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Some inequalities for the multilinear singular integrals with Lipschitz functions on weighted Morrey spaces

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Abstract
The aim of this paper is to prove the boundedness of the oscillation and variation operators for the multilinear singular integrals with Lipschitz functions on weighted Morrey spaces.

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1 Introduction

We first say that there exists a continuous function $K(x,y)$ defined on $\Omega = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x \neq y\}$ and $C > 0$ if K admits the following representation:

$$\left|K(x,y)\right| \leq \frac{C}{|x-y|}, \quad \forall (x,y) \in \Omega, \tag{1}$$

and for all $x, x_0, y \in \mathbb{R}$ with $|x-y| > 2|x-x_0|$

$$\begin{aligned} &\left|K(x,y) - K(x_0,y)\right| + \left|K(y,x) - K(y,x_0)\right| \\ &\quad \leq \frac{C}{|x-y|}\left(\frac{|x-x_0|}{|x-y|}\right)^{\beta}, \end{aligned} \tag{2}$$

where $1 > \beta > 0$. Then K is said to be a Calderón–Zygmund standard kernel.

Suppose that K satisfies (1) and (2). Then Zhang and Wu [12] considered the family of operators $T := \{T_\epsilon\}_{\epsilon>0}$ and a related the family of commutator operators $T_b := \{T_{\epsilon,b}\}_{\epsilon>0}$ generated by T_ϵ and b which are given by

$$T_\epsilon f(x) = \int_{|x-y|>\epsilon} K(x,y) f(y)\, dy \tag{3}$$

and

$$T_{\epsilon,b}f(x) = \int_{|x-y|>\epsilon} \big(b(x) - b(y)\big)K(x,y)f(y)\,dy. \tag{4}$$

In this sense, following [12], the definition of the oscillation operator of T is given by

$$\mathcal{O}(Tf)(x) := \left(\sum_{i=1}^{\infty} \sup_{t_{i+1}\le\epsilon_{i+1}<\epsilon_i\le t_i} \left|T_{\epsilon_{i+1}}f(x) - T_{\epsilon_i}f(x)\right|^2\right)^{\frac{1}{2}},$$

where $\{t_i\}$ is a decreasing fixed sequence of positive numbers converging to 0 and a related ρ-variation operator is defined by

$$\mathcal{V}_\rho(Tf)(x) := \sup_{\epsilon_i\searrow 0}\left(\sum_{i=1}^{\infty}\left|T_{\epsilon_{i+1}}f(x) - T_{\epsilon_i}f(x)\right|^\rho\right)^{\frac{1}{\rho}}, \quad \rho > 2,$$

where the supremum is taken over all sequences of real number $\{\epsilon_i\}$ decreasing to 0. We also take into account the operator

$$\mathcal{O}'(Tf)(x) := \left(\sum_{i=1}^{\infty} \sup_{t_{i+1}<\eta_i<t_i} \left|T_{t_{i+1}}f(x) - T_{\eta_i}f(x)\right|^2\right)^{\frac{1}{2}}.$$

On the other hand, it is obvious that

$$\mathcal{O}'(Tf) \approx \mathcal{O}(Tf).$$

That is,

$$\mathcal{O}'(Tf) \le \mathcal{O}(Tf) \le 2\mathcal{O}'(Tf).$$

Recently, Campbell et al. in [1] proved the oscillation and variation inequalities for the Hilbert transform in L^p $(1 < p < \infty)$ and then following [1], we denote by E the mixed norm Banach space of the two-variable function h defined on $\mathbb{R}\times\mathbb{N}$ such that

$$\|h\|_E \equiv \left(\sum_i \left(\sup_s \left|h(s,i)\right|\right)^2\right)^{1/2} < \infty.$$

Given $T := \{T_\epsilon\}_{\epsilon>0}$ is a family operators such that $\lim_{\epsilon\to 0} T_\epsilon f(x) = Tf(x)$ exists almost everywhere for certain class of functions f, where T_ϵ defined as (3). For a fixed decreasing sequence $\{t_i\}$ with $t_i \searrow 0$, let $J_i = (t_{i+1}, t_i]$ and define the E-valued operator $U(T) : f \to U(T)f$ given by

$$U(T)f(x) = \left\{T_{t_{i+1}}f(x) - T_s f(x)\right\}_{s\in J_i, i\in\mathbb{N}} = \left\{\int_{\{t_{i+1}<|x-y|<s\}} K(x,y)f(y)\,dy\right\}_{s\in J_i, i\in\mathbb{N}}.$$

