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Gel'fand-N-width in probabilistic setting

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Abstract

In this article, we first put forward a new definition of probabilistic Gel'fand-(N, δ)-width which is the classical Gel'fand-N-width in a probabilistic setting. Then we estimate the sharp order of the probabilistic Gel'fand-(N, δ)-width of finite-dimensional space. Furthermore, we obtain the exact order of probabilistic Gel'fand-(N, δ)-width of univariate Sobolev space by the discretization method according to the result of finite-dimensional space.

Keywords: Probabilistic Gel'fand-(N, δ)-width; Finite-dimensional space; Univariate Sobolev space

1 Introduction

It is well-known that the width plays an important role in computational complexity which is a core aspect of the theoretical basis of computer science. The relationship between the width and computational complexity can be found in the monograph [1] by the works of Traus and Waskikawski. It is worth noting that the relationship between Gel'fand-N-width and diameter of information has been stated in the worst case setting, and a series of perfect results of the width in worst case setting have been provided by Pinkus in [2]. Moreover, Maiorov has furthered the width theory by introducing the probabilistic Kolomogorov-(N, δ)-width and probabilistic linear-(N, δ)-width which are Kolomogorov-N-width and linear N-width in a probabilistic setting, respectively. Maiorov, Fang Gensun and Ye Peixin have studied probabilistic Kolomogorov-(N, δ)-width and probabilistic linear-(N, δ)-width of finite-dimensional space and univariate Sobolev space, and obtained some pretty conclusions in [3–6]. Chen Guanggui and Fang Gensun have discussed probabilistic Kolomogorov-(N, δ)-width and probabilistic linear-(N, δ)-width of multivariate Sobolev space with mixed derivative and obtained some useful results in [7, 8]. In 2010, Chen Guanggui, Nie Pengjuan and Luo Xinjian have putted forward the width of operator in a probabilistic setting by the first definition of probabilistic linear-(N, δ)-width of operator, and have estimated the exact order of probabilistic linear-(N, δ)-width of finite-dimensional diagonal operator in [9]. Later, on the basis of random process, Dai Feng and Wang Heping have discussed probabilistic linear-(N, δ)-width of finite-dimensional diagonal operator in [10], and have acquired profound result.

In this article, we continue the research above by proposing the definition of a probabilistic Gel'fand-(N, δ)-width, and estimate the sharp order of probabilistic Gel'fand-(N, δ)-width in both finite-dimensional space and univariate Sobolev space.

In order to present the relevant results, we first introduce some notations. Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the real number set, the integer set and the nonnegative integer set, respectively. Assume that c, c_i, c_i', $i = 0, 1, \ldots,$ are positive constants depending only on the parameters p, q, r, ρ. The notation $a(y) \asymp b(y)$ or $a(y) \ll b(y)$ for two positive functions $a(y)$ and $b(y)$ means that there exist constants c, c_1 and c_2 such that $c_1 \leq a(y)/b(y) \leq c_2$ or $a(y) \leq cb(y)$ for any $y \in D$.

This article is arranged as follows. In Sect. 2, we review notions of width and define probabilistic Gel'fand-(N, δ)-width. In Sect. 3, we introduce the probabilistic Gel'fand-(N, δ)-width of finite-dimensional space. In Sect. 4, we investigate the probabilistic Gel'fand-(N, δ)-width of univariate Sobolev space.

2 Gel'fand-*N*-width in a probabilistic setting

In this section, we first review notions of the width in worst case setting, then introduce the definitions of probabilistic Kolomogorov-(N, δ)-width and probabilistic linear-(N, δ)-width. Finally, we propose the new definition of the probabilistic Gel'fand-(N, δ)-width.

Definition 2.1 Suppose that X is a normed linear space equipped with a norm $\| \cdot \|$, W is a non-null subset of X, N is nonnegative integer. Then

$$d_N(W, X) := \inf_{F_N} \sup_{x \in W} \inf_{y \in F_N} \|x - y\|,$$

$$\lambda_N(W, X) := \inf_{T_N} \sup_{x \in W} \|x - T_N x\|,$$

$$d^N(W, X) := \inf_{L^N} \sup_{x \in W \cap L^N} \|x\|,$$

are called Kolomogorov-N-width, linear N-width, Gel'fand-N-width of W in X, respectively. Here, F_N runs through all linear subspaces of X with dimension at most N, T_N runs through all bounded linear operators on X with rank at most N, L^N runs all linear subspaces of X with codimension at most N.

The codimension is defined as follows.

A linear subspace L^N of the normed linear space X is called codimension N, if there are N linear independent continuous linear functionals $f_1, f_2, \ldots, f_N$ on X, such that

$$L^N = \left\{ x \in X : f_i(x) = 0, i = 0, 1, \ldots, N \right\}.$$

We simply write $\inf \emptyset = +\infty$.

