcal{U}(\rho)\mathcal{U}(\zeta). \tag{3.3}$$

Multiplying both sides of (3.3) with $\frac{1}{\Gamma(\kappa)}e^{-\xi(\Psi(\varrho)-\Psi(\rho))}(\Psi(\varrho) - \Psi(\rho))^{\kappa-1}\Psi'(\rho)$ and integrating the resulting estimate with respect to ρ from 0 to ϱ, we get

$$\begin{aligned} &\mathcal{U}(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^{\varrho} e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\big(\Psi(\varrho) - \Psi(\rho)\big)^{\kappa-1}\Psi'(\rho)\phi_2(\rho)\,d\rho \\ &\qquad + \phi_1(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^{\varrho} e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\big(\Psi(\varrho) - \Psi(\rho)\big)^{\kappa-1}\Psi'(\rho)\mathcal{U}(\rho)\,d\rho \\ &\quad \geq \phi_1(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^{\varrho} e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\big(\Psi(\varrho) - \Psi(\rho)\big)^{\kappa-1}\Psi'(\rho)\phi_2(\rho)\,d\rho \\ &\qquad + \mathcal{U}(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^{\varrho} e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\big(\Psi(\varrho) - \Psi(\rho)\big)^{\kappa-1}\Psi'(\rho)\mathcal{U}(\rho)\,d\rho, \end{aligned}$$

which with the aid of (2.7) becomes

$$\begin{aligned} &\mathcal{U}(\zeta){}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\rho) + \phi_1(\zeta){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho) \\ &\quad \geq \phi_1(\zeta){}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho) + \mathcal{U}(\zeta){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho). \end{aligned} \tag{3.4}$$

Again, multiplying both sides of (3.4) with $\frac{1}{\Gamma(\lambda)}e^{-\xi(\Psi(\varrho)-\Psi(\zeta))}(\Psi(\varrho) - \Psi(\zeta))^{\lambda-1}\Psi'(\zeta)$ and integrating the resulting estimate with respect to ζ from 0 to ϱ, we obtain

$$\begin{aligned} &{}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\rho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\phi_1(\varrho) \\ &\quad \geq {}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\phi_1(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho), \end{aligned} \tag{3.5}$$

which proves the desired assertion (3.2). □

Theorem 3.2 *Suppose that the two positive functions $\mathcal{U}$ and $\mathcal{V}$ are defined on $[0,\infty[$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0)=0$. Assume that (3.1) holds and let ψ_1, ψ_2 be two integrable functions defined on $[0,\infty[$ such that*

$$\psi_1(\varrho) \le \mathcal{V}(\varrho) \le \psi_2(\varrho), \quad \varrho \in [0,\infty[. \tag{3.6}$$

Then, for $\varrho > 0$ and $\kappa, \lambda > 0$, the following four inequalities hold:

$$\begin{aligned} &{}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) + {}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\kappa,\xi}\psi_1(\varrho) \\ &\quad \ge {}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\psi_1(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &{}^{\Psi}\mathcal{J}^{\lambda,\xi}\phi_1(\varrho)^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa,\xi}\psi_2(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho) \\ &\quad \ge {}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_1(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\psi_2(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho), \end{aligned} \tag{3.8}$$

$$\begin{aligned} &{}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\psi_2(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) \\ &\quad \ge {}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\psi_2(\varrho), \end{aligned} \tag{3.9}$$

$$\begin{aligned} &{}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_1(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\psi_1(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) \\ &\quad \ge {}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_1(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda,\xi}\psi_1(\varrho). \end{aligned} \tag{3.10}$$

Proof To prove inequality (3.7), using (3.1) and (3.6) for $\rho, \zeta \in [0,\infty[$ yields

$$\bigl(\phi_2(\rho) - \mathcal{U}(\rho)\bigr)\bigl(\mathcal{V}(\zeta) - \psi_1(\zeta)\bigr) \ge 0.$$

It follows that

$$\phi_2(\rho)\mathcal{V}(\zeta) + \mathcal{U}(\rho)\psi_1(\zeta) \ge \phi_2(\rho)\psi_1(\zeta) + \mathcal{U}(\rho)\mathcal{V}(\zeta). \tag{3.11}$$

Multiplying both sides of inequality (3.11) with $\frac{1}{\Gamma(\kappa)}e^{-\xi(\Psi(\varrho)-\Psi(\rho))}(\Psi(\varrho) - \Psi(\rho))^{\kappa-1}\Psi'(\rho)$ and integrating the resulting estimate with respect to ρ from 0 to ϱ, we get

$$\begin{aligned} &\mathcal{V}(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^{\varrho} e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\bigl(\Psi(\varrho) - \Psi(\rho)\bigr)^{\kappa-1}\Psi'(\rho)\phi_2(\rho)\,d\rho \\ &\qquad + \psi_1(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^{\varrho} e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\bigl(\Psi(\varrho) - \Psi(\rho)\bigr)^{\kappa-1}\Psi'(\rho)\mathcal{U}(\rho)\,d\rho \\ &\quad \ge \psi_1(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^{\varrho} e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\bigl(\Psi(\varrho) - \Psi(\rho)\bigr)^{\kappa-1}\Psi'(\rho)\phi_2(\rho)\,d\rho \\ &\qquad + \mathcal{V}(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^{\varrho} e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\bigl(\Psi(\varrho) - \Psi(\rho)\bigr)^{\kappa-1}\Psi'(\rho)\mathcal{U}(\rho)\,d\rho, \end{aligned}$$

which with the aid of (2.7) becomes

$$\mathcal{V}(\zeta)^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho) + \psi_1(\zeta)^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho) \ge \psi_1(\zeta)^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho) + \mathcal{V}(\zeta)^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho). \tag{3.12}$$

