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A class of Hilbert-type multiple integral inequalities with the kernel of generalized homogeneous function and its applications

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Abstract

Let $x = (x_1, x_2, \ldots, x_n)$, and let $K(u(x), v(y))$ satisfy $u(rx) = ru(x)$, $v(ry) = rv(y)$, $K(ru, v) = r^{\lambda\lambda_1} K(u, r^{-\frac{\lambda_1}{\lambda_2}} v)$, and $K(u, rv) = r^{\lambda\lambda_2} K(r^{-\frac{\lambda_2}{\lambda_1}} u, v)$. In this paper, we obtain a necessary and sufficient condition and the best constant factor for the Hilbert-type multiple integral inequality with kernel $K(u(x), v(y))$ and discuss its applications in the theory of operators.

MSC: 26D15

Keywords: Generalized homogeneous kernel; Hilbert-type multiple integral inequality; Necessary and sufficient condition; The best constant factor; Bounded operator; Operator norm

1 Preliminary

Let $n \geq 1$, $x = (x_1, x_2, \ldots, x_n)$, $\|x\|_\rho = (x_1^\rho + \cdots + x_n^\rho)^{1/\rho}$, and $\mathbf{R}_+^n = \{x = (x_1, \ldots, x_n) : x_1 > 0, \ldots, x_n > 0\}$.

Define the function space

$$L_{\omega(x)}^p(\mathbf{R}_+^n) = \left\{ f(x) \geq 0 : \|f\|_{p,\omega(x)} = \left(\int_{\mathbf{R}_+^n} f^p(x)\omega(x)\,dx \right)^{\frac{1}{p}} < +\infty \right\}.$$

Definition 1 Let λ, λ_1, and λ_2 be constants, and let $u(x)$, $v(y)$ and $K(u, v)$ satisfy: for all $r > 0$, $u(rx) = ru(x)$, $v(ry) = rv(y)$, and

$$K(ru, v) = r^{\lambda\lambda_1} K\left(u, r^{-\frac{\lambda_1}{\lambda_2}} v\right), \qquad K(u, rv) = r^{\lambda\lambda_2} K\left(r^{-\frac{\lambda_2}{\lambda_1}} u, v\right).$$

Then we call $K(u(x), v(y))$ a generalized homogeneous function with parameters $(\lambda, \lambda_1, \lambda_2)$. Obviously, $K(u(x), v(y))$ is a homogeneous function of order $\lambda\lambda_1$ when $\lambda_1 = \lambda_2$.

If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we call the inequality

$$\int_{\mathbf{R}_+^n} \int_{\mathbf{R}_+^n} K\left(u(x), v(y)\right) f(x) g(y)\,dx\,dy \leq M \|f\|_{p,u^\alpha(x)} \|g\|_{q,v^\beta(y)} \tag{1.1}$$

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the Hilbert-type multiple integral inequality with $f \in L^p_{u^\alpha(x)}(\mathbf{R}^n_+)$ and $g \in L^q_{v^\beta(y)}(\mathbf{R}^n_+)$.

Define the integral operator T with kernel $K(u(x), v(y))$ as follows:

$$T(f)(y) = \int_{\mathbf{R}^n_+} K\big(u(x), v(y)\big) f(x)\,dx, \quad y \in \mathbf{R}^n_+. \tag{1.2}$$

If there exists a constant M such that

$$\big\| T(f)\big\|_{p,\omega_2(y)} \le M\|f\|_{p,\omega_1(x)}, \quad f \in L^p_{\omega_1(x)}\big(\mathbf{R}^n_+\big),$$

then T is called a bounded operator from $L^p_{\omega_1}(\mathbf{R}^n_+)$ to $L^p_{\omega_2}(\mathbf{R}^n_+)$. If T is a bounded operator from $L^p_{\omega_1}(\mathbf{R}^n_+)$ to itself, then we call T a bounded operator in $L^p_{\omega_1}(\mathbf{R}^n_+)$. The operator norm of T is defined as

$$\|T\| = \inf M = \sup_{f \in L^p_{\omega_1}(\mathbf{R}^n_+)} \frac{\|T(f)\|_{p,\omega_2}}{\|f\|_{p,\omega_1}}.$$

By (1.2) inequality (1.1) can be rewritten as

$$\int_{\mathbf{R}^n_+} T(f)(y) g(y)\,dy \le M\|f\|_{p,u^\alpha(x)}\|g\|_{q,v^\beta(y)}.$$

It is not hard to prove that this inequality is equivalent to

$$\big\| T(f)\big\|_{p,v^{\beta(1-p)}(y)} \le M\|f\|_{p,u^\alpha(x)}. \tag{1.3}$$

In this paper, we discuss a necessary and sufficient condition and the best constant factor for the Hilbert-type multiple integral inequality with the integral kernel of the generalized homogeneous function $K(u(x), v(y))$. Our research is of some theoretical and application value for the research of Hilbert-type inequalities. Further, these results are used to study the boundedness and norm of the operator. Related studies can be found in [1–16].

Lemma 1 *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n \ge 1$, $\lambda > 0$, $\lambda_1\lambda_2 > 0$, *and let a nonnegative measurable function* $K(u(x), v(y))$ *be a generalized homogeneous function with parameters* $(\lambda, \lambda_1, \lambda_2)$. *Denote*

$$W_1 = \int_{\mathbf{R}^n_+} \big[v(t)\big]^{-\frac{\beta+n}{q}} K\big(1, v(t)\big)\,dt,$$
$$W_2 = \int_{\mathbf{R}^n_+} \big[u(t)\big]^{-\frac{\alpha+n}{p}} K\big(u(t), 1\big)\,dt.$$

Then

$$\omega_1(x) = \int_{\mathbf{R}^n_+} \big[v(y)\big]^{-\frac{\beta+n}{q}} K\big(u(x), v(y)\big)\,dy = \big[u(x)\big]^{\lambda\lambda_1 - \frac{\lambda_1}{\lambda_2}(\frac{\beta+n}{q} - n)} W_1,$$
$$\omega_2(y) = \int_{\mathbf{R}^n_+} \big[u(x)\big]^{-\frac{\alpha+n}{p}} K\big(u(x), v(y)\big)\,dx = \big[v(y)\big]^{\lambda\lambda_2 - \frac{\lambda_2}{\lambda_1}(\frac{\alpha+n}{p} - n)} W_2.$$