Definition 2.2 Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be normed linear spaces, and X can be imbedded continuously into Y. Then

$$d_N(X, Y) := d_N\big(B(X), Y\big),$$

$$\lambda_N(X, Y) := \inf_{T_N} \sup_{x \in B(X)} \|x - T_N x\|_Y,$$

$$d^N(X, Y) := \inf_{L^N} \sup_{x \in B(X) \cap L^N} \|x\|_Y,$$

are regarded as Kolomogorov-N-width, linear N-width and Gel'fand-N-width of X in Y, respectively. Here, T_N is taken over all continuous linear operators from X into Y of rank at most N, L^N is taken over all subspaces of X of the codimension at most N, $B(X)$ is the unit ball of X.

There are a series of useful properties of the width in worst case setting which are as follows.

Proposition 2.1 ([2]) *Let X and Y be normed linear spaces, and X can be imbedded continuously into Y. Then*

  1. $d_N(X,Y) \le \lambda_N(X,Y), d^N(X,Y) \le \lambda_N(X,Y)$.
  2. *If X is the Hilbert space, then*

$$d^N(X,Y) = \lambda_N(X,Y).$$

More details about the width in worst case setting can be consulted in [2]. Now, we tend to introduce the width in a probabilistic setting. Maiorov [3, 4] has proposed the definitions of Kolomogorov-N-width and linear N-width in a probabilistic setting.

Definition 2.3 Suppose that X is a normed linear space, $W \subset X$, and $\mathcal{B}$ is a Borel field formed by all opened sets in W, μ is a probabilistic measure on $\mathcal{B}$, $\delta \in (0,1]$, then

$$d_{N,\delta}(W,\mu,X) = \inf_{G_\delta} d_N(W \backslash G_\delta, X)$$

and

$$\lambda_{N,\delta}(W,\mu,X) = \inf_{G_\delta} \lambda_N(W \backslash G_\delta, X)$$

are called probabilistic Kolomogorov-(N,δ)-width and probabilistic linear-(N,δ)-width of W in X with measure μ, respectively. Here G_δ is taken over all subsets in $\mathcal{B}$ with the measure at most δ.

Remark 2.1 Comparing Definition 2.1 with Definition 2.3, we can obviously find that the N-width in a probabilistic setting is just the N-width in a worst case setting by eliminating the subset whose measure is at most δ.

In order to define the Gel'fand-N-width in a probabilistic setting, we first introduce the following obvious result.

Let H be a Hilbert space with the probabilistic measure μ, F be a closed subspace in H. Let $F^\perp$ denote the orthogonal complement of F. Any $x \in H$ can be decomposed uniquely in the form

$$x = y + z, \quad \text{where } y \in F, z \in F^\perp.$$

The element y will be denoted by Px and P is called projection operator upon F. For any Borel set G_F in F, let

$$\mu_F(G_F) := \mu\big(\{x \in H : Px \in G_F\}\big). \tag{1}$$

Then μ_F is a probabilistic measure on F.

   With Definitions 2.1, 2.2, 2.3 and Remark 2.1, we now can put foreword the new defini-
tion of Gel'fand-N-width in a probabilistic setting.

Definition 2.4 Let H be a Hilbert space, $(X, \| \cdot \|)$ be a normed linear space, and H can be
imbedded continuously into X, μ be a probabilistic measure on H, $\delta \in (0, 1]$. Then

$$d_\delta^N(H, \mu, X) = \inf_{G_\delta} \inf_{L^N} \sup_{x \in (H \setminus G_\delta) \cap L^N} \|x\|$$

is called probabilistic Gel'fand-(N, δ)-width of H in X with the measure μ. Here L^N runs
over all linear subspaces of H with codimension at most N. G_δ is taken over all subsets of H
with the measure at most δ, and satisfies the following condition: for any closed subspace
F in H,

$$\mu_F(G_\delta \cap F) \leq \delta. \tag{2}$$

Here μ_F refers to Eq. (1).

Remark 2.2 Here we add the condition (2) in order to make sure that $(H \setminus G_\delta) \cap L^N$ has
enough elements.

   There is a very useful relationship between the probabilistic linear-(N, δ)-width and the
probabilistic Gel'fand-(N, δ)-width.

Theorem 2.1 *Suppose H is a Hilbert space, $(X, \| \cdot \|)$ is a normed linear space, H can be
imbedded continuously into X, $\delta \in (0, 1]$, μ is a probabilistic measure on H. Then*

$$\lambda_{N,\delta}(H, \mu, X) \leq d_\delta^N(H, \mu, X).$$

Proof $\forall \varepsilon > 0$, by the definition of $d_\delta^N(H, \mu, X)$, there is subspace L^N of H with codimension
at most N, and subset $Q \subset H$, for which

$$\mu(Q) \geq 1 - \delta, \qquad \mu_{L^N}(Q \cap L^N) \geq 1 - \delta,$$

$$\sup_{x \in Q \cap L^N} \|x\|_X \leq d_\delta^N(H, \mu, X) + \varepsilon.$$

   We have $\mu(Q') \geq 1 - \delta$, where $Q' = \{x \in H : P_{L^N} x \in Q \cap L^N\}$, P_{L^N} is a projection operator
on L^N.
   Let $T_N = I - P_{L^N}$. Then T_N is a bounded linear operator from H into X with the rank at
most N.
   It is clear that