Then

$$\mathcal{O}'(Tf)(x) = \|U(T)f(x)\|_E = \|\{T_{t_{i+1}}f(x) - T_s f(x)\}_{s\in J_i, i\in\mathbb{N}}\|_E$$
$$= \left\| \left\{ \int_{\{t_{i+1}<|x-y|<s\}} K(x,y)f(y)\,dy \right\}_{s\in J_i, i\in\mathbb{N}} \right\|_E.$$

Let $\Phi = \{\beta : \beta = \{\epsilon_i\}, \epsilon_i \in \mathbb{R}, \epsilon_i \searrow 0\}$. We denote by F_ρ the mixed norm space of two-variable functions $g(i,\beta)$ such that

$$\|g\|_{F_\rho} \equiv \sup_\beta \left(\sum_i |g(i,\beta)|^\rho \right)^{1/\rho}.$$

We also take into account the F_ρ-valued operator $V(T) : f \to V(T)f$ such that

$$V(T)f(x) = \{T_{\epsilon_{i+1}}f(x) - T_{\epsilon_i}f(x)\}_{\beta=\{\epsilon_i\}\in\Phi}.$$

Thus,

$$V_\rho(T)f(x) = \|V(T)f(x)\|_{F_\rho}.$$

Given m is a positive integer, and b is a function on $\mathbb{R}$. Let $R_{m+1}(b;x,y)$ be the $m+1$th order Taylor series remainder of b at x about y, that is,

$$R_{m+1}(b;x,y) = b(x) - \sum_{\gamma\le m} \frac{1}{\gamma!} b^{(\gamma)}(y)(x-y)^\gamma.$$

In this paper, we consider the family of operators $T^b := \{T_\epsilon^b\}_{\epsilon>0}$ given by [6], where T_ϵ^b are the multilinear singular integral operators of T_ϵ as follows:

$$T_\epsilon^b f(x) = \int_{|x-y|>\epsilon} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y)f(y)\,dy. \tag{5}$$

Thus, if $m = 0$, then T_ϵ^b is just the commutator of T_ϵ and b, which is given by (4). But, if $m > 0$, then T_ϵ^b are non-trivial generation of the commutators.

The theory of multilinear analysis was received extensive studies in the last 3 decades (see [2, 5] for example). Hu and Wang [6] proved that the weighted (L^p, L^q)-boundedness of the oscillation and variation operators for T^b when the mth derivative of b belongs to the homogeneous Lipschitz space $\dot{\Lambda}_\beta$. In this sense, we recall the definition of homogeneous Lipschitz space $\dot{\Lambda}_\beta$ as follows.

Definition 1 (Homogeneous Lipschitz space) Let $0 < \beta \le 1$. The homogeneous Lipschitz space $\dot{\Lambda}_\beta$ is defined by

$$\dot{\Lambda}_\beta(\mathbb{R}) = \left\{ b : \|b\|_{\dot{\Lambda}_\beta} = \sup_{x,h\in\mathbb{R}, h\neq 0} \frac{|b(x+h) - b(x)|}{|h|^\beta} < \infty \right\}.$$

Obviously, if $\beta > 1$, then $\dot{\Lambda}_\beta(\mathbb{R})$ only includes constant. So we restrict $0 < \beta \le 1$.

Now, we recall the definitions of basic spaces such as Morrey, weighted Lebesgue, weighted Morrey spaces and consider the relationship between these spaces.

Besides the Lebesgue space $L^q(\mathbb{R})$, the Morrey space $M_p^q(\mathbb{R})$ is another important function space with definition as follows.

Definition 2 (Morrey space) For $1 \leq p \leq q < \infty$, the Morrey space $M_p^q(\mathbb{R})$ is the collection of all measurable functions f whose Morrey space norm is

$$\|f\|_{M_p^q(\mathbb{R})} = \sup_{\substack{I\subset\mathbb{R} \\ I:Interval}} \frac{1}{|I|^{\frac{1}{p}-\frac{1}{q}}} \|f\chi_I\|_{L_p(\mathbb{R})} < \infty.$$

Remark 1 If $p = q$, then

$$\|f\|_{M_q^q(\mathbb{R})} = \|f\|_{L^q(\mathbb{R})}.$$

If $q < p$, then $M_p^q(\mathbb{R})$ is strictly larger than $L^q(\mathbb{R})$. For example, $f(x) := |x|^{-\frac{1}{q}} \in M_p^q(\mathbb{R})$ but $f(x) := |x|^{-\frac{1}{q}} \notin L^q(\mathbb{R})$.