Again, multiplying both sides of (3.12) with $\frac{1}{\Gamma(\lambda)}e^{-\xi(\Psi(\varrho)-\Psi(\zeta))}(\Psi(\varrho)-\Psi(\zeta))^{\lambda-1}\Psi'(\zeta)$, integrating the resulting estimate with respect to ζ from 0 to ϱ and using (2.7), we obtain

$$\begin{aligned}&{}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho)+{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\psi_1(\varrho)\\&\quad\geq{}^{\Psi}\mathcal{J}^{\kappa,\xi}\phi_2(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\psi_1(\varrho)+{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho),\end{aligned}$$

which completes the desired assertion (3.7). Inequalities (3.8)–(3.10) can be proved by using the following identities:

$$\begin{aligned}&\bigl(\psi_2(\rho)-\mathcal{V}(\rho)\bigr)\bigl(\mathcal{U}(\zeta)-\phi_1(\zeta)\bigr)\geq 0,\\&\bigl(\phi_2(\rho)-\mathcal{U}(\rho)\bigr)\bigl(\mathcal{V}(\zeta)-\psi_2(\zeta)\bigr)\leq 0,\end{aligned}$$

and

$$\bigl(\phi_1(\rho)-\mathcal{U}(\rho)\bigr)\bigl(\mathcal{V}(\zeta)-\psi_1(\zeta)\bigr)\leq 0,$$

respectively. □

4 Certain other inequalities via GTF-integral concerning another function

Certain other types of inequalities which involving generalized tempered fractional (GTF) integral (2.7) are presented in this section.

Theorem 4.1 *Suppose that the two positive functions $\mathcal{U}$ and $\mathcal{V}$ are defined on $[0,\infty[$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0)=0$. If $p_1,q_1>1$ are such that $\frac{1}{p_1}+\frac{1}{q_1}=1$, then, for $\varrho>0$, the following inequalities hold*:

$$\begin{aligned}&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{p_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}^{p_1}(\varrho)+\frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}^{q_1}(\varrho)\\&\quad\geq{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho)\mathcal{U}(\varrho),\qquad(4.1)\\&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{p_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}^{q_1}(\varrho)+\frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}^{p_1}(\varrho)\\&\quad\geq{}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}^{q_1-1}(\varrho)\mathcal{U}^{p_1-1}(\varrho){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho),\qquad(4.2)\\&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{p_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}^{2}(\varrho)+\frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}^{2}(\varrho)\\&\quad\geq{}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}^{\frac{2}{p_1}}(\varrho)\mathcal{V}^{\frac{2}{q_1}}(\varrho){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho),\qquad(4.3)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{2}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}^{q_1}(\varrho)+\frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{2}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}^{p_1}(\varrho)\\&\quad\geq{}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}^{p_1-1}(\varrho)\mathcal{V}^{q_1-1}(\varrho){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{\frac{2}{p_1}}(\varrho)\mathcal{V}^{\frac{2}{q_1}}(\varrho).\qquad(4.4)\end{aligned}$$

Proof Consider the well-known Young's inequality [44]:

$$\frac{1}{p_1}\mathfrak{a}^{p_1}+\frac{1}{q_1}\mathfrak{b}^{q_1}\geq \mathfrak{a}\mathfrak{b},\qquad \mathfrak{a},\mathfrak{b}>0,\qquad \frac{1}{p_1}+\frac{1}{q_1}=1. \tag{4.5}$$

Applying (4.5) for $\mathfrak{a}=\mathcal{U}(\rho)\mathcal{V}(\zeta)$ and $\mathfrak{b}=\mathcal{U}(\zeta)\mathcal{V}(\rho)$, $\rho,\zeta>0$, we have

$$\frac{1}{p_1}\big(\mathcal{U}(\rho)\mathcal{V}(\zeta)\big)^{p_1}+\frac{1}{q_1}\big(\mathcal{U}(\zeta)\mathcal{V}(\rho)\big)^{q_1}\geq \big(\mathcal{U}(\rho)\mathcal{V}(\zeta)\big)\big(\mathcal{U}(\zeta)\mathcal{V}(\rho)\big). \tag{4.6}$$

Multiplying both sides of inequality (4.6) with $\frac{1}{\Gamma(\kappa)}e^{-\xi(\Psi(\varrho)-\Psi(\rho))}(\Psi(\varrho)-\Psi(\rho))^{\kappa-1}\Psi'(\rho)$ and integrating the resulting estimate with respect to ρ from 0 to ϱ, we get

$$\begin{aligned}&\frac{\mathcal{V}^{p_1}(\zeta)}{p_1\Gamma(\kappa)}\int_0^{\varrho}e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\big(\Psi(\varrho)-\Psi(\rho)\big)^{\kappa-1}\Psi'(\rho)\mathcal{U}^{p_1}(\rho)\,d\rho\\&\quad+\frac{\mathcal{U}^{q_1}(\zeta)}{q_1\Gamma(\kappa)}\int_0^{\varrho}e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\big(\Psi(\varrho)-\Psi(\rho)\big)^{\kappa-1}\Psi'(\rho)\mathcal{V}^{q_1}(\rho)\,d\rho\\&\quad\geq\frac{\mathcal{U}(\zeta)\mathcal{V}(\zeta)}{\Gamma(\kappa)}\int_0^{\varrho}e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\big(\Psi(\varrho)-\Psi(\rho)\big)^{\kappa-1}\Psi'(\rho)\mathcal{U}(\rho)\mathcal{V}(\rho)\,d\rho,\end{aligned}$$

which in view of (2.7) becomes

$$\frac{\mathcal{V}^{p_1}(\zeta)}{p_1}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{p_1}(\varrho)+\frac{\mathcal{U}^{q_1}(\zeta)}{q_1}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1}(\varrho)\geq\mathcal{U}(\zeta)\mathcal{V}(\zeta){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho). \tag{4.7}$$