$$\lambda_{N,\delta}(H, \mu, X) \leq \sup_{x \in Q'} \|x - T_N x\|_X$$

$$= \sup_{x \in Q'} \|P_{L^N} x\|_X$$

$$= \sup_{x \in Q \cap L^N} \|x\|_X$$

$$\leq d_\delta^N(H, \mu, X) + \varepsilon.$$

Since ε is arbitrary, one has

$$\lambda_{N,\delta}(H, \mu, X) \leq d_\delta^N(H, \mu, X). \qquad \square$$

3 Gel'fand-*N*-width of finite-dimensional space in a probabilistic setting

In this section, we will discuss the Gel'fand-N-width of finite-dimensional space in a probabilistic setting. We first review the finite-dimensional space.

Let l_p^m $(1 \leq p \leq \infty)$ be a m-dimensional normed linear space of vector $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ with the norm

$$\|x\|_{l_p^m} = \begin{cases} (\sum_{i=1}^m |x_i|^p)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \text{Max}_{1 \leq i \leq m} |x_i|, & p = \infty. \end{cases}$$

We denote by $B_p^m(\rho) := \{x \in l_p^m : \|x\|_{l_q^m} \leq \rho\}$ a ball with the radius ρ in l_q^m, and $B_p^m := B_p^m(1)$.

It is clear that l_2^m is a Hilbert space with inner product

$$\langle x, y \rangle = \sum_{i=1}^m x_i y_i,$$

where $x = (x_1, x_2, \ldots, x_m), y = (y_1, y_2, \ldots, y_m) \in R^m$.

There are several useful results about linear width and Gel'fand width of finite-dimensional space in worst case setting as follows.

Theorem 3.1 ([2]) *Let* $N \leq m$, $1 \leq p \leq 2$, $1 \leq q \leq \infty$. *Then*

$$\lambda_N(B_2^m, l_p^m) = d^N(B_2^m, l_p^m) = (m - N)^{\frac{1}{p} - \frac{1}{2}},$$

$$\lambda_N(B_q^m, l_q^m) = d^N(B_q^m, l_q^m) = 1.$$

Considering the space of l_2^m with the standard Gaussian measure $\gamma = \gamma_m$, which is defined as

$$\gamma(G) = (2\pi)^{-\frac{m}{2}} \int_G \exp\left(-\frac{1}{2}\|x\|_{l_2^m}^2\right) dx,$$

where G is any Borel subset in l_2^m. Obviously, $\gamma(l_2^m) = 1$.

Maiorov, Fang Gensun and Ye Peixin have obtained the sharp order of linear width of finite-dimensional space in probabilistic setting which can be summarized as follows.

Theorem 3.2 ([4–6]) *Let* $2N \leq m$, $\delta \in (0, \frac{1}{2}]$. *Then*:
(1) *For* $1 \leq q < 2$,

$$\lambda_{N,\delta}(l_2^m, \gamma, l_q^m) \asymp m^{\frac{1}{q} - \frac{1}{2}} \sqrt{m + \ln \frac{1}{\delta}}.$$

(2) *For $2 \le q < \infty$,*

$$\lambda_{N,\delta}\left(l_2^m, \gamma, l_q^m\right) \asymp m^{\frac{1}{q}} + \sqrt{\ln \frac{1}{\delta}}.$$

(3) *For $q = \infty$,*

$$\lambda_{N,\delta}\left(l_2^m, \gamma, l_\infty^m\right) \asymp \sqrt{\ln \frac{(m-n)}{\delta}}.$$

Here the upper bounds only need the condition of $N \le m$.

For discussing the sharp order of Gel'fand-(N, δ)-width of finite-dimensional space, we introduce two special Borel sets in finite-dimensional space.

Lemma 3.1 ([3]) *For any $\delta \in (0, \frac{1}{2}]$, there exists an absolute positive constant c_0 such that*

$$\gamma\left(\left\{x \in R^m : \|x\|_2 \ge c_0\left(\sqrt{m} + \sqrt{\ln \frac{1}{\delta}}\right)\right\}\right) \le \delta.$$

Lemma 3.2 ([4]) *For $2 \le q < \infty$ and any $\delta \in (0, \frac{1}{2}]$, there exists a positive constant c_q depending only on the q such that*

$$\gamma\left(\left\{x \in R^m : \|x\|_{l_q^m} \ge c_q\left(m^{\frac{1}{q}} + \sqrt{\ln \frac{1}{\delta}}\right)\right\}\right) \le \delta.$$

We now start to estimate the exact order of the Gel'fand-(N, δ)-width of finite-dimensional space.