Proof Since $K(u(x),v(y))$ is a generalized homogeneous function with parameters $(\lambda,\lambda_1,\lambda_2)$, we have

$$\begin{aligned}\omega_1(x) &= \int_{\mathbf{R}_+^n} u^{\lambda\lambda_1}(x)\big[v(y)\big]^{-\frac{\beta+n}{q}} K\big(1,u^{-\frac{\lambda_1}{\lambda_2}}(x)v(y)\big)\,dy \\ &= \int_{\mathbf{R}_+^n} u^{\lambda\lambda_1}(x)\big[v(y)\big]^{-\frac{\beta+n}{q}} K\big(1,v\big(u^{-\frac{\lambda_1}{\lambda_2}}(x)y\big)\big)\,dy \\ &= u^{\lambda\lambda_1}(x)\int_{\mathbf{R}_+^n}\big[u^{\frac{\lambda_1}{\lambda_2}}(x)v(t)\big]^{-\frac{\beta+n}{q}} K\big(1,v(t)\big)u^{\frac{n\lambda_1}{\lambda_2}}(x)\,dt \\ &= \big[u(x)\big]^{\lambda\lambda_1-\frac{\lambda_1}{\lambda_2}(\frac{\beta+n}{q}-n)}\int_{\mathbf{R}_+^n}\big[v(t)\big]^{-\frac{\beta+n}{q}} K\big(1,v(t)\big)\,dt \\ &= \big[u(x)\big]^{\lambda\lambda_1-\frac{\lambda_1}{\lambda_2}(\frac{\beta+n}{q}-n)} W_1.\end{aligned}$$

By the same method we can obtain $\omega_2(y) = [v(y)]^{\lambda\lambda_2-\frac{\lambda_2}{\lambda_1}(\frac{\alpha+n}{p}-n)} W_2$. □

Lemma 2 ([17]) *Let $p_i > 0$, $a_i > 0$, $\alpha_i > 0$ $(i = 1,2,\dots,n)$, and let $\psi(u)$ be measurable. Then*

$$\begin{aligned}&\int\cdots\int_{x_1>0,\dots,x_n>0;\sum_{i=1}^n(\frac{x_i}{a_i})^{\alpha_i}\le 1}\psi\left(\sum_{i=1}^n\left(\frac{x_i}{a_i}\right)^{\alpha_i}\right)x_1^{p_1-1}\cdots x_n^{p_n-1}\,dx_1\cdots dx_n \\ &\quad= \frac{a_1^{p_1}\cdots a_n^{p_n}\Gamma(\frac{p_1}{\alpha_1})\cdots\Gamma(\frac{p_n}{\alpha_n})}{\alpha_1\cdots\alpha_n\Gamma(\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n})}\int_0^1\psi(t)t^{\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}-1}\,dt,\end{aligned}$$

where Γ is the gamma function.

2 Main results

Theorem 1 *Let $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $n \ge 1$, $\rho > 0$, $\lambda > 0$, $\lambda_1\lambda_2 > 0$, let there exist positive constants C_1 and C_2 such that $C_1\|x\|_\rho \le u(x) \le C_2\|x\|_\rho$, $C_1\|y\|_\rho \le v(y) \le C_2\|y\|_\rho$, let a nonnegative measurable function $K(u(x),v(y))$ be a generalized homogeneous function with parameters $(\lambda,\lambda_1,\lambda_2)$, and let the convergent integrals W_1 and W_2 be defined as in Lemma 1. Then we have*:

(i) *There exists a constant M such that the Hilbert-type multiple integral inequality in (1.1) holds if and only if $\frac{\lambda_2\alpha-n\lambda_1}{p}+\frac{\lambda_1\beta-n\lambda_2}{q} = \lambda\lambda_1\lambda_2$.*

(ii) *The best constant factor in (1.1) is $\inf M = W_1^{\frac{1}{p}}W_2^{\frac{1}{q}}$.*

Proof Let $\Omega(a<b) = \{x = (x_1,\dots,x_n) : a < \|x\|_\rho < b\}$.

(i) Suppose there exists a constant M such that (1.1) holds. Denote $l = \frac{\lambda_2\alpha-n\lambda_1}{p}+\frac{\lambda_1\beta-n\lambda_2}{q}-\lambda\lambda_1\lambda_2$. First, we let $\lambda_1 > 0$, $\lambda_2 > 0$. For $l > 0$ and $\varepsilon > 0$ sufficiently small, we set

$$f(x) = \begin{cases}[u(x)]^{(-\alpha-n+\lambda_1\varepsilon)/p}, & 0 < \|x\|_\rho < 1,\\ 0, & \|x\|_\rho \ge 1.\end{cases}$$

$$g(y) = \begin{cases}[v(y)]^{(-\beta-n+\lambda_2\varepsilon)/q}, & 0 < \|y\|_\rho < 1,\\ 0, & \|y\|_\rho \ge 1.\end{cases}$$

Thus we have

$$\|f\|_{p,u^{\alpha}(x)}\|g\|_{q,v^{\beta}(y)} = \left(\int_{\Omega(0<1)}\bigl[u(x)\bigr]^{-n+\lambda_1\varepsilon}\,dx\right)^{\frac{1}{p}}\left(\int_{\Omega(0<1)}\bigl[v(y)\bigr]^{-n+\lambda_2\varepsilon}\,dy\right)^{\frac{1}{q}}. \tag{2.1}$$

In view of $\lambda_1 > 0$, $\lambda_2 > 0$, $C_1\|x\|_\rho \le u(x) \le C_2\|x\|_\rho$, $C_1\|y\|_\rho \le v(y) \le C_2\|y\|_\rho$, the two integrals in (2.1) are all convergent.