On the other hand, for a given weight function w and any interval I, we also denote the Lebesgue measure of I by $|I|$ and set weighted measure

$$w(I) = \int_I w(x)\,dx.$$

For $0 < p < \infty$, the weighted Lebesgue space $L_p(w) \equiv L_p(\mathbb{R}, w)$ is defined by the norm

$$\|f\|_{L_p(w)} = \left(\int_{\mathbb{R}} |f(x)|^p w(x)\,dx\right)^{\frac{1}{p}} < \infty.$$

A weight w is said to belong to the Muckenhoupt class A_p for $1 < p < \infty$ such that

$$\begin{aligned}[][w]_{A_p} &:= \sup_I [w]_{A_p(I)} \\ &= \sup_I \left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{1-p'}\,dx\right)^{p-1} < \infty, \end{aligned} \tag{6}$$

where $p' = \frac{p}{p-1}$. The condition (6) is called the A_p-condition, and the weights which satisfy it are called A_p-weights. The expression $[w]_{A_p}$ is also called characteristic constant of w.

Here and after, A_p denotes the Muckenhoupt classes (see [5, 7]). The A_p class of weights characterizes the $L_p(w)$ boundedness of the maximal function as Muckenhoupt [9] established in the 1970s. Subsequent work of Muckenhoupt [9] himself Muckenhoupt and Wheeden [10, 11], Coifman and Fefferman [3] was devoted to exploring the connection of the A_p class with weighted estimates for singular integrals. However, it was not until the 2000s that the quantitative dependence on the so called A_p constant, namely $[w]_{A_p}$, became a trending topic.

When $p = 1$, $w \in A_1$ if there exists $C > 1$ such that, for almost every x,

$$Mw(x)\,dx \leq Cw(x) \tag{7}$$

and the infimum of C satisfying the inequality (7) is denoted by $[w]_{A_1}$, where M is the classical Hardy–Littlewood maximal operator.

When $p = \infty$, we define $A_\infty(\mathbb{R}) = \bigcup_{1\leq p<\infty} A_p(\mathbb{R})$. That is, the A_∞ constant is given by

$$\begin{aligned} [w]_{A_\infty} &:= \sup_I [w]_{A_\infty(I)} \\ &= \sup_I \int_I M(\chi_I w)(x)\,dx, \end{aligned}$$

where we utilize the notation $M(\chi_I w)$ to denote the Hardy–Littlewood maximal function of a function $\chi_I w$ by

$$M(\chi_I w)(x) := \sup_I \frac{1}{|I|}\int_I \bigl|\chi_I w(x)\bigr|\,dx.$$

A weight function w belongs to $A_{p,q}$ (Muckenhoupt–Wheeden class) for $1 < p < q < \infty$ if

$$\begin{aligned} [w]_{A_{p,q}} &:= \sup_I [w]_{A_{p,q}(I)} \\ &= \sup_I \biggl(\frac{1}{|I|}\int_I w(x)^q\,dx\biggr)^{\frac{1}{q}} \biggl(\frac{1}{|I|}\int_I w(x)^{-p'}\,dx\biggr)^{\frac{1}{p'}} < \infty. \end{aligned} \tag{8}$$

From the definition of $A_{p,q}$, we know that $w(x) \in A_{p,q}(\mathbb{R})$ implies $w(x)^q \in A_q(\mathbb{R})$ and $w(x)^p \in A_p(\mathbb{R})$.

Now, we begin with some lemmas. These lemmas are very necessary for the proof of the main result.

Lemma 1 ([4]) *If $w \in A_p$, $p \geq 1$, then there exists a constant $C > 0$ such that*

$$w(2I) \leq Cw(I).$$

for any interval I.

More precisely, for all $\lambda > 1$ we have

$$w(\lambda I) \leq C\lambda^p w(I),$$

where C is a constant independent of I or λ and $w(I) = \int_I w(x)\,dx$.