Theorem 3.3 *Let $2N \le m$, $\delta \in (0, \frac{1}{2}]$. Then:*
(1) *For $1 \le q < 2$,*

$$d_\delta^N\left(l_2^m, \gamma, l_q^m\right) \asymp m^{\frac{1}{q}-\frac{1}{2}}\sqrt{m + \ln \frac{1}{\delta}}.$$

(2) *For $2 \le q < \infty$,*

$$d_\delta^N\left(l_2^m, \gamma, l_q^m\right) \asymp m^{\frac{1}{q}} + \sqrt{\ln \frac{1}{\delta}}.$$

Here the upper bounds hold if $N \le m$.

Remark 3.1 Moreover, we conjecture that if $2N \le m$, $\delta \in (0, \frac{1}{2}]$, then $d_\delta^N(l_2^m, \gamma, l_\infty^m) \asymp \sqrt{\ln \frac{(m-N)}{\delta}}$.

Proof It is obvious that the lower bound holds by the Theorem 2.1 and Theorem 3.2.
　Now,we estimate the upper bound.
(1) For $1 \le q < 2$.

Let $G_\delta = \{x \in \mathbb{R}^m : \|x\|_{l_2^m} \geq c_0(\sqrt{m} + \sqrt{\ln \frac{1}{\delta}})\}$, where c_0 is the constant of Lemma 3.1. By Lemma 3.1, $\gamma(G_\delta) \leq \delta$. It is clear that G_δ which is the subset in l_2^m satisfies condition in Definition 2.4. From the definition of the Gel'fand-(N, δ)-width, we have

$$d_\delta^N(l_2^m, \gamma, l_q^m) \leq d^N(l_2^m \backslash Q, l_q^m) = d^N\left(B_2^m\left(C_0\left(m^{\frac{1}{2}} + \sqrt{\ln \frac{1}{\delta}}\right)\right), l_q^m\right)$$

$$\leq C_0\left(m^{\frac{1}{2}} + \sqrt{\ln \frac{1}{\delta}}\right) d^N(B_2^m, l_q^m)$$

$$\leq C_0\left(m^{\frac{1}{2}} + \sqrt{\ln \frac{1}{\delta}}\right)(m - N)^{\frac{1}{q} - \frac{1}{2}}$$

$$\ll m^{\frac{1}{q} - \frac{1}{2}}\sqrt{m + \ln \frac{1}{\delta}}.$$

(2) For $2 \leq q < \infty$.

Let $G_\delta = \{x \in \mathbb{R}^m : \|x\|_{l_q^m} \geq c_q(m^{\frac{1}{q}} + \sqrt{\ln \frac{1}{\delta}})\}$, where c_q is the constant of Lemma 3.2. By Lemma 3.2, $\gamma(G_\delta) \leq \delta$. Obviously, G_δ also satisfies the condition of Definition 2.4. Using the definition of Gel'fand-(N, δ)-width, we have

$$d_\delta^N(l_2^m, \gamma, l_q^m) \leq d^N(l_q^m \backslash Q, l_q^m) = d^N\left(B_q^m\left(C_q\left(m^{\frac{1}{q}} + \sqrt{\ln \frac{1}{\delta}}\right)\right), l_q^m\right)$$

$$= C_q\left(m^{\frac{1}{q}} + \sqrt{\ln \frac{1}{\delta}}\right) d^N(B_q^m, l_q^m)$$

$$= C_q\left(m^{\frac{1}{q}} + \sqrt{\ln \frac{1}{\delta}}\right)$$

$$\ll m^{\frac{1}{q}} + \sqrt{\ln \frac{1}{\delta}}. \qquad \square$$

4 Gel'fand-(N, δ)-width of univariate Sobolev space

In this section, we estimate the exact order of Gel'fand-(N, δ)-width of univariate Sobolev space.

Denote by $L_q(\mathbb{T})$, $1 \leq q \leq \infty$, the classical q-integral Lebesgue space of 2π-periodic functions with the usual norm, $\|\cdot\|_{L_q} := \|\cdot\|_{L_q(\mathbb{T})}$. It is clear that $L_2(\mathbb{T})$ is a Hilbert space with inner product

$$\langle x, y \rangle = \frac{1}{2\pi} \int_0^{2\pi} x(t)\overline{y(t)}\, dt, \quad x, y \in L_2(\mathbb{T}).$$

For any $x \in L_2(\mathbb{T})$, the Fourier series of x can be regarded as

$$x(t) = \sum_{k=-\infty}^{\infty} \hat{x}(k)e^{ikt}$$

$$= \sum_{k=-\infty}^{\infty} \hat{x}(k)e_k(t),$$

where $\hat{x}(k) = \frac{1}{2\pi}\int_{\mathbb{T}} x(t)e^{ikt}\, dt$, $e_k(t) = e^{ikt}$.

For arbitrary $r \in \mathbb{R}$, we define the rth order derivative of x in the sense of Weyl by

$$x^{(r)}(t) = (D^r x)(t)$$
$$:= \sum_{n \in \mathbb{Z}} \hat{x}(k)(ik)^r e_k(t),$$

where $(ik)^r = |k|^r e^{\frac{i\pi r}{2} \operatorname{sgn} k}$.