Also, since $-\frac{\lambda_1}{\lambda_2} < 0$ and $(C_2\|x\|_\rho)^{-\frac{\lambda_1}{\lambda_2}} \le u^{-\frac{\lambda_1}{\lambda_2}}(x) \le (C_1\|x\|_\rho)^{-\frac{\lambda_1}{\lambda_2}}$, we have

$$\begin{aligned}
&\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^n} K\bigl(u(x),v(y)\bigr)f(x)g(y)\,dx\,dy\\
&= \int_{\Omega(0<1)}\bigl[u(x)\bigr]^{(-\alpha-n+\lambda_1\varepsilon)/p}\left(\int_{\Omega(0<1)} K\bigl(u(x),v(y)\bigr)\bigl[v(y)\bigr]^{(-\beta-n+\lambda_2\varepsilon)/q}\,dy\right)dx\\
&= \int_{\Omega(0<1)}\bigl[u(x)\bigr]^{\lambda\lambda_1+(-\alpha-n+\lambda_1\varepsilon)/p}\left(\int_{\Omega(0<1)} K\bigl(1,v\bigl(u^{-\frac{\lambda_1}{\lambda_2}}(x)y\bigr)\bigr)\bigl[v(y)\bigr]^{(-\beta-n+\lambda_2\varepsilon)/q}\,dy\right)dx\\
&= \int_{\Omega(0<1)}\bigl[u(x)\bigr]^{\lambda\lambda_1+(-\alpha-n+\lambda_1\varepsilon)/p}\\
&\quad\times\left(\int_{\Omega(0<u^{-\frac{\lambda_1}{\lambda_2}}(x))} K\bigl(1,v(t)\bigr)\bigl[u^{\frac{\lambda_1}{\lambda_2}}(x)v(t)\bigr]^{(-\beta-n+\lambda_2\varepsilon)/q}u^{\frac{n\lambda_1}{\lambda_2}}(x)\,dt\right)dx\\
&= \int_{\Omega(0<1)}\bigl[u(x)\bigr]^{-n+\lambda_1\varepsilon-\frac{l}{\lambda_2}}\left(\int_{\Omega(0<u^{-\frac{\lambda_1}{\lambda_2}}(x))} K\bigl(1,v(t)\bigr)\bigl[v(t)\bigr]^{-\frac{\beta+n-\lambda_2\varepsilon}{q}}\,dt\right)dx\\
&\ge \int_{\Omega(0<1)}\bigl[u(x)\bigr]^{-n+\lambda_1\varepsilon-\frac{l}{\lambda_2}}\left(\int_{\Omega(0<(C_2\|x\|_\rho)^{-\frac{\lambda_1}{\lambda_2}})} K\bigl(1,v(t)\bigr)\bigl[v(t)\bigr]^{-\frac{\beta+n-\lambda_2\varepsilon}{q}}\,dt\right)dx\\
&\ge \int_{\Omega(0<1)}\bigl[u(x)\bigr]^{-n+\lambda_1\varepsilon-\frac{l}{\lambda_2}}\,dx\int_{\Omega(0<C_2^{-\frac{\lambda_1}{\lambda_2}})} K\bigl(1,v(t)\bigr)\bigl[v(t)\bigr]^{-\frac{\beta+n-\lambda_2\varepsilon}{q}}\,dt.
\end{aligned}$$

Combining this with (1.1) and (2.1), we get

$$\begin{aligned}
&\int_{\Omega(0<1)}\bigl[u(x)\bigr]^{-n+\lambda_1\varepsilon-\frac{l}{\lambda_2}}\,dx\int_{\Omega(0<C_2^{-\frac{\lambda_1}{\lambda_2}})} K\bigl(1,v(t)\bigr)\bigl[v(t)\bigr]^{-\frac{\beta+n-\lambda_2\varepsilon}{q}}\,dt\\
&\quad\le M\left(\int_{\Omega(0<1)}\bigl[u(x)\bigr]^{-n+\lambda_1\varepsilon}\,dx\right)^{\frac{1}{p}}\left(\int_{\Omega(0<1)}\bigl[v(y)\bigr]^{-n+\lambda_2\varepsilon}\,dy\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.2}$$

Since $l > 0$ and ε is sufficiently small, $-n + \lambda_1\varepsilon - \frac{l}{\lambda_2} < -n$, and additionally $C_1\|x\|_\rho \le u(x) \le C_2\|x\|_\rho$, then $\int_{\Omega(0<1)}[u(x)]^{-n+\lambda_1\varepsilon-\frac{l}{\lambda_2}}\,dx = +\infty$. So (2.2) is a contradiction to $l > 0$.

If $l < 0$, let $\varepsilon > 0$ be sufficient small. Then we set

$$f(x) = \begin{cases} [u(x)]^{(-\alpha-n-\lambda_1\varepsilon)/p}, & \|x\|_\rho > 1,\\ 0 & \text{otherwise,}\end{cases}$$

$$g(y) = \begin{cases} [v(y)]^{(-\beta-n-\lambda_2\varepsilon)/q}, & \|y\|_\rho > 1,\\ 0 & \text{otherwise.}\end{cases}$$

Similarly, we can get

$$\int_{\Omega(1<+\infty)} \bigl[v(y)\bigr]^{-n-\lambda_2\varepsilon-\frac{l}{\lambda_1}}\,dy \int_{\Omega(C_1^{-\frac{\lambda_2}{\lambda_1}}<+\infty)} K\bigl(u(t),1\bigr)\bigl[u(t)\bigr]^{-\frac{\alpha+\beta+\lambda_1\varepsilon}{p}}\,dt$$
$$\leq M\biggl(\int_{\Omega(1<+\infty)} \bigl[u(x)\bigr]^{-n-\lambda_1\varepsilon}\,dx\biggr)^{\frac{1}{p}}\biggl(\int_{\Omega(1<+\infty)} \bigl[v(y)\bigr]^{-n-\lambda_2\varepsilon}\,dy\biggr)^{\frac{1}{q}}. \tag{2.3}$$

Since $C_1\|x\|_\rho \leq u(x) \leq C_2\|x\|_\rho$, $C_1\|y\|_\rho \leq v(y) \leq C_2\|y\|_\rho$, $l<0$, $\lambda_1>0$, $\lambda_2>0$, and $\varepsilon>0$ is sufficient small, the right-hand side of (2.3) converges; also, $\int_{\Omega(1<+\infty)}[v(y)]^{-n-\lambda_2\varepsilon-\frac{l}{\lambda_1}}\,dy$ diverges, and thus (2.3) is a contradiction to $l<0$.

In conclusion, when $\lambda_1>0$, $\lambda_2>0$, then we have $l=0$, that is, $\frac{\lambda_2\alpha-n\lambda_1}{p}+\frac{\lambda_1\beta-n\lambda_2}{q}=\lambda\lambda_1\lambda_2$.