Lemma 2 ([2]) *Let b be a function on $\mathbb{R}$ and $b^{(m)} \in L_u(\mathbb{R})$ with $m \in \mathbb{N}$ for any $u > 1$. Then*

$$\bigl|R_m(b;x,y)\bigr| \leq C|x-y|^m \biggl(\frac{1}{|I(x,y)|}\int_{I(x,y)} \bigl|b^{(m)}(z)\bigr|^u dz\biggr)^{\frac{1}{u}}, \quad C > 0,$$

where $I(x,y)$ is the interval $(x-5|x-y|, x+5|x-y|)$.

Lemma 3 ([6]) *Let $K(x,y)$ satisfies (1) and (2), $\rho > 2$, and $T := \{T_\epsilon\}_{\epsilon>0}$ and $T^b := \{T^b_\epsilon\}_{\epsilon>0}$ be given by (3) and (5), respectively. If $\mathcal{O}(T)$ and $\mathcal{V}_\rho(T)$ are bounded on $L_{p_0}(\mathbb{R},dx)$ for some*

$1 < p_0 < \infty$, *and* $b^{(m)} \in \dot{\Lambda}_\beta$ *with* $m \in \mathbb{N}$ *for* $0 < \beta < 1$, *then*

$$\left\| \mathcal{O}'(T^b) \right\|_{L_q(w^q)} \le \left\| \mathcal{O}(T^b) \right\|_{L_q(w^q)} \le C\|b\|_{\dot{\Lambda}_\beta} \|f\|_{L_p(w^p)}, \quad C > 0, \tag{9}$$

and

$$\left\| \mathcal{V}_\rho(T^b) \right\|_{L_q(w^q)} \le C\|b\|_{\dot{\Lambda}_\beta} \|f\|_{L_p(w^p)}, \quad C > 0$$

for any $1 < p < \frac{1}{\beta}$ *with* $\frac{1}{q} = \frac{1}{p} - \beta$ *and* $w \in A_{p,q}$.

Next, in 2009, the weighted Morrey space $L_{p,\kappa}(w)$ was defined by Komori and Shirai [7] as follows.

Definition 3 (Weighted Morrey space) Let $1 \le p < \infty$, $0 < \kappa < 1$ and w be a weight function. Then the weighted Morrey space $L_{p,\kappa}(w) \equiv L_{p,\kappa}(\mathbb{R}, w)$ is defined by

$$L_{p,\kappa}(w) \equiv L_{p,\kappa}(\mathbb{R}, w) = \left\{ f \in L_{p,w}^{\mathrm{loc}}(\mathbb{R}) : \|f\|_{L_{p,\kappa}(w)} = \sup_I w(I)^{-\frac{\kappa}{p}} \|f\|_{L_{p,w}(I)} < \infty \right\}.$$

Remark 2 If $\kappa = 0$, then

$$\|f\|_{L_{p,0}(w)} = \|f\|_{L_p(w)}.$$

When $w \equiv 1$ and $\kappa = 1 - \frac{p}{q}$ with $1 < p \le q < \infty$, then

$$\|f\|_{L_{p,1-\frac{p}{q}}(1)} = \|f\|_{M_p^q(\mathbb{R})}.$$

Finally, we recall the definition of the weighted Morrey space with two weights as follows.

Definition 4 (Weighted Morrey space with two weights) Let $1 \le p < \infty$ and $0 < \kappa < 1$. Then for two weights u and v, the weighted Morrey space $L_{p,\kappa}(u, v) \equiv L_{p,\kappa}(\mathbb{R}, u, v)$ is defined by

$$L_{p,\kappa}(u, v) \equiv L_{p,\kappa}(\mathbb{R}, u, v) = \left\{ f \in L_{p,u}^{\mathrm{loc}}(\mathbb{R}) : \|f\|_{L_{p,\kappa}(w)} = \sup_I v(I)^{-\frac{\kappa}{p}} \|f\|_{L_{p,u}(I)} < \infty \right\}.$$

It is obvious that

$$L_{p,\kappa}(w, w) \equiv L_{p,\kappa}(w).$$

In 2016, Zhang and Wu [12] gave the boundedness of the oscillation and variation operators for Calderón–Zygmund singular integrals and the corresponding commutators on the weighted Morrey spaces. In 2017, Hu and Wang [6] established the weighted (L^p, L^q)-inequalities of the variation and oscillation operators for the multilinear Calderón–Zygmund singular integral with a Lipschitz function in $\mathbb{R}$. Inspired of these results [6, 12], we investigate the boundedness of the oscillation and variation operators for the family of the multilinear singular integral defined by (5) on weighted Morrey spaces when the mth derivative of b belongs to the homogeneous Lipschitz space $\dot{\Lambda}_\beta$ in this work.