Let

$$W_2^r(\mathbb{T}) := \left\{ x \in L_2(\mathbb{T}) : x^{(r)} \in L_2(\mathbb{T}), \hat{x}(0) = 0 \right\}.$$

It is well-known that $W_2^r(\mathbb{T})$ is a Hilbert space with the inner product

$$\langle x, y \rangle_r := \langle x^{(r)}, y^{(r)} \rangle, \quad x, y \in W_2^r(T),$$

and the norm can be obtained

$$\|x\|_{W_2^r(\mathbb{T})}^2 = \langle x^{(r)}, x^{(r)} \rangle.$$

By the Parseval equality, we can obtain

$$\|x\|_{W_2^r(\mathbb{T})} = \left(\sum_{k \in \mathbb{Z}_0} |(ik)^r c_k|^2 \right)^{1/2},$$

where $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$.

$W_2^r(\mathbb{T})$ is named as univariate Sobolev space. It is well-known that $W_2^r(\mathbb{T})$ $(r > \frac{1}{2})$ can be embedded continuously into the $L_q(\mathbb{T})$, $1 \le q \le \infty$. Numerous approximation characteristic of the univariate Sobolev space, such as Kolomogorov-N-width, linear N-width in worst case setting, probabilistic setting and average setting, have been referred to in the literature.

Now we equipped $W_2^r(\mathbb{T})$ with a Gaussian measure μ whose mean is zero and correlation operator C_μ has eigenfunctions $e_k(t)$ and eigenvalues

$$\lambda_k = |k|^{-\rho}, \quad \rho > 1, k \in \mathbb{Z}_0. \tag{3}$$

That is,

$$C_\mu e_k = \lambda_k e_k, \quad \forall k \in \mathbb{Z}_0.$$

Let $y_1, \ldots, y_n$ be any orthogonal system of functions in $L_2(\mathbb{T})$, $\sigma_j = \langle C_\mu y_j, y_j \rangle$, $j = 1, \ldots, n$, and B be an arbitrary Borel subset of l_2^n. Then the Gaussian measure μ on the cylindrical subset G in the space $W_2^r(\mathbb{T})$ is given by

$$\mu(G) = \prod_{j=1}^n (2\pi \sigma_j)^{-1/2} \int_B \exp\left(-\sum_{j=1}^n \frac{|u_j|^2}{2\sigma_j} \right) du_1 \cdots du_n,$$

where

$$G = \left\{ x \in W_2^r(\mathbb{T}) : \left(\langle x, y_1^{(-r)} \rangle_r, \ldots, \langle x, y_n^{(-r)} \rangle_r \right) \in B \right\}.$$

More detailed information about the Gaussian measure in Banach space can be found in the books by Kuo [11], Ledoux and Talagrand [12].

Maiorov, Fang Gensun and Ye Peixin have researched the exact order of the Kolomogorov-(N, δ)-width and the linear-(N, δ)-width of univariate Sobolev space $W_2^r(\mathbb{T})$.

Theorem 4.1 ([3–6]) *Let $r > \frac{1}{2}$, $\delta \in (0, \frac{1}{2}]$, $1 \le q \le \infty$, $N \in \mathbb{N}$. Then*

$$d_{N,\delta}\left(W_2^r(\mathbb{T}), \mu, L_q(\mathbb{T}) \right) \asymp N^{-(r+\frac{\rho-1}{2})} \sqrt{1 + \frac{1}{N} \ln \frac{1}{\delta}},$$

and

$$\lambda_{N,\delta}\left(W_2^r(\mathbb{T}), \mu, L_q(\mathbb{T}) \right) \asymp \begin{cases} N^{-(r+\frac{\rho-1}{2})} \sqrt{1 + \frac{1}{N} \ln \frac{1}{\delta}}, & 1 \le q < 2, \\ N^{-(r+\frac{\rho-1}{2})}(1 + N^{-\frac{1}{q}} \sqrt{\ln \frac{1}{\delta}}), & 2 \le q < \infty, \\ N^{-(r+\frac{\rho-1}{2})} \sqrt{\ln \frac{N}{\delta}}, & q = \infty. \end{cases}$$

We now discuss the order of Gel'fand-(N, δ)-width of $W_2^r(\mathbb{T})$ in $L_q(\mathbb{T})$. It is the main result of this section.

Theorem 4.2 *Let $r > \frac{1}{2}$, $\delta \in (0, \frac{1}{2}]$, $1 \le q < \infty$, $N \in \mathbb{N}$. Then*

$$d_\delta^N\left(W_2^r(\mathbb{T}), \mu, L_q(\mathbb{T}) \right) \asymp \begin{cases} N^{-(r+\frac{\rho-1}{2})} \sqrt{1 + \frac{1}{N} \ln \frac{1}{\delta}}, & 1 \le q < 2, \\ N^{-(r+\frac{\rho-1}{2})}(1 + N^{-\frac{1}{q}} \sqrt{\ln \frac{1}{\delta}}), & 2 \le q < \infty. \end{cases}$$

According to Theorem 2.1 and Theorem 4.1, we can easily obtain the lower bound of Theorem 4.2. Now, we just need to estimate the upper bound of Theorem 4.2 by the discretization method. At first, we introduce some notations and results.