Again, suppose $\lambda_1<0$, $\lambda_2<0$. If $l>0$, then taking $\varepsilon>0$ sufficiently small, we set

$$f(x)=\begin{cases}[u(x)]^{(-\alpha-n+\lambda_1\varepsilon)/p}, & \|x\|_\rho>1,\\ 0 & \text{otherwise,}\end{cases}$$
$$g(y)=\begin{cases}[v(y)]^{(-\beta-n+\lambda_2\varepsilon)/q}, & \|y\|_\rho>1,\\ 0 & \text{otherwise.}\end{cases}$$

We thus have

$$\|f\|_{p,u^\alpha(x)}\|g\|_{q,v^\beta(y)}=\biggl(\int_{\Omega(1<+\infty)} \bigl[u(x)\bigr]^{-n+\lambda_1\varepsilon}\,dx\biggr)^{\frac{1}{p}}\biggl(\int_{\Omega(1<+\infty)} \bigl[v(y)\bigr]^{-n+\lambda_2\varepsilon}\,dy\biggr)^{\frac{1}{q}}. \tag{2.4}$$

Meanwhile, using $C_1\|x\|_\rho \leq u(x) \leq C_2\|x\|_\rho$, $(C_2\|x\|_\rho)^{-\frac{\lambda_1}{\lambda_2}} \leq u^{-\frac{\lambda_1}{\lambda_2}} \leq (C_1\|x\|_\rho)^{-\frac{\lambda_1}{\lambda_2}}$, we have

$$\begin{aligned}
&\int_{R_+^n}\int_{R_+^n} K\bigl(u(x),v(y)\bigr)f(x)g(y)\,dx\,dy\\
&\quad=\int_{\Omega(1<+\infty)}\bigl[u(x)\bigr]^{(-\alpha-n+\lambda_1\varepsilon)/p}\biggl(\int_{\Omega(1<+\infty)} K\bigl(u(x),v(y)\bigr)\bigl[v(y)\bigr]^{(-\beta-n+\lambda_2\varepsilon)/q}\,dy\biggr)\,dx\\
&\quad=\int_{\Omega(1<+\infty)}\bigl[u(x)\bigr]^{\lambda\lambda_1+(-\alpha-n+\lambda_1\varepsilon)/p}\\
&\qquad\times\biggl(\int_{\Omega(1<+\infty)} K\bigl(1,v\bigl(u^{-\frac{\lambda_1}{\lambda_2}}(x)y\bigr)\bigr)\bigl[v(y)\bigr]^{(-\beta-n+\lambda_2\varepsilon)/q}\,dy\biggr)\,dx\\
&\quad=\int_{\Omega(1<+\infty)}\bigl[u(x)\bigr]^{\lambda\lambda_1-\frac{\alpha+n-\lambda_1\varepsilon}{p}}\\
&\qquad\times\biggl(\int_{\Omega(u^{-\frac{\lambda_1}{\lambda_2}}(x)<+\infty)} K\bigl(1,v(t)\bigr)\bigl[u^{\frac{\lambda_1}{\lambda_2}}(x)v(t)\bigr]^{-\frac{\beta+n-\lambda_2\varepsilon}{q}}u^{\frac{n\lambda_1}{\lambda_2}}(x)\,dt\biggr)\,dx\\
&\quad\geq\int_{\Omega(1<+\infty)}\bigl[u(x)\bigr]^{-n+\lambda_1\varepsilon-\frac{l}{\lambda_2}}\biggl(\int_{\Omega((C_1\|x\|_\rho)^{-\frac{\lambda_1}{\lambda_2}}<+\infty)} K\bigl(1,v(t)\bigr)\bigl[v(t)\bigr]^{-\frac{\beta+n-\lambda_2\varepsilon}{q}}\,dt\biggr)\,dx\\
&\quad\geq\int_{\Omega(1<+\infty)}\bigl[u(x)\bigr]^{-n+\lambda_1\varepsilon-\frac{l}{\lambda_2}}\,dx\int_{\Omega(C_1^{-\frac{\lambda_1}{\lambda_2}}<+\infty)} K\bigl(1,v(t)\bigr)\bigl[v(t)\bigr]^{-\frac{\beta+n-\lambda_2\varepsilon}{q}}\,dt.
\end{aligned}$$

Combining this with (1.1) and (2.4), we obtain

$$\int_{\Omega(1<+\infty)}\big[u(x)\big]^{-n+\lambda_1\varepsilon-\frac{l}{\lambda_2}}\,dx\int_{\Omega(C_1^{-\frac{\lambda_1}{\lambda_2}}<+\infty)}K\big(1,v(t)\big)\big[v(t)\big]^{-\frac{\beta+n-\lambda_2\varepsilon}{q}}\,dt$$
$$\le M\biggl(\int_{\Omega(1<+\infty)}\big[u(x)\big]^{-n+\lambda_1\varepsilon}\,dx\biggr)^{\frac{1}{p}}\biggl(\int_{\Omega(1<+\infty)}\big[v(y)\big]^{-n+\lambda_2\varepsilon}\,dy\biggr)^{\frac{1}{q}}. \tag{2.5}$$

Since the two integrals of the right-hand side of (2.5) converge, but the integral

$$\int_{\Omega(1<+\infty)}\big[u(x)\big]^{-n+\lambda_1\varepsilon-\frac{l}{\lambda_2}}\,dx$$

diverges, (2.5) is a contradiction to $l>0$.

If $l<0$ and $\varepsilon>0$ is sufficiently small, then we set

$$f(x)=\begin{cases}[u(x)]^{(-\alpha-n-\lambda_1\varepsilon)/p}, & 0<\|x\|_\rho<1,\\ 0 & \text{otherwise},\end{cases}$$

$$g(y)=\begin{cases}[v(y)]^{(-\beta-n-\lambda_2\varepsilon)/q}, & 0<\|y\|_\rho<1,\\ 0 & \text{otherwise}.\end{cases}$$

Similarly, we can get

$$\int_{\Omega(0<1)}\big[v(y)\big]^{-n-\lambda_2\varepsilon-\frac{l}{\lambda_1}}\,dy\int_{\Omega(0<C_2^{-\frac{\lambda_2}{\lambda_1}})}K\big(u(t),1\big)\big[u(t)\big]^{-\frac{\alpha+\beta+\lambda_1\varepsilon}{p}}\,dt$$
$$\le M\biggl(\int_{\Omega(0<1)}\big[u(x)\big]^{-n-\lambda_1\varepsilon}\,dx\biggr)^{\frac{1}{p}}\biggl(\int_{\Omega(0<1)}\big[v(y)\big]^{-n-\lambda_2\varepsilon}\,dy\biggr)^{\frac{1}{q}}. \tag{2.6}$$

We now easily get that both integrals on the right-hand side of (2.6) converge, but

$$\int_{\Omega(0<1)}\big[v(y)\big]^{-n-\lambda_2\varepsilon-\frac{l}{\lambda_1}}\,dy$$

diverges, and thus (2.6) is a contradiction to $l<0$.