Now, multiplying both sides of (4.7) with $\frac{1}{\Gamma(\lambda)}e^{-\xi(\Psi(\varrho)-\Psi(\zeta))}(\Psi(\varrho)-\Psi(\zeta))^{\lambda-1}\Psi'(\zeta)$, integrating the resulting estimate with respect to ζ from 0 to ϱ and using (2.7), we obtain

$$\begin{aligned}&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{p_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}^{p_1}(\varrho)+\frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}^{q_1}(\varrho){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1}(\varrho)\\&\quad\geq{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho),\end{aligned}$$

which is the desired assertion (4.1). The inequalities (4.2), (4.3) and (4.4) can be obtained by employing the similar procedure and setting the following parameters in (4.5), respectively:

$$\mathfrak{a}=\frac{\mathcal{U}(\rho)}{\mathcal{U}(\zeta)},\qquad \mathfrak{b}=\frac{\mathcal{V}(\rho)}{\mathcal{V}(\zeta)},\quad \mathcal{U}(\zeta),\mathcal{V}(\zeta)\neq 0, \tag{4.8}$$

$$\mathfrak{a}=\mathcal{U}(\rho)\mathcal{V}^{\frac{2}{p_1}}(\zeta),\qquad \mathfrak{b}=\mathcal{U}^{\frac{2}{q_1}}(\zeta)\mathcal{V}(\rho), \tag{4.9}$$

and

$$\mathfrak{a}=\mathcal{U}^{\frac{2}{p_1}}(\rho)\mathcal{U}(\zeta),\qquad \mathfrak{b}=\mathcal{V}^{\frac{2}{q_1}}(\rho)\mathcal{V}(\zeta). \tag{4.10}$$

□

Theorem 4.2 *Suppose that the two positive functions $\mathcal{U}$ and $\mathcal{V}$ are defined on $[0,\infty[$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0)=0$. If $p_1,q_1>1$ are such that $\frac{1}{p_1}+\frac{1}{q_1}=1$, then, for*

$\varrho > 0$, *the following inequalities hold*:

$$\begin{aligned} &p_1{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) + q_1{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho) \\ &\quad \geq {}^{\Psi}\mathcal{J}^{\kappa,\xi}\big(\mathcal{U}^{p_1}(\varrho)\mathcal{V}^{q_1}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda,\xi}\big(\mathcal{U}^{q_1}(\varrho)\mathcal{V}^{p_1}(\varrho)\big), \qquad (4.11) \\ &p_1{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{p_1-1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\big(\mathcal{U}(\varrho)\mathcal{V}^{q_1}(\varrho)\big) + q_1{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1-1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\big(\mathcal{U}^{q_1}(\varrho)\mathcal{V}(\varrho)\big) \\ &\quad \geq {}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}^{q}(\varrho){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{p}(\varrho), \qquad (4.12) \\ &p_1{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}^{\frac{2}{p_1}}(\varrho) + q_1{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}^{\frac{2}{q_1}}(\varrho) \\ &\quad \geq {}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}^{p_1}(\varrho)\mathcal{V}(\varrho){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1}(\varrho)\mathcal{U}^{2}(\varrho), \qquad (4.13) \end{aligned}$$

and

$$\begin{aligned} &p_1{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{\frac{2}{p_1}}(\varrho)\mathcal{V}^{q_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}^{q_1-1}(\varrho) + q_1{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1-1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}^{\frac{2}{q_1}}(\varrho)\mathcal{V}^{p_1}(\varrho) \\ &\quad \geq {}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}^{2}(\varrho){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{2}(\varrho). \qquad (4.14) \end{aligned}$$

Proof Consider the following well-known $\mathcal{AM}-\mathcal{GM}$ inequality:

$$p_1\mathfrak{a} + q_1\mathfrak{b} \geq \mathfrak{a}^{p_1}\mathfrak{b}^{q_1}, \quad \forall \mathfrak{a}, \mathfrak{b} \geq 0, p_1 + q_1 = 1. \qquad (4.15)$$

Applying (4.15) for $\mathfrak{a} = \mathcal{U}(\rho)\mathcal{V}(\zeta)$ and $\mathfrak{b} = \mathcal{U}(\zeta)\mathcal{V}(\rho)$, $\rho, \zeta > 0$, we have

$$p_1\mathcal{U}(\rho)\mathcal{V}(\zeta) + q_1\mathcal{U}(\zeta)\mathcal{V}(\rho) \geq \big(\mathcal{U}(\rho)\mathcal{V}(\zeta)\big)^{p_1}\big(\mathcal{U}(\zeta)\mathcal{V}(\rho)\big)^{q_1}. \qquad (4.16)$$

Multiplying both sides of inequality (4.16) with $\frac{1}{\Gamma(\kappa)}e^{-\xi(\Psi(\varrho)-\Psi(\rho))}(\Psi(\varrho)-\Psi(\rho))^{\kappa-1}\Psi'(\rho)$ and integrating the resulting estimate with respect to ρ from 0 to ϱ, we get

$$\begin{aligned} &p_1\mathcal{V}(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^{\varrho} e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\big(\Psi(\varrho)-\Psi(\rho)\big)^{\kappa-1}\Psi'(\rho)\mathcal{U}(\rho)\,d\rho \\ &\qquad + q_1\mathcal{U}(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^{\varrho} e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\big(\Psi(\varrho)-\Psi(\rho)\big)^{\kappa-1}\Psi'(\rho)\mathcal{V}(\rho)\,d\rho \\ &\quad \geq \mathcal{V}^{p_1}(\zeta)\mathcal{U}^{q_1}(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^{\varrho} e^{-\xi(\Psi(\varrho)-\Psi(\rho))}\big(\Psi(\varrho)-\Psi(\rho)\big)^{\kappa-1}\Psi'(\rho)\mathcal{U}^{p_1}(\rho)\mathcal{V}^{q_1}(\rho)\,d\rho, \end{aligned}$$

which in view of (2.7) becomes

$$p_1\mathcal{V}(\zeta){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho) + q_1\mathcal{U}(\zeta){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}(\varrho) \geq \mathcal{V}^{p_1}(\zeta)\mathcal{U}^{q_1}(\zeta){}^{\Psi}\mathcal{J}^{\kappa,\xi}\big(\mathcal{U}^{p_1}(\varrho)\mathcal{V}^{q_1}(\varrho)\big). \qquad (4.17)$$