Throughout this paper, C always means a positive constant independent of the main parameters involved, and may change from one occurrence to another. We also use the notation $F \lesssim G$ to mean $F \leq CG$ for an appropriate constant $C > 0$, and $F \approx G$ to mean $F \lesssim G$ and $G \lesssim F$.

2 Main result

We now formulate our main result as follows.

Theorem 1 *Let $K(x,y)$ satisfies (1) and (2), $\rho > 2$, and $T := \{T_\epsilon\}_{\epsilon>0}$ and $T^b := \{T_\epsilon^b\}_{\epsilon>0}$ be given by (3) and (5), respectively. If $\mathcal{O}(T)$ and $\mathcal{V}_\rho(T)$ are bounded on $L_{p_0}(\mathbb{R},dx)$ for some $1 < p_0 < \infty$, and $b^{(m)} \in \dot{\Lambda}_\beta$ with $m \in \mathbb{N}$ for $0 < \beta < 1$, then $\mathcal{O}(T^b)$ and $\mathcal{V}_\rho(T^b)$ are bounded from $L_{p,\kappa}(w^p,w^q)$ to $L_{p,\frac{\kappa q}{p}}(w^q)$ for any $1 < p < \frac{1}{\beta}$, $\frac{1}{q} = \frac{1}{p} - \beta$, $0 < \kappa < \frac{p}{q}$ and $w \in A_{p,q}$.*

Corollary 1 ([12]) *Let $K(x,y)$ satisfies (1) and (2), $\rho > 2$, and $T := \{T_\epsilon\}_{\epsilon>0}$ and $T_b := \{T_{\epsilon,b}\}_{\epsilon>0}$ be given by (3) and (4), respectively. If $\mathcal{O}(T)$ and $\mathcal{V}_\rho(T)$ are bounded on $L_{p_0}(\mathbb{R},dx)$ for some $1 < p_0 < \infty$, and $b \in \dot{\Lambda}_\beta$ for $0 < \beta < 1$, then $\mathcal{O}(T_b)$ and $\mathcal{V}_\rho(T_b)$ are bounded from $L_{p,\kappa}(w^p,w^q)$ to $L_{p,\frac{\kappa q}{p}}(w^q)$ for any $1 < p < \frac{1}{\beta}$, $\frac{1}{q} = \frac{1}{p} - \beta$, $0 < \kappa < \frac{p}{q}$ and $w \in A_{p,q}$.*

2.1 The proof of Theorem 1

Proof We consider the proof related to $\mathcal{O}(T^b)$ firstly. Fix an interval $I = (x_0 - l, x_0 + l)$, and we write as $f = f_1 + f_2$, where $f_1 = f\chi_{2I}$, χ_{2I} denotes the characteristic function of $2I$. Thus, it is sufficient to show that the conclusion

$$\begin{aligned} \|\mathcal{O}'(T^b f)(x)\|_{L_{p,\frac{\kappa q}{p}}(w^q)} &\leq \|\mathcal{O}'(T^b f_1)(x)\|_{L_{p,\frac{\kappa q}{p}}(w^q)} + \|\mathcal{O}'(T^b f_1)(x)\|_{L_{p,\frac{\kappa q}{p}}(w^q)} \\ &\lesssim \|b\|_{\dot{\Lambda}_\beta} \|f\|_{L_{p,\kappa}(w^p,w^q)} \end{aligned}$$

holds for every interval $I \subset \mathbb{R}$. Then

$$\begin{aligned} &\left(\int_I \left|\mathcal{O}'(T^b f)(x)\right|^q w^q(x)\,dx\right)^{\frac{1}{q}} \\ &\quad\leq \left(\int_I \left|\mathcal{O}'(T^b f_1)(x)\right|^q w^q(x)\,dx\right)^{\frac{1}{q}} + \left(\int_I \left|\mathcal{O}'(T^b f_2)(x)\right|^q w^q(x)\,dx\right)^{\frac{1}{q}} \\ &\quad=: F_1 + F_2. \end{aligned}$$