For natural number k, set

$$S_k = \left\{ n \in \mathbb{Z}_0 : 2^{k-1} \le |n| < 2^k, k \in N \right\}, \qquad F_k = \text{span}\left\{ e^{\text{int}} : n \in S_k \right\}.$$

It is obvious that

$$|S_k| = 2^k, \qquad \dim F_k = |S_k| = 2^k, \tag{4}$$

where $|A|$ denotes the cardinality of A.

For arbitrary natural number k and $x = \sum_{n \in \mathbb{Z}_0} c_n e^{\text{int}}$, denote

$$F_k = \text{span}\{ e_n \mid n \in S_k \}$$

and

$$\Delta_k x(t) = \sum_{n \in S_k} c_n e^{\text{int}}.$$

Lemma 4.1 ([13]) *Let $r \in \mathbb{R}$, $1 < q < \infty$ and $x \in F_k$. Then*

$$2^{rk} \|x\|_{L_q} \asymp \|x^{(r)}\|_{L_q}, \qquad \|x\|_{L_q} \asymp 2^{-\frac{k}{q}} \left\| x(t_j)_{j=1}^{2^k} \right\|_{l_q^{|S_k|}},$$

where $t_j = \frac{2\pi j}{2^k}$, $j = 1, 2, \ldots, 2^k$.

For arbitrary $x \in F_k$, by Lemma 4.1, we have

$$\|x\|_{L_q} \asymp 2^{-rk} \|x^{(r)}\|_{L_q} \asymp 2^{-rk-\frac{k}{q}} \left\| x^{(r)}(t_j)_{j=1}^{2^k} \right\|_{l_q^{|S_k|}}. \tag{5}$$

In order to establish the discrete theorem to estimate the upper bound, we consider the polynomial in the space F_k

$$\varphi_{k,j}(t) = \sum_{n \in S_k} e^{in(t-t_j)}, \quad j \in 1, 2, \ldots, 2^k.$$

We have

$$(D^r x)(t_j) = \langle D^r x, \varphi_{k,j} \rangle, \quad x \in F_k.$$

Plugging this into Eq. (5), we obtain

$$\|x\|_{L_q} \asymp 2^{-rk-\frac{k}{q}} \left\| \left\{ \langle D^r x, \varphi_{k,j} \rangle \right\}_{j=1}^{m_k} \right\|_{l_q^{|S_k|}}. \tag{6}$$

For any $k \in \mathbb{N}$, we consider the following mapping:

$$I_k: \quad F_k \to l_q^{|S_k|}, \qquad x \mapsto \left\{ \langle D^r x, \varphi_{k,j} \rangle \right\}_{j=1}^{2^k}.$$

By (6), I_k is linear isomorph from the space F_k into the space $l_q^{|S_k|}$.

For any $k \in \mathbb{N}$, let

$$\sigma_{k,j} =: \langle C_\mu \varphi_{k,j}(t), \varphi_{k,j}(t) \rangle = \sum_{n \in \mathbb{Z}_0} |n|^{-\rho} |\langle \varphi_{k,j}(t), e^{\text{int}} \rangle| = \sum_{n \in S_k} |n|^{-\rho} \asymp 2^{-k(\rho-1)}, \quad j = 1, \ldots, 2^k,$$

and

$$\sigma := \sigma_{k,j}, \quad j = 1, \ldots, 2^k.$$

Then there is $c' > 0$ such that $\sigma = c' 2^{-k(\rho-1)}$.

Theorem 4.3 *Suppose $1 < q < \infty$, $r > \frac{1}{2}$, $\delta \in (0, \frac{1}{2}]$, $N \in \mathbb{N}$. Let $\{N_k\}$ and $\{\delta_k\}$ be nonnegative integer sequence and nonnegative real sequence, respectively, in which $\sum_{k \in \mathbb{Z}} N_k \leq N$ and $\sum_{k \in \mathbb{Z}} \delta_k \leq \delta$. Then*

$$d_\delta^N\left(W_2^r(\mathbb{T}), \mu, L_q(\mathbb{T}) \right) \ll \sum_{k \in Z} 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{q}} d_{\delta_k}^{N_k}\left(l_2^{|S_k|}, \gamma, l_q^{|S_k|} \right).$$

Proof For any $k \in \mathbb{N}$. From Theorem 3.3, there is a positive constant c'_q such that

$$d_{\delta_k}^{N_k}\left(l_2^{|S_k|}, \gamma, l_q^{|S_k|}\right) = \begin{cases} c'_q |S_k|^{\frac{1}{q}-\frac{1}{2}} \sqrt{|S_k| + \ln\frac{1}{\delta_k}}, & 1 \le q < 2, \\ c'_q\left(|S_k|^{\frac{1}{q}} + \sqrt{\ln\frac{1}{\delta_k}}\right), & 2 \le q < \infty. \end{cases}$$