Again, multiplying both sides of (4.17) with $\frac{1}{\Gamma(\lambda)}e^{-\xi(\Psi(\varrho)-\Psi(\zeta))}(\Psi(\varrho)-\Psi(\zeta))^{\lambda-1}\Psi'(\zeta)$, integrating the resulting estimate with respect to ζ from 0 to ϱ and using (2.7), we obtain the desired assertion (4.11),

$$\begin{aligned} &p_1{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) + q_1{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}(\varrho){}^{\Psi}\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho) \\ &\quad \geq {}^{\Psi}\mathcal{J}^{\kappa,\xi}\big(\mathcal{U}^{p_1}(\varrho)\mathcal{V}^{q_1}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda,\xi}\big(\mathcal{V}^{p_1}(\varrho)\mathcal{U}^{q_1}(\varrho)\big). \end{aligned}$$

The inequalities (4.12), (4.13) and (4.14) can be easily obtained by following the same procedure and using the following parameters in (4.15), respectively. We have

$$\mathfrak{a} = \frac{\mathcal{U}(\zeta)}{\mathcal{U}(\rho)}, \qquad \mathfrak{b} = \frac{\mathcal{V}(\rho)}{\mathcal{V}(\zeta)}, \quad \mathcal{U}(\rho), \mathcal{V}(\zeta) \neq 0, \tag{4.18}$$

$$\mathfrak{a} = \mathcal{U}(\rho)\mathcal{U}^{\frac{2}{p_1}}(\zeta), \qquad \mathfrak{b} = \mathcal{U}^{\frac{2}{q_1}}(\zeta)\mathcal{V}(\rho), \tag{4.19}$$

and

$$\mathfrak{a} = \frac{\mathcal{U}^{\frac{2}{p_1}}(\rho)}{\mathcal{V}(\zeta)}, \qquad \mathfrak{b} = \frac{\mathcal{U}^{\frac{2}{q_1}}(\zeta)}{\mathcal{V}(\rho)}, \quad \mathcal{V}(\rho), \mathcal{V}(\zeta) \neq 0. \tag{4.20}$$ □

Theorem 4.3 *Suppose that the two positive functions $\mathcal{U}$ and $\mathcal{V}$ are defined on $[0,\infty[$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0)=0$. Let $p_1, q_1 > 1$ be such that $\frac{1}{p_1}+\frac{1}{q_1}=1$. Suppose*

$$\mathcal{K} = \min_{0\leq\rho\leq\varrho} \frac{\mathcal{U}(\rho)}{\mathcal{V}(\rho)} \quad \text{and} \quad \mathcal{H} = \max_{0\leq\rho\leq\varrho} \frac{\mathcal{U}(\rho)}{\mathcal{V}(\rho)}. \tag{4.21}$$

Then, for $\varrho > 0$, the following inequalities hold:

$$0 \leq {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^2(\varrho)\,{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^2(\varrho) \leq \frac{(\mathcal{K}+\mathcal{H})^2}{4\mathcal{K}\mathcal{H}}\big({}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho)\big)^2, \tag{4.22}$$

$$\begin{aligned} 0 &\leq \sqrt{{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^2(\varrho)\,{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^2(\varrho)} - \big({}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho)\big) \\ &\leq \frac{\sqrt{\mathcal{H}}-\sqrt{\mathcal{K}}}{2\sqrt{\mathcal{K}\mathcal{H}}}\big({}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho)\big), \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} 0 &\leq {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^2(\varrho)\,{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^2(\varrho) - \big({}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho)\big)^2 \\ &\leq \frac{\mathcal{H}-\mathcal{K}}{4\mathcal{K}\mathcal{H}}\big({}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho)\big)^2. \end{aligned} \tag{4.24}$$

Proof From (4.21), we have

$$\left(\frac{\mathcal{U}(\rho)}{\mathcal{V}(\rho)} - \mathcal{K}\right)\left(\mathcal{H} - \frac{\mathcal{U}(\rho)}{\mathcal{V}(\rho)}\right)\mathcal{V}^2(\rho) \geq 0, \quad 0 \leq \rho \leq \varrho.$$

It follows that

$$\mathcal{U}^2(\rho) + \mathcal{K}\mathcal{H}\mathcal{U}^2(\rho) \leq (\mathcal{K}+\mathcal{H})\mathcal{U}(\rho)\mathcal{V}(\rho). \tag{4.25}$$

Multiplying both sides of inequality (4.25) with $\frac{1}{\Gamma(\kappa)}e^{-\xi(\Psi(\varrho)-\Psi(\rho))}(\Psi(\varrho)-\Psi(\rho))^{\kappa-1}\Psi'(\rho)$ and integrating the resulting estimate with respect to ρ from 0 to ϱ and using (2.7), we obtain

$${}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^2(\varrho) + \mathcal{K}\mathcal{H}\,{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^2(\varrho) \leq (\mathcal{K}+\mathcal{H})\,{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho). \tag{4.26}$$

Now, since $\mathcal{K}\mathcal{H} > 0$ and

$$\left(\sqrt{{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^2(\varrho)}\sqrt{\mathcal{K}\mathcal{H}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^2(\varrho)}\right)^2 \geq 0,$$

it follows that

$$2\sqrt{{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^2(\varrho)}\sqrt{\mathcal{K}\mathcal{H}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^2(\varrho)} \leq {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^2(\varrho) + \mathcal{K}\mathcal{H}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^2(\varrho). \tag{4.27}$$

Hence, by using (4.26) and (4.27), we have

$$4\mathcal{K}\mathcal{H}{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^2(\varrho){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^2(\varrho) \leq (\mathcal{K}+\mathcal{H})^2\left({}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho)\right)^2, \tag{4.28}$$

which gives the desired assertion (4.22).