To sum up, when $\lambda_1<0$, $\lambda_2<0$, we also have $l=0$, that is, $\frac{\lambda_2\alpha-n\lambda_1}{p}+\frac{\lambda_1\beta-n\lambda_2}{q}=\lambda\lambda_1\lambda_2$.

On the contrary, if $\frac{\lambda_2\alpha-n\lambda_1}{p}+\frac{\lambda_1\beta-n\lambda_2}{q}=\lambda\lambda_1\lambda_2$, then let $a=\frac{\alpha}{pq}+\frac{n}{pq}$, $b=\frac{\beta}{pq}+\frac{n}{pq}$. By the Hölder inequality and Lemma 1 we have

$$\begin{aligned}&\int_{R_+^n}\int_{R_+^n}K\big(u(x),v(y)\big)f(x)g(y)\,dx\,dy\\ &=\int_{R_+^n}\int_{R_+^n}\biggl[f(x)\frac{u^a(x)}{v^b(y)}\biggr]\biggl[g(y)\frac{v^b(y)}{u^a(x)}\biggr]K\big(u(x),v(y)\big)\,dx\,dy\\ &\le\biggl(\int_{R_+^n}\int_{R_+^n}f^p(x)\frac{u^{ap}(x)}{v^{bp}(y)}K\big(u(x),v(y)\big)\,dx\,dy\biggr)^{\frac{1}{p}}\\ &\quad\times\biggl(\int_{R_+^n}\int_{R_+^n}g^q(y)\frac{v^{bq}(y)}{u^{aq}(x)}K\big(u(x),v(y)\big)\,dx\,dy\biggr)^{\frac{1}{q}}\end{aligned}$$

$$= \left(\int_{R_+^n} \left[u(x)\right]^{\frac{\alpha+n}{q}} f^p(x)\omega_1(x)\,dx\right)^{\frac{1}{p}} \left(\int_{R_+^n} \left[\nu(y)\right]^{\frac{\beta+n}{p}} g^q(y)\omega_2(y)\,dy\right)^{\frac{1}{q}}$$

$$= W_1^{\frac{1}{p}} W_2^{\frac{1}{q}} \left(\int_{R_+^n} \left[u(x)\right]^{\frac{\alpha+n}{q}+\lambda\lambda_1-\frac{\lambda_1}{\lambda_2}(\frac{\beta+n}{q}-n)} f^p(x)\,dx\right)^{\frac{1}{p}}$$

$$\times \left(\int_{R_+^n} \left[\nu(y)\right]^{\frac{\beta+n}{p}+\lambda\lambda_2-\frac{\lambda_2}{\lambda_1}(\frac{\alpha+n}{p}-n)} g^q(y)\,dy\right)^{\frac{1}{q}}$$

$$= W_1^{\frac{1}{p}} W_2^{\frac{1}{q}} \left(\int_{R_+^n} u^\alpha(x) f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_{R_+^n} \nu^\beta(y) g^q(y)\,dy\right)^{\frac{1}{q}}$$

$$= W_1^{\frac{1}{p}} W_2^{\frac{1}{q}} \|f\|_{p,u^\alpha(x)} \|g\|_{q,\nu^\beta(y)}.$$

Taking arbitrary $M \geq W_1^{\frac{1}{p}} W_2^{\frac{1}{q}}$, inequality (1.1) holds.

(ii) Suppose inequality (1.1) holds. If $\inf M \neq W_1^{\frac{1}{p}} W_2^{\frac{1}{q}}$, then there exists a constant $M_0 < W_1^{\frac{1}{p}} W_2^{\frac{1}{q}}$ such that

$$\int_{R_+^n} \int_{R_+^n} K\big(u(x),\nu(y)\big) f(x)g(y)\,dx\,dy \leq M_0 \|f\|_{p,u^\alpha(x)} \|g\|_{q,\nu^\beta(y)} \tag{2.7}$$

for all $f \in L^p_{u^\alpha(x)}(R_+^n)$ and $g \in L^q_{\nu^\beta(y)}(R_+^n)$.

Let $\varepsilon > 0$ and $\delta > 0$ be sufficient small. We take

$$f(x) = \begin{cases} [u(x)]^{(-\alpha-n-|\lambda_1|\varepsilon)/p}, & \|x\|_\rho > \delta, \\ 0 & \text{otherwise,} \end{cases}$$

$$g(y) = \begin{cases} [\nu(y)]^{(-\beta-n-|\lambda_2|\varepsilon)/q}, & \|y\|_\rho > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\|f\|_{p,u^\alpha(x)} \|g\|_{q,\nu^\beta(y)} = \left(\int_{\Omega(\delta<+\infty)} \left[u(x)\right]^{-n-|\lambda_1|\varepsilon}\,dx\right)^{\frac{1}{p}} \left(\int_{\Omega(1<+\infty)} \left[\nu(y)\right]^{-n-|\lambda_2|\varepsilon}\,dy\right)^{\frac{1}{q}}. \tag{2.8}$$

Since $\frac{\lambda_2\alpha-n\lambda_1}{p} + \frac{\lambda_1\beta-n\lambda_2}{q} = \lambda\lambda_1\lambda_2$ and $\nu^{-\frac{\lambda_2}{\lambda_1}}(y) \leq (C_1\|y\|_\rho)^{-\frac{\lambda_2}{\lambda_1}}$, we have

$$\int_{R_+^n} \int_{R_+^n} K\big(u(x),\nu(y)\big) f(x)g(y)\,dx\,dy$$

$$= \int_{\Omega(1<+\infty)} \left[\nu(y)\right]^{(-\beta-n-|\lambda_2|\varepsilon)/q} \left(\int_{\Omega(\delta<+\infty)} K\big(u(x),\nu(y)\big)\left[u(x)\right]^{(-\alpha-n-|\lambda_1|\varepsilon)/p}\,dx\right) dy$$

$$= \int_{\Omega(1<+\infty)} \left[\nu(y)\right]^{\lambda\lambda_2-\frac{\beta+n+|\lambda_2|\varepsilon}{q}} \left(\int_{\Omega(\delta<+\infty)} K\big(u(\nu^{-\frac{\lambda_2}{\lambda_2}}(y)x),1\big)\left[u(x)\right]^{-\frac{\alpha+n+|\lambda_1|\varepsilon}{p}}\,dx\right) dy$$

$$= \int_{\Omega(1<+\infty)} \left[\nu(y)\right]^{\lambda\lambda_2-\frac{\beta+n+|\lambda_2|\varepsilon}{q}}$$