Let

$$Q_k = \left\{ y \in l_q^{|S_k|} : \|y\|_{l_q^{|S_k|}} > c'' c_q'^{-1} d_{\delta_k}^{N_k} \right\},$$

where $d_{\delta_k}^{N_k} =: d_{\delta_k}^{N_k}(l_2^{|S_k|}, \gamma, l_q^{|S_k|})$, $c'' = c_0$, if $1 \le q \le 2$, and $c'' = c_q$, if $2 \le q \le \infty$. For c_0 and c_q see Lemma 3.1 and Lemma 3.2, respectively. By

$$\|x\|_{l_q^{|S_k|}} \le |S_k|^{\frac{1}{q}-\frac{1}{2}} \|x\|_{l_2^{|S_k|}}, \quad 1 \le q \le 2,$$

and Theorem 3.3, we can conclude that if $y \in Q_k$, then

$$y \in \left\{ x \in \mathbb{R}^{|S_k|} : \|x\|_{l_2^{|S_k|}} > c_0\left(\sqrt{|S_k|} + \sqrt{\ln\frac{1}{\delta_k}}\right) \right\}, \quad 1 \le q \le 2,$$

$$y \in \left\{ x \in \mathbb{R}^{|S_k|} : \|x\|_{l_q^{|S_k|}} > c_q\left(|S_k|^{\frac{1}{q}} + \sqrt{\ln\frac{1}{\delta_k}}\right) \right\}, \quad 2 \le q < \infty.$$

From Lemma 3.1 and Lemma 3.2, we have

$$\gamma(Q_k) \le \delta_k.$$

It is clear that Q_k satisfies the condition (2) in Definition 2.4.

Let L^k be a subspace in $l_2^{|S_k|}$ with codimension at most N_k. Then

$$\|y\|_{l_q^{|S_k|}} \le c'' c_q'^{-1} d_{\delta_k}^{N_k}, \quad y \in L^k \cap \left(l_2^{|S_k|} \setminus Q_k\right).$$

For any $x \in W_2^r(\mathbb{T})$, by (5), there is a constant $c''' > 0$ such that

$$\|\Delta_k x\|_{L_q} \le c''' 2^{-rk-\frac{k}{q}} \left\| \left\{ \langle D^r x, \varphi_{k,j} \rangle \right\}_{j=1}^{2^k} \right\|_{l_q^{|S_k|}}. \tag{7}$$

Consider the set of $W_2^r(\mathbb{T})$

$$G_k = \left\{ x \in W_2^r(\mathbb{T}) : \|\Delta_k x\|_{L_q} > c'' c_q'^{-1} c''' 2^{-rk-\frac{k}{q}} \sigma^{\frac{1}{2}} d_{\delta_k}^{N_k} \right\}.$$

Then

$$\begin{aligned} \mu(G_k) &\le \mu\left(\left\{ x \in W_2^r(\mathbb{T}) : \left\| \left\{ \langle D^r x, \varphi_{k,j} \rangle \right\}_{j=1}^{|S_k|} \right\|_{l_q^{|S_k|}} > c'' c_q'^{-1} \sigma^{\frac{1}{2}} d_{\delta_k}^{N_k} \right\}\right) \\ &= \gamma\left\{ y \in \mathbb{R}^{|S_k|} : \left\| \sigma^{\frac{1}{2}} y \right\|_{l_q^{|S_k|}} > c'' c_q'^{-1} \sigma^{\frac{1}{2}} d_{\delta_k}^{N_k} \right\} \\ &= \gamma\left\{ y \in \mathbb{R}^{|S_k|} : \|y\|_{l_q^{|S_k|}} > c'' c_q'^{-1} d_{\delta_k}^{N_k} \right\} \end{aligned}$$

$$= \gamma(Q_k)$$

$$\leq \delta_k.$$

It is clear that the subspace $F_k := D^{-r}I_k^{-1}L^k$ has codimension at most N_k in $W_2^r(\mathbb{T})$ and

$$\|\Delta_k x\|_{L_q} \ll 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{q}} d_{\delta_k}^{N_k}\left(l_2^{|S_k|}, \gamma, l_q^{|S_k|}\right), \quad x \in \left(W_2^r(\mathbb{T}) \setminus G_k\right) \cap F_k. \tag{8}$$

Consider the set $G = \bigcup_{k=1}^{\infty} G_k$ and the subspace $F^N = \sum_k F^k \subset W_2^r(\mathbb{T})$, where the sum is a direct sum. We obtain

$$\mu(G) \leq \sum_k \mu(G_k) \leq \sum_k \delta_k \leq \delta, \qquad \mathrm{codim}\, F^N \leq \sum_k N_k \leq N.$$

By the definition of the Gel'fand-(N, δ)-width and (8), we have

$$\begin{aligned}
d_\delta^N\left(W_2^r(\mathbb{T}), \mu, L_q(\mathbb{T})\right) &\leq \sup_{x \in (W_2^r(T) \setminus G) \cap F^N} \|x\|_{L_q} \\
&\leq \sup_{x \in (W_2^r(\mathbb{T}) \setminus G) \cap F^N} \sum_k \|\Delta x\|_{L_q} \\
&\leq \sum_k \sup_{x \in (W_2^r(\mathbb{T}) \setminus G_k) \cap F_k} \|\Delta x\|_{L_q} \\
&\ll \sum_k 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{q}} d_{\delta_k}^{N_k}\left(l_2^{|S_k|}, \gamma, l_q^{|S_k|}\right).
\end{aligned} \qquad \square$$

In order to estimate the upper bound of Theorem 4.3, we also need the following lemma.