Now, from (4.28), we have

$$\sqrt{{}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}^2(\varrho){}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{V}^2(\varrho)} \leq \frac{\mathcal{K}+\mathcal{H}}{2\sqrt{\mathcal{K}\mathcal{H}}}\left({}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho)\right). \tag{4.29}$$

Subtraction of $({}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho))$ from (4.29) yields the desired assertion (4.23). Similarly, we can easily prove the assertion (4.24) from (4.22). □

5 Special cases

This section is devoted to certain special cases of the main result obtained in Sects. 3 and 4.

(I) Applying Theorem 3.1 for $\Psi(\varrho) = \varrho$, we attain the following result for a one sided tempered fractional integral.

Corollary 5.1 *Suppose that we have the function $\mathcal{U} \in L_1[0,\infty[$ and let ϕ_1 and ϕ_2 be two integrable functions defined on $[0,\infty[$ such that*

$$\phi_1(\varrho) \leq \mathcal{U}(\varrho) \leq \phi_2(\varrho), \quad \varrho \in [0,\infty[.$$

Then, for $\varrho > 0$, $\kappa, \lambda > 0$, we have

$$\begin{aligned}&\mathcal{J}^{\kappa,\xi}\phi_2(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho) + \mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{J}^{\lambda,\xi}\phi_1(\varrho)\\ &\quad \geq \mathcal{J}^{\kappa,\xi}\phi_2(\varrho)\mathcal{J}^{\lambda,\xi}\phi_1(\varrho) + \mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho).\end{aligned}$$

(II) Applying Theorem 3.1 for $\xi = 1$, we attain the following result for the one sided generalized Riemann–Liouville fractional integral proved earlier by Kacar et al. [37].

Corollary 5.2 *Suppose that the function $\mathcal{U} \in \chi^p_{\Psi}(0,\infty)$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0) = 0$. Moreover, let ϕ_1 and ϕ_2 be two integrable functions defined on $[0,\infty[$ such that*

$$\phi_1(\varrho) \leq \mathcal{U}(\varrho) \leq \phi_2(\varrho), \quad \varrho \in [0,\infty[.$$

Then, for $\varrho > 0$, $\kappa, \lambda > 0$, *we have*

$$\begin{aligned} &{}^{\Psi}\mathcal{J}^{\kappa}\phi_2(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\mathcal{U}(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\phi_1(\varrho) \\ &\quad \geq {}^{\Psi}\mathcal{J}^{\kappa}\phi_2(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\phi_1(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\mathcal{U}(\varrho). \end{aligned}$$

(III) Applying Theorem 3.1 for $\Psi(\varrho) = \varrho$ and $\xi = 1$, we attain the following result for one sided Riemann–Liouville fractional integral proved earlier by Tariboon et al. [45].

Corollary 5.3 *Suppose that the function* $\mathcal{U} \in L_1[0,\infty[$ *and let* ϕ_1 *and* ϕ_2 *be two integrable functions defined on* $[0,\infty[$ *such that*

$$\phi_1(\varrho) \leq \mathcal{U}(\varrho) \leq \phi_2(\varrho), \quad \varrho \in [0,\infty[.$$

Then, for $\varrho > 0$, $\kappa, \lambda > 0$, *we have*

$$\begin{aligned} &\mathcal{J}^{\kappa}\phi_2(\varrho)\mathcal{J}^{\lambda}\mathcal{U}(\varrho) + \mathcal{J}^{\kappa}\mathcal{U}(\varrho)\mathcal{J}^{\lambda}\phi_1(\varrho) \\ &\quad \geq \mathcal{J}^{\kappa}\phi_2(\varrho)\mathcal{J}^{\lambda}\phi_1(\varrho) + \mathcal{J}^{\kappa}\mathcal{U}(\varrho)\mathcal{J}^{\lambda}\mathcal{U}(\varrho). \end{aligned}$$

The special cases of Theorem 3.2 are presented by the following corollaries.

(I) Setting $\Psi(\varrho) = \varrho$, we attain the following result for one sided tempered fractional integral.

Corollary 5.4 *Suppose that the two positive functions* $\mathcal{U}$ *and* $\mathcal{V}$ *are defined on* $[0,\infty[$. *Assume that* (3.1) *holds and* ψ_1, ψ_2 *are two integrable functions on* $[0,\infty[$ *such that*

$$\psi_1(\varrho) \leq \mathcal{V}(\varrho) \leq \psi_2(\varrho), \quad \varrho \in [0,\infty[.$$

Then, for $\varrho > 0$ *and* $\kappa, \lambda > 0$, *the following four inequalities hold*:

$$\begin{aligned} &\mathcal{J}^{\kappa,\xi}\phi_2(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) + \mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho)\mathcal{J}^{\kappa,\xi}\psi_1(\varrho) \\ &\quad \geq \mathcal{J}^{\kappa,\xi}\phi_2(\varrho)\mathcal{J}^{\lambda,\xi}\psi_1(\varrho) + \mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho), \\ &\mathcal{J}^{\lambda,\xi}\phi_1(\varrho)\mathcal{J}^{\kappa,\xi}\mathcal{V}(\varrho) + \mathcal{J}^{\kappa,\xi}\psi_2(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{U}(\varrho) \\ &\quad \geq \mathcal{J}^{\kappa,\xi}\phi_1(\varrho)\mathcal{J}^{\lambda,\xi}\psi_2(\varrho) + \mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho), \\ &\mathcal{J}^{\kappa,\xi}\phi_2(\varrho)\mathcal{J}^{\lambda,\xi}\psi_2(\varrho) + \mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) \\ &\quad \geq \mathcal{J}^{\kappa,\xi}\phi_2(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) + \mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{J}^{\lambda,\xi}\psi_2(\varrho), \\ &\mathcal{J}^{\kappa,\xi}\phi_1(\varrho)\mathcal{J}^{\lambda,\xi}\psi_1(\varrho) + \mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) \\ &\quad \geq \mathcal{J}^{\kappa,\xi}\phi_1(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho) + \mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{J}^{\lambda,\xi}\psi_1(\varrho). \end{aligned}$$

(II) Applying Theorem 3.2 for $\xi = 1$, we attain the following result for the one sided generalized Riemann–Liouville fractional integral proved earlier by Kacar et al. [37].