$$\times \left(\int_{\Omega(\delta\nu^{-\frac{\lambda_2}{\lambda_1}}(y)<+\infty)} K\big(u(t),1\big)\left[\nu^{\frac{\lambda_2}{\lambda_1}}(y)u(t)\right]^{-\frac{\alpha+n+|\lambda_1|\varepsilon}{p}} \nu^{\frac{n\lambda_2}{\lambda_1}}(y)\,dt\right) dy$$

$$\begin{aligned}
&= \int_{\Omega(1<+\infty)} \bigl[\nu(y)\bigr]^{-n-|\lambda_2|\varepsilon} \biggl(\int_{\Omega(\delta\nu^{-\frac{\lambda_2}{\lambda_1}}(y)<+\infty)} K\bigl(u(t),1\bigr)\bigl[u(t)\bigr]^{-\frac{\alpha+n+|\lambda_1|\varepsilon}{p}}\,dt\biggr)\,dy \\
&\geq \int_{\Omega(1<+\infty)} \bigl[\nu(y)\bigr]^{-n-|\lambda_2|\varepsilon} \biggl(\int_{\Omega(\delta(C_1\|y\|_\rho)^{-\frac{\lambda_2}{\lambda_1}}<+\infty)} K\bigl(u(t),1\bigr)\bigl[u(t)\bigr]^{-\frac{\alpha+n+|\lambda_1|\varepsilon}{p}}\,dt\biggr)\,dy \\
&\geq \int_{\Omega(1<+\infty)} \bigl[\nu(y)\bigr]^{-n-|\lambda_2|\varepsilon} \int_{\Omega(\delta C_1^{-\frac{\lambda_2}{\lambda_1}}<+\infty)} K\bigl(u(t),1\bigr)\bigl[u(t)\bigr]^{-\frac{\alpha+n+|\lambda_1|\varepsilon}{p}}\,dt.
\end{aligned}$$

Combining this with (2.7) and (2.8), we obtain

$$\begin{aligned}
&\int_{\Omega(1<+\infty)} \bigl[\nu(y)\bigr]^{-n-|\lambda_2|\varepsilon}\,dy \int_{\Omega(\delta C_1^{-\frac{\lambda_2}{\lambda_1}}<+\infty)} K\bigl(u(t),1\bigr)\bigl[u(t)\bigr]^{-\frac{\alpha+n+|\lambda_1|\varepsilon}{p}}\,dt \\
&\quad\leq M_0\biggl(\int_{\Omega(\delta<+\infty)} \bigl[u(x)\bigr]^{-n-|\lambda_1|\varepsilon}\,dx\biggr)^{\frac{1}{p}} \biggl(\int_{\Omega(1<+\infty)} \bigl[\nu(y)\bigr]^{-n-|\lambda_2|\varepsilon}\,dy\biggr)^{\frac{1}{q}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\biggl(\int_{\Omega(1<+\infty)} \bigl[\nu(y)\bigr]^{-n-|\lambda_2|\varepsilon}\,dy\biggr)^{\frac{1}{p}} \int_{\Omega(\delta C_1^{-\frac{\lambda_2}{\lambda_1}}<+\infty)} K\bigl(u(t),1\bigr)\bigl[u(t)\bigr]^{-\frac{\alpha+n+|\lambda_1|\varepsilon}{p}}\,dt \\
&\quad\leq M_0\biggl(\int_{\Omega(\delta<+\infty)} \bigl[u(x)\bigr]^{-n-|\lambda_1|\varepsilon}\,dx\biggr)^{\frac{1}{p}}.
\end{aligned}\tag{2.9}$$

We also take

$$f(x) = \begin{cases} [u(x)]^{(-\alpha-n-|\lambda_1|\varepsilon)/p}, & \|x\|_\rho > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$g(y) = \begin{cases} [\nu(y)]^{(-\beta-n-|\lambda_2|\varepsilon)/q}, & \|y\|_\rho > \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we can get

$$\begin{aligned}
&\biggl(\int_{\Omega(1<+\infty)} \bigl[u(x)\bigr]^{-n-|\lambda_1|\varepsilon}\,dx\biggr)^{\frac{1}{q}} \int_{\Omega(\delta C_1^{-\frac{\lambda_2}{\lambda_1}}<+\infty)} K\bigl(1,\nu(t)\bigr)\bigl[\nu(t)\bigr]^{-\frac{\beta+n+|\lambda_2|\varepsilon}{q}}\,dt \\
&\quad\leq M_0\biggl(\int_{\Omega(\delta<+\infty)} \bigl[\nu(y)\bigr]^{-n-|\lambda_2|\varepsilon}\,dy\biggr)^{\frac{1}{q}}.
\end{aligned}\tag{2.10}$$

By (2.9) and (2.10) we have

$$\begin{aligned}
&\biggl(\int_{\Omega(\delta C_1^{-\frac{\lambda_1}{\lambda_2}}<+\infty)} K\bigl(1,\nu(t)\bigr)\bigl[\nu(t)\bigr]^{-\frac{\beta+n+|\lambda_2|\varepsilon}{q}}\,dt\biggr)^{\frac{1}{p}} \\
&\quad\times\biggl(\int_{\Omega(\delta C_1^{-\frac{\lambda_2}{\lambda_1}}<+\infty)} K\bigl(u(t),1\bigr)\bigl[u(t)\bigr]^{-\frac{\alpha+n+|\lambda_1|\varepsilon}{p}}\,dt\biggr)^{\frac{1}{q}} \\
&\quad\leq M_0\biggl(\frac{\int_{\Omega(\delta<+\infty)}[u(x)]^{-n-|\lambda_1|\varepsilon}\,dx}{\int_{\Omega(1<+\infty)}[u(x)]^{-n-|\lambda_1|\varepsilon}\,dx}\biggr)^{\frac{1}{pq}} \biggl(\frac{\int_{\Omega(\delta<+\infty)}[\nu(y)]^{-n-|\lambda_2|\varepsilon}\,dy}{\int_{\Omega(1<+\infty)}[\nu(y)]^{-n-|\lambda_2|\varepsilon}\,dy}\biggr)^{\frac{1}{pq}}
\end{aligned}$$

$$= M_0\left(1+\frac{\int_{\Omega(\delta<1)}[u(x)]^{-n-|\lambda_1|\varepsilon}\,dx}{\int_{\Omega(1<+\infty)}[u(x)]^{-n-|\lambda_1|\varepsilon}\,dx}\right)^{\frac{1}{pq}}\left(1+\frac{\int_{\Omega(\delta<1)}[v(y)]^{-n-|\lambda_2|\varepsilon}\,dy}{\int_{\Omega(1<+\infty)}[v(y)]^{-n-|\lambda_2|\varepsilon}\,dy}\right)^{\frac{1}{pq}}. \quad (2.11)$$