Lemma 4.2 ([13]) *Let N be natural number set, $k' = [\log_2 N]$. For arbitrary $k \in \mathbb{N}$, let*

$$N_k = \begin{cases} |S_k|, & k \leq k', \\ [N2^{k'-k}], & k > k', \end{cases}$$

$$\delta_k = \begin{cases} 0, & k \leq k', \\ \delta 2^{k'-k}, & k > k'. \end{cases}$$

Then $\{N_k\}$, $\{\delta_k\}$ satisfy the condition of Theorem 4.3, where $[x]$ denotes the largest integer no larger than x.

Now we prove Theorem 4.2.

Proof According to Theorem 2.1 and Theorem 4.1, we can easily obtain the lower bound of Theorem 4.2. We now just need to prove the upper bound.

(1) For $1 \leq q \leq 2$, by the definition of $\|\cdot\|_{L_q}$,

$$d_\delta^N\left(W_2^r(\mathbb{T}), \mu, L_q(\mathbb{T})\right) \leq d_\delta^N\left(W_2^r(\mathbb{T}), \mu, L_2(\mathbb{T})\right).$$

Considering $d_0^j(l_2^j, \gamma, l_2^j) = 0$, by Theorem 4.3, Lemma 4.2 and Theorem 3.3, we have

$$d_\delta^N\left(W_2^r(\mathbb{T}), \mu, L_q(\mathbb{T})\right) \leq d_\delta^N\left(W_2^r(\mathbb{T}), \mu, L_2(\mathbb{T})\right)$$

$$\ll \sum_k 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{2}} d_{\delta_k}^{N_k}\left(l_2^{|S_k|},\gamma,l_2^{|S_k|}\right)$$

$$\ll \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{2}} d_{\delta_k}^{N_k}\left(l_2^{|S_k|},\gamma,l_2^{|S_k|}\right)$$

$$\ll \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{2}} \sqrt{|S_k|+\ln\frac{1}{\delta_k}}$$

$$\ll \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{2}}\sqrt{2^k} + \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{2}}\sqrt{\ln\frac{2^{k-k'}}{\delta}}$$

$$\ll \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k} + \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{2}}\sqrt{k-k'}\sqrt{\ln\frac{1}{\delta}}$$

$$\ll 2^{-(r+\frac{\rho-1}{2})k'} + 2^{-(r+\frac{\rho-1}{2})k'-\frac{k'}{2}}\sqrt{\ln\frac{1}{\delta}}$$

$$\ll 2^{-(r+\frac{\rho-1}{2})k'}\left(1+2^{-\frac{k'}{2}}\sqrt{\ln\frac{1}{\delta}}\right)$$

$$\ll N^{-(r+\frac{\rho-1}{2})}\sqrt{1+\frac{1}{N}\ln\frac{1}{\delta}}.$$

(2) For $2<q<\infty$, by Theorem 4.3, Lemma 4.2 and Theorem 3.3, we have

$$d_\delta^N\left(W_2^r(\mathbb{T}),\mu,L_q(\mathbb{T})\right) \ll \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{q}} d_{\delta_k}^{N_k}\left(l_2^{|S_k|},\gamma,l_q^{|S_k|}\right)$$

$$\ll \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{q}}\left(|S_k|^{\frac{1}{q}}+\sqrt{\ln\frac{1}{\delta_k}}\right)$$

$$\ll \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{q}}2^{\frac{k}{q}} + \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{q}}\sqrt{\ln\frac{2^{k-k'}}{\delta}}$$

$$\ll \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k} + \sum_{k>k'} 2^{-(r+\frac{\rho-1}{2})k-\frac{k}{q}}\sqrt{k-k'}\sqrt{\ln\frac{1}{\delta}}$$

$$\ll 2^{-(r+\frac{\rho-1}{2})k'} + 2^{-(r+\frac{\rho-1}{2})k'-\frac{k'}{q}}\sqrt{\ln\frac{1}{\delta}}$$

$$\ll 2^{-(r+\frac{\rho-1}{2})k'}\left(1+2^{-\frac{k'}{q}}\sqrt{\ln\frac{1}{\delta}}\right)$$

$$\ll N^{-(r+\frac{\rho-1}{2})}\left(1+N^{-\frac{1}{q}}\sqrt{\ln\frac{1}{\delta}}\right). \qquad \square$$

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