Corollary 5.5 *Suppose that the two positive functions* $\mathcal{U}$ *and* $\mathcal{V}$ *are defined on* $[0,\infty[$ *and assume that the function* Ψ *is positive, monotone and increasing on* $[0,\infty[$ *and its derivative*

Ψ' is continuous on $[0,\infty[$ with $\Psi(0) = 0$. Assume that (3.1) holds and ψ_1, ψ_2 are two integrable functions defined on $[0,\infty[$ such that

$$\psi_1(\varrho) \le \mathcal{V}(\varrho) \le \psi_2(\varrho), \quad \varrho \in [0,\infty[.$$

Then, for $\varrho > 0$ and $\kappa,\lambda > 0$, the following four inequalities hold:

$$\begin{aligned}
&{}^{\Psi}\mathcal{J}^{\kappa}\phi_2(\varrho)^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}(\varrho) + {}^{\Psi}\mathcal{J}^{\lambda}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\kappa}\psi_1(\varrho)\\
&\quad \ge {}^{\Psi}\mathcal{J}^{\kappa}\phi_2(\varrho)^{\Psi}\mathcal{J}^{\lambda}\psi_1(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}(\varrho),\\
&{}^{\Psi}\mathcal{J}^{\lambda}\phi_1(\varrho)^{\Psi}\mathcal{J}^{\kappa}\mathcal{V}(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa}\psi_2(\varrho)^{\Psi}\mathcal{J}^{\lambda}\mathcal{U}(\varrho)\\
&\quad \ge {}^{\Psi}\mathcal{J}^{\kappa}\phi_1(\varrho)^{\Psi}\mathcal{J}^{\lambda}\psi_2(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}(\varrho),\\
&{}^{\Psi}\mathcal{J}^{\kappa}\phi_2(\varrho)^{\Psi}\mathcal{J}^{\lambda}\psi_2(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}(\varrho)\\
&\quad \ge {}^{\Psi}\mathcal{J}^{\kappa}\phi_2(\varrho)^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda}\psi_2(\varrho),\\
&{}^{\Psi}\mathcal{J}^{\kappa}\phi_1(\varrho)^{\Psi}\mathcal{J}^{\lambda}\psi_1(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}(\varrho)\\
&\quad \ge {}^{\Psi}\mathcal{J}^{\kappa}\phi_1(\varrho)^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}(\varrho) + {}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}(\varrho)^{\Psi}\mathcal{J}^{\lambda}\psi_1(\varrho).
\end{aligned}$$

(III) Applying Theorem 3.2 for $\Psi(\varrho) = \varrho$ and $\xi = 1$, we attain the following result for the one sided Riemann–Liouville fractional integral proved earlier by Tariboon et al. [45].

Corollary 5.6 *Suppose that the functions $\mathcal{U}$ and $\mathcal{V}$ are two positive functions defined on $[0,\infty[$. Assume that (3.1) holds and ψ_1, ψ_2 are two integrable functions defined on $[0,\infty[$ such that*

$$\psi_1(\varrho) \le \mathcal{V}(\varrho) \le \psi_2(\varrho), \quad \varrho \in [0,\infty[.$$

Then, for $\varrho > 0$ and $\kappa,\lambda > 0$, then the following four inequalities hold:

$$\begin{aligned}
&\mathcal{J}^{\kappa}\phi_2(\varrho)\mathcal{J}^{\lambda}\mathcal{V}(\varrho) + \mathcal{J}^{\lambda}\mathcal{U}(\varrho)\mathcal{J}^{\kappa}\psi_1(\varrho)\\
&\quad \ge \mathcal{J}^{\kappa}\phi_2(\varrho)\mathcal{J}^{\lambda}\psi_1(\varrho) + \mathcal{J}^{\kappa}\mathcal{U}(\varrho)\mathcal{J}^{\lambda}\mathcal{V}(\varrho),\\
&\mathcal{J}^{\lambda}\phi_1(\varrho)\mathcal{J}^{\kappa}\mathcal{V}(\varrho) + \mathcal{J}^{\kappa}\psi_2(\varrho)\mathcal{J}^{\lambda}\mathcal{U}(\varrho)\\
&\quad \ge \mathcal{J}^{\kappa}\phi_1(\varrho)\mathcal{J}^{\lambda}\psi_2(\varrho) + \mathcal{J}^{\kappa}\mathcal{U}(\varrho)\mathcal{J}^{\lambda}\mathcal{V}(\varrho),\\
&\mathcal{J}^{\kappa}\phi_2(\varrho)\mathcal{J}^{\lambda}\psi_2(\varrho) + \mathcal{J}^{\kappa}\mathcal{U}(\varrho)\mathcal{J}^{\lambda}\mathcal{V}(\varrho)\\
&\quad \ge \mathcal{J}^{\kappa}\phi_2(\varrho)\mathcal{J}^{\lambda}\mathcal{V}(\varrho) + \mathcal{J}^{\kappa}\mathcal{U}(\varrho)\mathcal{J}^{\lambda}\psi_2(\varrho),\\
&\mathcal{J}^{\kappa}\phi_1(\varrho)\mathcal{J}^{\lambda}\psi_1(\varrho) + \mathcal{J}^{\kappa}\mathcal{U}(\varrho)\mathcal{J}^{\lambda}\mathcal{V}(\varrho)\\
&\quad \ge \mathcal{J}^{\kappa}\phi_1(\varrho)\mathcal{J}^{\lambda}\mathcal{V}(\varrho) + \mathcal{J}^{\kappa}\mathcal{U}(\varrho)\mathcal{J}^{\lambda}\psi_1(\varrho).
\end{aligned}$$

The following corollaries represent special cases of Theorem 4.1.