Since $C_1\|x\|_\rho \le u(x) \le C_2\|x\|_\rho$, $\int_{\Omega(\delta<1)}[u(x)]^{-n}\,dx$ is a usual integral, but $\int_{\Omega(1<+\infty)}[u(x)]^{-n}\,dx$ diverges, and thus

$$\lim_{\varepsilon\to 0^+}\frac{\int_{\Omega(\delta<1)}[u(x)]^{-n-|\lambda_1|\varepsilon}\,dx}{\int_{\Omega(1<+\infty)}[u(x)]^{-n-|\lambda_1|\varepsilon}\,dx}=0.$$

In the same way, we have

$$\lim_{\varepsilon\to 0^+}\frac{\int_{\Omega(\delta<1)}[v(y)]^{-n-|\lambda_2|\varepsilon}\,dy}{\int_{\Omega(1<+\infty)}[v(y)]^{-n-|\lambda_2|\varepsilon}\,dy}=0.$$

Letting $\varepsilon \to 0^+$ in (2.11), we get

$$\left(\int_{\Omega(\delta C_1^{-\frac{\lambda_1}{\lambda_2}}<+\infty)} K\bigl(1,v(t)\bigr)\bigl[v(t)\bigr]^{-\frac{\beta+n}{q}}\,dt\right)^{\frac{1}{p}} \times \left(\int_{\Omega(\delta C_1^{-\frac{\lambda_2}{\lambda_1}}<+\infty)} K\bigl(u(t),1\bigr)\bigl[u(t)\bigr]^{-\frac{\alpha+n}{p}}\,dt\right)^{\frac{1}{q}} \le M_0.$$

Letting $\delta \to 0^+$, we obtain

$$W_1^{\frac{1}{p}}W_2^{\frac{1}{q}} = \left(\int_{\Omega(0<+\infty)} K\bigl(1,v(t)\bigr)\bigl[v(t)\bigr]^{-\frac{\beta+n}{q}}\,dt\right)^{\frac{1}{p}} \times \left(\int_{\Omega(0<+\infty)} K\bigl(u(t),1\bigr)\bigl[u(t)\bigr]^{-\frac{\alpha+n}{p}}\,dt\right)^{\frac{1}{q}} \le M_0.$$

This is a contradiction, and hence $\inf M = W_1^{\frac{1}{p}}W_2^{\frac{1}{q}}$. □

Theorem 2 *Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $n\ge 1$, $\lambda>0$, $\lambda_1\lambda_2>0$, $\gamma=(1-p)\beta$, and let there exist constants C_1 and C_2 such that $C_1\|x\|_\rho \le u(x)\le C_2\|x\|_\rho$ and $C_1\|y\|_\rho \le v(y)\le C_2\|y\|_\rho$. Let a nonnegative measurable function $K(u(x),v(y))$ be a generalized homogeneous function for parameters $(\lambda,\lambda_1,\lambda_2)$. Let the operator T be defined by (1.2), and let W_1 and W_2 defined by Lemma 1 be also convergent. Then*

(i) *T is a bounded operator from $L^p_{u^\alpha(x)}(R^n_+)$ to $L^p_{v^\gamma(y)}(R^n_+)$ if and only if $\frac{1}{p}[\lambda_2(\alpha+n)-\lambda_1(\gamma+n)] = n\lambda_2+\lambda\lambda_1\lambda_2$.*

(ii) *If T is a bounded operator from $L^p_{u^\alpha(x)}(R^n_+)$ to $L^p_{v^\gamma(y)}(R^n_+)$, then the operator norm of T is $\|T\| = W_1^{\frac{1}{p}}W_2^{\frac{1}{q}}$.*

Proof Since $\frac{1}{p}+\frac{1}{q}=1$, $\beta=\frac{\gamma}{1-p}$, $\frac{\lambda_2\alpha-n\lambda_1}{p}+\frac{\lambda_1\beta-n\lambda_2}{q}=\lambda\lambda_1\lambda_2$ leads to $\frac{1}{p}[\lambda_2(\alpha+n)-\lambda_1(\gamma+n)] = n\lambda_2+\lambda\lambda_1\lambda_2$, and since equality (1.1) is equivalent to (1.3), Theorem 2 holds by Theorem 1. □

3 Applications

Theorem 3 *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n \geq 1$, $\rho > 0$, $\lambda > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $a_i > 0$, $b_i > 0$, $\alpha < n(p-1)$, $\beta < n(q-1)$, $u(x) = (\sum_{i=1}^n a_i x_i^\rho)^{1/\rho}$, *and* $v(y) = (\sum_{i=1}^n b_i y_i^\rho)^{1/\rho}$. *Then:*

(i) *There exists a constant M such that*

$$\int_{R_+^n}\int_{R_+^n} \frac{1}{(u^{\lambda_1}(x) + v^{\lambda_2}(y))^\lambda} f(x)g(y)\,dx\,dy \leq M\|f\|_{p,u^\alpha(x)}\|g\|_{q,v^\beta(y)} \tag{3.1}$$

if and only if $\frac{n\lambda_1-\lambda_2\alpha}{p} + \frac{n\lambda_2-\lambda_1\beta}{q} = \lambda\lambda_1\lambda_2$, *where* $f \in L^p_{u^\alpha(x)}(R_+^n)$ *and* $g \in L^q_{v^\beta(y)}(R_+^n)$.