First, we use (9) to estimate F_1, and we obtain

$$\begin{aligned} F_1 &= \left(\int_I \left|\mathcal{O}'(T^b f_1)(x)\right|^q w^q(x)\,dx\right)^{\frac{1}{q}} \lesssim \|b\|_{\dot{\Lambda}_\beta} \|f_1\|_{L_p(w^p)} \\ &= \|b\|_{\dot{\Lambda}_\beta} \left(\frac{1}{w^q(2I)^\kappa} \int_{2I} \left|f(x)\right|^p w^p(x)\,dx\right)^{\frac{1}{p}} w^q(2I)^{\frac{\kappa}{p}} \\ &\lesssim \|b\|_{\dot{\Lambda}_\beta} \|f\|_{L_{p,\kappa}(w^p,w^q)}^p w^q(I)^{\frac{\kappa}{p}}. \end{aligned}$$

Thus,

$$\left\|\mathcal{O}'\left(T^b f_1\right)(x)\right\|_{L_{p,\frac{\kappa q}{p}}(w^q)} \lesssim \|b\|_{\dot{\Lambda}_\beta} \|f\|_{L_{p,\kappa}(w^p,w^q)}. \tag{10}$$

Second, for $x \in I$, $k = 1,2,\dots,m \in \mathbb{N}$, let $A_k = \{y : 2^k l \le |y-x| < 2^{k+1} l\}$, $B_k = \{y : |y-x| < 2^{k+1} l\}$, and

$$b_k(z) = b(z) - \frac{1}{m!}\left(b^{(m)}\right)_{B_k} z^m.$$

By [2], for any $y \in A_k$, it is obvious that

$$R_{m+1}(b;x,y) = R_{m+1}(b_k;x,y).$$

Moreover, since $b \in \dot{\Lambda}_\beta$, for $y \in A_k$, we get

$$\begin{aligned}\left|b^{(m)}(y) - \left(b^{(m)}\right)_{B_k}\right| &\le \frac{1}{|B_k|}\int_{B_k} \left|b^{(m)}(y) - b^{(m)}(z)\right| dz \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} \left(2^k l\right)^\beta. \end{aligned} \tag{11}$$

Hence, by Lemma 2 and (11)

$$\begin{aligned} R_m(b_k;x,y) &\lesssim |x-y|^m \left(\frac{1}{|I(x,y)|}\int_{I(x,y)} \left|b^{(m)}(z)\right|^u dz\right)^{\frac{1}{u}} \\ &\lesssim |x-y|^m \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} \left(2^k l\right)^\beta. \end{aligned}$$

Also, following [12], we have

$$\left\|\{\chi_{\{t_i+1<|x-y|<u\}}\}_{u\in J_i, i\in\mathbb{N}}\right\|_A \le 1.$$

Thus, the estimate of F_2 can be obtained as follows:

$$\begin{aligned} \left|\mathcal{O}'\left(T^b f_2\right)(x)\right| &= \left\|U\left(T^b f_2\right)(x)\right\| \\ &= \left\|\left\{\int_{\{t_i+1<|x-y|<u\}} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f_2(y)\,dy\right\}\right\|_A \\ &\le \int_{\mathbb{R}} \left\|\{\chi_{\{t_i+1<|x-y|<u\}}\}_{u\in J_i, i\in\mathbb{N}}\right\|_A \left|\frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f_2(y)\right| dy \\ &\le \int_{\mathbb{R}} \left|\frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f_2(y)\right| dy \\ &\lesssim \int_{|x-y|>2l} \left|\frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f(y)\right| dy \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{2^k l}\int_{A_k} \left(\left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} \left(2^k l\right)^\beta + \left|b^{(m)}(y) - \left(b^{(m)}\right)_{B_k}\right|\right) \left|f(y)\right| dy \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} \sum_{k=1}^{\infty} \frac{1}{(2^k l)^{1-\beta}} \int_{A_k} \left|f(y)\right| dy \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty}\frac{1}{2^k l}\int_{A_k}\left|b^{(m)}(y)-\left(b^{(m)}\right)_{B_k}\right|\left|f(y)\right|dy\\
&=G_1+G_2.
\end{aligned}
$$