(I) Setting $\Psi(\varrho) = \varrho$, we attain the following result for a one sided tempered fractional integral.

Corollary 5.7 *Suppose that the functions $\mathcal{U}$ and $\mathcal{V}$ are two positive functions defined on $[0,\infty[$ and let $p_1, q_1 > 1$ be such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Then, for $\varrho > 0$, the following inequalities hold*:

$$\begin{aligned}
&\frac{1}{p_1}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{p_1}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}^{p_1}(\varrho) + \frac{1}{q_1}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{U}^{q_1}(\varrho)\\
&\quad \geq \mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}(\varrho)\mathcal{U}(\varrho),\\
&\frac{1}{p_1}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{p_1}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}^{q_1}(\varrho) + \frac{1}{q_1}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{U}^{p_1}(\varrho)\\
&\quad \geq \mathcal{J}^{\lambda,\xi}\mathcal{V}^{q_1-1}(\varrho)\mathcal{U}^{p_1-1}(\varrho)\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho),\\
&\frac{1}{p_1}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{p_1}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}^{2}(\varrho) + \frac{1}{q_1}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{q_1}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{U}^{2}(\varrho)\\
&\quad \geq \mathcal{J}^{\lambda,\xi}\mathcal{U}^{\frac{2}{p_1}}(\varrho)\mathcal{V}^{\frac{2}{q_1}}(\varrho)\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho)\mathcal{V}(\varrho),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{p_1}\mathcal{J}^{\kappa,\xi}\mathcal{U}^{2}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{V}^{q_1}(\varrho) + \frac{1}{q_1}\mathcal{J}^{\kappa,\xi}\mathcal{V}^{2}(\varrho)\mathcal{J}^{\lambda,\xi}\mathcal{U}^{p_1}(\varrho)\\
&\quad \geq \mathcal{J}^{\lambda,\xi}\mathcal{U}^{p_1-1}(\varrho)\mathcal{V}^{q_1-1}(\varrho)\mathcal{J}^{\kappa,\xi}\mathcal{U}^{\frac{2}{p_1}}(\varrho)\mathcal{V}^{\frac{2}{q_1}}(\varrho).
\end{aligned}$$

(II) Applying Theorem 4.1 for $\xi = 1$, we attain the following new result for a one sided generalized Riemann–Liouville fractional integral.

Corollary 5.8 *Suppose that the functions $\mathcal{U}$ and $\mathcal{V}$ are two positive functions defined on $[0,\infty[$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0) = 0$. If $p_1, q_1 > 1$ are such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$, then, for $\varrho > 0$, the following inequalities hold*:

$$\begin{aligned}
&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}^{p_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}^{p_1}(\varrho) + \frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\kappa}\mathcal{V}^{q_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\mathcal{U}^{q_1}(\varrho)\\
&\quad \geq {}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}(\varrho)\mathcal{V}(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}(\varrho)\mathcal{U}(\varrho),\\
&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}^{p_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}^{q_1}(\varrho) + \frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\kappa}\mathcal{V}^{q_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\mathcal{U}^{p_1}(\varrho)\\
&\quad \geq {}^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}^{q_1-1}(\varrho)\mathcal{U}^{p_1-1}(\varrho){}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}(\varrho)\mathcal{V}(\varrho),\\
&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}^{p_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}^{2}(\varrho) + \frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\kappa}\mathcal{V}^{q_1}(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\mathcal{U}^{2}(\varrho)\\
&\quad \geq {}^{\Psi}\mathcal{J}^{\lambda}\mathcal{U}^{\frac{2}{p_1}}(\varrho)\mathcal{V}^{\frac{2}{q_1}}(\varrho){}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}(\varrho)\mathcal{V}(\varrho),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}^{2}(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\mathcal{V}^{q_1}(\varrho) + \frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\kappa}\mathcal{V}^{2}(\varrho){}^{\Psi}\mathcal{J}^{\lambda}\mathcal{U}^{p_1}(\varrho)\\
&\quad \geq {}^{\Psi}\mathcal{J}^{\lambda}\mathcal{U}^{p_1-1}(\varrho)\mathcal{V}^{q_1-1}(\varrho){}^{\Psi}\mathcal{J}^{\kappa}\mathcal{U}^{\frac{2}{p_1}}(\varrho)\mathcal{V}^{\frac{2}{q_1}}(\varrho).
\end{aligned}$$

In a similar way, we can obtain the special cases of Theorems 4.2 and 4.3 by applying similar procedures.

6 Particular cases

Here, we present certain new particular cases of our main result by employing the connection of GTF-integral (2.7) with the classical Riemann–Liouville expression containing another function in the kernel.

Li et al. [39] defined the connection of a tempered fractional integral (2.3) with the Riemann–Liouville fractional integral by

$$ {}_a\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho) = e^{-\xi\varrho}\,{}_a\mathcal{J}^{\kappa}\big[e^{\xi\varrho}\mathcal{U}(\varrho)\big]. $$

Here, we propose the following connection of the GTF-integral (2.7) with the generalized Riemann–Liouville fractional integral as

$$ {}^{\Psi}\mathcal{J}^{\kappa,\xi}\mathcal{U}(\varrho) = e^{-\xi\Psi(\varrho)}\,{}^{\Psi}\mathcal{J}^{\kappa}\big[e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big], \tag{6.1} $$

where ${}^{\Psi}\mathcal{J}^{\kappa}[e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)]$ is the generalized Riemann–Liouville fractional integral concerning another function.

Applying the above connection (6.1) to Theorem 3.1, one can get the following new result in terms of the generalized Riemann–Liouville fractional integral in the sense of another function.

Theorem 6.1 *Suppose that the function $\mathcal{U} \in \chi^{p}_{\Psi}(0,\infty)$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0) = 0$. Moreover, let ϕ_1 and ϕ_2 be two integrable functions defined on $[0,\infty[$ such that*

$$ \phi_1(\varrho) \le \mathcal{U}(\varrho) \le \phi_2(\varrho), \quad \varrho \in [0,\infty[. \tag{6.2} $$

Then, for $\varrho > 0$, $\kappa,\lambda > 0$, we have

$$ \begin{aligned} &{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\phi_2(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big) + {}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\phi_1(\varrho)\big) \\ &\quad\ge {}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\phi_2(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\phi_1(\varrho)\big) + {}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big). \end{aligned} $$

Applying (6.1) to Theorem 3.2, one can obtain easily the following new results.

Theorem 6.2 *Suppose that the two positive functions $\mathcal{U}$ and $\mathcal{V}$ are defined on $[0,\infty[$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0) = 0$. Assume that (6.2) holds and ψ_1, ψ_2 are two integrable functions defined on $[0,\infty[$ such that*

$$ \psi_1(\varrho) \le \mathcal{V}(\varrho) \le \psi_2(\varrho), \quad \varrho \in [0,\infty[. \tag{6.3} $$

Then, for $\varrho > 0$ and $\kappa,\lambda > 0$, the following four inequalities hold:

$$ \begin{aligned} &{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\phi_2(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}(\varrho)\big) + {}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\psi_1(\varrho)\big) \\ &\quad\ge {}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\phi_2(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\psi_1(\varrho)\big) + {}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}(\varrho)\big), \end{aligned} $$

$$
\begin{aligned}
&{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\phi_1(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}(\varrho)\big)+{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\psi_2(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big)\\
&\quad\geq{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\phi_1(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\psi_2(\varrho)\big)+{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}(\varrho)\big),\\
&{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\phi_2(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\psi_2(\varrho)\big)+{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}(\varrho)\big)\\
&\quad\geq{}^{\Psi}\mathcal{J}^{\kappa,\xi}\big(e^{\xi\Psi(\varrho)}\phi_2(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}(\varrho)\big)+{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\psi_2(\varrho)\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\phi_1(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\psi_1(\varrho)\big)+{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}(\varrho)\big)\\
&\quad\geq{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\phi_1(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}(\varrho)\big)+{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\psi_1(\varrho)\big).
\end{aligned}
$$

One can obtain the following new result of Theorem 4.1 in terms of the generalized Riemann–Liouville fractional integral in the sense of another function by utilizing (6.1).

Theorem 6.3 *Suppose that the two positive functions $\mathcal{U}$ and $\mathcal{V}$ are defined on $[0,\infty[$ and assume that the function Ψ is positive, monotone and increasing on $[0,\infty[$ and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0)=0$. If $p_1,q_1>1$ is such that $\frac{1}{p_1}+\frac{1}{q_1}=1$, then, for $\varrho>0$, the following inequalities hold*:

$$
\begin{aligned}
&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}^{p_1}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}^{p_1}(\varrho)\big)+\frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}^{q_1}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}^{q_1}(\varrho)\big)\\
&\quad\geq{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\mathcal{V}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}(\varrho)\mathcal{U}(\varrho)\big),\\
&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}^{p_1}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}^{q_1}(\varrho)\big)+\frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}^{q_1}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}^{p_1}(\varrho)\big)\\
&\quad\geq{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}^{q_1-1}(\varrho)\mathcal{U}^{p_1-1}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\mathcal{V}(\varrho)\big),\\
&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}^{p_1}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}^{2}(\varrho)\big)+\frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}^{q_1}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}^{2}(\varrho)\big)\\
&\quad\geq{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}^{\frac{2}{p_1}}(\varrho)\mathcal{V}^{\frac{2}{q_1}}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}(\varrho)\mathcal{V}(\varrho)\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{p_1}{}^{\Psi}\mathcal{J}^{\kappa}\big(e^{\xi\Psi(\varrho)}\mathcal{U}^{2}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}^{q_1}(\varrho)\big)+\frac{1}{q_1}{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{V}^{2}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}^{p_1}(\varrho)\big)\\
&\quad\geq{}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}^{p_1-1}(\varrho)\mathcal{V}^{q_1-1}(\varrho)\big){}^{\Psi}\mathcal{J}^{\lambda}\big(e^{\xi\Psi(\varrho)}\mathcal{U}^{\frac{2}{p_1}}(\varrho)\mathcal{V}^{\frac{2}{q_1}}(\varrho)\big).
\end{aligned}
$$

Similarly, we can obtain particular new results of Theorems 4.2 and 4.3 in terms of the generalized Riemann–Liouville fractional integral in the sense of another function by utilizing (6.1). Also, one can easily obtain certain new results of Theorems presented in Sect. 6 by utilizing the special cases discussed in Remark 2.2.

7 Conclusion

In this paper, we presented various types of inequalities such as Grüss-type inequalities and certain other inequalities by employing a generalized tempered fractional (GTF)-integral in the sense of another function Ψ. Furthermore, we have discussed several special cases by using Remark 2.2. Also, we proposed a connection between the GTF-integral with the classical Riemann–Liouville fractional integral and derived certain new results in terms of the Riemann–Liouville fractional integral concerning another function. One can easily obtain several other types of inequalities, such as Hadamard fractional integral inequalities and generalized fractional conformable inequalities by utilizing Remark 2.2. Moreover, certain new inequalities can be derived by utilizing the inequalities discussed in Sect. 6.

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Author details
[1]Department of Mathematics, Shaheed Benazir Bhutto University, Sheringal, Pakistan. [2]Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawaser, Saudi Arabia. [3]Department of Mathematics, Government College University, Faisalabad, Pakistan. [4]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Kingdom of Saudi Arabia. [5]Department of Medical Research, China Medical University, Taichung, Taiwan. [6]Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan.

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