For G_1, since

$$
\left(\int_{A_k} w(y)^{-p'}\,dy\right)^{\frac{1}{p'}} \lesssim w^q(B_k)^{-\frac{1}{q}}|B_k|^{\frac{1}{p'}+\frac{1}{q}}
$$

with $1<p<\frac{1}{\beta}$, $\frac{1}{q}=\frac{1}{p}-\beta$ and using Hölder's inequality, we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{(2^k l)^{1-\beta}}\int_{A_k}\left|f(y)\right|dy\\
&\quad\lesssim\sum_{k=1}^{\infty}\frac{1}{(2^k l)^{1-\beta}}\left(\int_{A_k}\left|f(y)\right|^p w^p(y)\,dy\right)^{\frac{1}{p}}\left(\int_{A_k} w(y)^{-p'}\,dy\right)^{\frac{1}{p'}}\\
&\quad\lesssim\|f\|_{L_{p,\kappa}(w^p,w^q)}\sum_{k=1}^{\infty}\frac{(2^k l)^{\frac{1}{p'}+\frac{1}{q}}}{(2^k l)^{1-\beta}}w^q(B_k)^{\frac{\kappa}{p}-\frac{1}{q}}\\
&\quad\lesssim\|f\|_{L_{p,\kappa}(w^p,w^q)}\sum_{k=1}^{\infty}w^q(B_k)^{\frac{\kappa}{p}-\frac{1}{q}}.
\end{aligned}
\tag{12}
$$

Since $w\in A_{p,q}$, we have $w^q\in A_\infty$. Thus, Lemma 1 implies $w^q(B_k)\le(C)^k w^q(I)$, $C>1$, i.e.,

$$
\sum_{k=1}^{\infty}w^q(B_k)^{\frac{\kappa}{p}-\frac{1}{q}}\lesssim w^q(I)^{\frac{\kappa}{p}-\frac{1}{q}}\sum_{k=1}^{\infty}C^{\frac{\kappa}{p}-\frac{1}{q}}\lesssim w^q(I)^{\frac{\kappa}{p}-\frac{1}{q}}
\tag{13}
$$

with $\frac{\kappa}{p}-\frac{1}{q}<0$. This implies

$$
G_1\lesssim\left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\|f\|_{L_{p,\kappa}(w^p,w^q)}w^q(I)^{\frac{\kappa}{p}-\frac{1}{q}}.
\tag{14}
$$

Let $y\in A_k$. For G_2, by (11), (12) and (13) we get

$$
\begin{aligned}
G_2&\lesssim\left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\sum_{k=1}^{\infty}\frac{1}{(2^{k+1}l)^{1-\beta}}\int_{A_k}\left|f(y)\right|dy\\
&\lesssim\left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\|f\|_{L_{p,\kappa}(w^p,w^q)}w^q(I)^{\frac{\kappa}{p}-\frac{1}{q}}.
\end{aligned}
\tag{15}
$$

Thus, by (14) and (15), we obtain

$$
\begin{aligned}
F_2&=\left(\int_I\left|\mathcal{O}'\left(T^b f_2\right)(x)\right|^q w^q(x)\,dx\right)^{\frac{1}{q}}\\
&\lesssim\left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\|f\|_{L_{p,\kappa}(w^p,w^q)}w^q(I)^{\frac{\kappa}{p}-\frac{1}{q}}w^q(I)^{\frac{1}{q}}\\
&=\left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\|f\|_{L_{p,\kappa}(w^p,w^q)}w^q(I)^{\frac{\kappa}{p}}.
\end{aligned}
$$

Thus,

$$\left\|\mathcal{O}'\left(T^b f_2\right)(x)\right\|_{L_{p,\frac{\kappa q}{p}}(w^q)} \lesssim \|b\|_{\dot{\Lambda}_\beta}\|f\|_{L_{p,\kappa}(w^p,w^q)}. \tag{16}$$

As a result, by (10) and (16), we get

$$\left\|\mathcal{O}'\left(T^b f\right)(x)\right\|_{L_{p,\frac{\kappa q}{p}}(w^q)} \lesssim \|b\|_{\dot{\Lambda}_\beta}\|f\|_{L_{p,\kappa}(w^p,w^q)}.$$

Similarly, $\mathcal{V}_\rho(T^b)$ has the same estimate as above (here we omit the details), thus the inequality

$$\left\|\mathcal{V}_\rho\left(T^b f\right)(x)\right\|_{L_{p,\frac{\kappa q}{p}}(w^q)} \lesssim \|b\|_{\dot{\Lambda}_\beta}\|f\|_{L_{p,\kappa}(w^p,w^q)}$$

is valid.

Therefore, Theorem 1 is completely proved. □

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