(ii) *If inequality* (3.1) *holds, then its best constant factor is*

$$\inf M = \left(\prod_{i=1}^n a_i^{-\frac{1}{\rho}}\right)^{\frac{1}{q}}\left(\prod_{i=1}^n b_i^{-\frac{1}{\rho}}\right)^{\frac{1}{p}} \frac{\Gamma^n(\frac{1}{\rho})}{\rho^{n-1}\Gamma(\lambda)\Gamma(\frac{n}{\rho})}\Gamma\left(\frac{1}{\lambda_1}\left(\frac{n}{q}-\frac{\alpha}{p}\right)\right)\Gamma\left(\frac{1}{\lambda_2}\left(\frac{n}{p}-\frac{\beta}{q}\right)\right).$$

Proof Set $K(u(x),v(y)) = \frac{1}{(u^{\lambda_1}(x)+v^{\lambda_2}(y))^\lambda}$. Then $K(u(x),v(y))$ is a generalized homogeneous function for parameters $(\lambda,-\lambda_1,-\lambda_2)$, and $\frac{n\lambda_1-\lambda_2\alpha}{p} + \frac{n\lambda_2-\lambda_1\beta}{q} = \lambda\lambda_1\lambda_2$ is equivalent to $\frac{(-\lambda_2)\alpha-n(-\lambda_1)}{p} + \frac{(-\lambda_1)\beta-n(-\lambda_2)}{q} = \lambda(-\lambda_1)(-\lambda_2)$. Further, we have $\lambda - \frac{1}{\lambda_2}(\frac{n}{p}-\frac{\beta}{q}) = \frac{1}{\lambda_1}(\frac{n}{q}-\frac{\alpha}{p})$, and $\frac{n}{p}-\frac{\beta}{q} > 0$ and $\frac{n}{q}-\frac{\alpha}{p} > 0$ when $\alpha < n(p-1)$ and $\beta < n(q-1)$. By Lemma 1 we have

$$\begin{aligned}
W_1 &= \int_{R_+^n}\left[v(t)\right]^{-\frac{\beta+n}{q}}K\left(1,v(t)\right)dt \\
&= \int_{R_+^n}\frac{1}{[1+(\sum_{i=1}^n b_i t_i^\rho)^{\lambda_2/\rho}]^\lambda}\left(\sum_{i=1}^n b_i t_i^\rho\right)^{-\frac{\beta+n}{q\rho}}dt \\
&= \prod_{i=1}^n b_i^{-\frac{1}{\rho}}\int_{R_+^n}\frac{1}{[1+(\sum_{i=1}^n x_i^\rho)^{\lambda_2/\rho}]^\lambda}\left(\sum_{i=1}^n x_i^\rho\right)^{-\frac{\beta+n}{q\rho}}dx \\
&= \prod_{i=1}^n b_i^{-\frac{1}{\rho}}\lim_{r\to+\infty}\int\cdots\int_{x_i>0,x_1^\rho+\cdots+x_n^\rho\leq r^\rho}\frac{1}{[1+r^{\lambda_2}(\sum_{i=1}^n(\frac{x_i}{r})^\rho)^{\lambda_2/\rho}]^\lambda} \\
&\quad\times r^{-\frac{\beta+n}{q}}\left(\prod_{i=1}^n\left(\frac{x_i}{r}\right)^\rho\right)^{-\frac{\beta+n}{q\rho}}x_1^{1-1}\cdots x_n^{1-1}\,dx_1\cdots dx_2 \\
&= \prod_{i=1}^n b_i^{-\frac{1}{\rho}}\lim_{r\to+\infty}r^{-\frac{\beta+n}{q}}\frac{r^n\Gamma^n(\frac{1}{\rho})}{\rho^n\Gamma(\frac{n}{\rho})}\int_0^1\frac{1}{(1+r^{\lambda_2}u^{\lambda_2/\rho})^\lambda}u^{-\frac{\beta+n}{q\rho}}u^{\frac{n}{\rho}-1}du \\
&= \prod_{i=1}^n b_i^{-\frac{1}{\rho}}\frac{\Gamma^n(\frac{1}{\rho})}{\rho^{n-1}\Gamma(\frac{n}{\rho})\lambda_2}\int_0^\infty\frac{1}{(1+t)^\lambda}t^{\frac{1}{\lambda_2}(\frac{n}{p}-\frac{\beta}{q})-1}dt \\
&= \prod_{i=1}^n b_i^{-\frac{1}{\rho}}\frac{\Gamma^n(\frac{1}{\rho})}{\lambda_2\rho^{n-1}\Gamma(\frac{n}{\rho})}B\left(\frac{1}{\lambda_2}\left(\frac{n}{p}-\frac{\beta}{q}\right),\lambda-\frac{1}{\lambda_2}\left(\frac{n}{p}-\frac{\beta}{q}\right)\right) \\
&= \prod_{i=1}^n b_i^{-\frac{1}{\rho}}\frac{\Gamma^n(\frac{1}{\rho})}{\lambda_2\rho^{n-1}\Gamma(\frac{n}{\rho})\Gamma(\lambda)}\Gamma\left(\frac{1}{\lambda_2}\left(\frac{n}{p}-\frac{\beta}{q}\right)\right)\Gamma\left(\frac{1}{\lambda_1}\left(\frac{n}{q}-\frac{\alpha}{p}\right)\right).
\end{aligned}$$

In the same way, we get

$$W_2 = \int_{R_+^n} \left[u(x)\right]^{-\frac{\alpha+n}{p}} K\left(u(t),1\right) dt$$
$$= \prod_{i=1}^{n} a_i^{-\frac{1}{\rho}} \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1 \rho^{n-1} \Gamma(\frac{n}{\rho}) \Gamma(\lambda)} \Gamma\left(\frac{1}{\lambda_1}\left(\frac{n}{q} - \frac{\alpha}{p}\right)\right) \Gamma\left(\frac{1}{\lambda_2}\left(\frac{n}{p} - \frac{\beta}{q}\right)\right).$$

Thus

$$W_1^{\frac{1}{p}} W_2^{\frac{1}{q}} = \left(\prod_{i=1}^{n} a_i^{-\frac{1}{\rho}}\right)^{\frac{1}{q}} \left(\prod_{i=1}^{n} b_i^{-\frac{1}{\rho}}\right)^{\frac{1}{p}} \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}} \rho^{n-1} \Gamma(\lambda) \Gamma(\frac{n}{\rho})}$$
$$\times \Gamma\left(\frac{1}{\lambda_1}\left(\frac{n}{q} - \frac{\alpha}{p}\right)\right) \Gamma\left(\frac{1}{\lambda_2}\left(\frac{n}{p} - \frac{\beta}{q}\right)\right).$$

Hence Theorem 3 holds by Theorem 1. □

Corollary 1 *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n \geq 1$, $\rho > 0$, $\lambda > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $u(x) = (\sum_{i=1}^{n} x_i^{\rho})^{\frac{1}{\rho}}$, *and* $v(y) = (\sum_{i=1}^{n} y_i^{\rho})^{\frac{1}{\rho}}$. *Then*:

(i) *The operator* T *defined by*

$$T(f)(y) = \int_{R_+^n} \frac{1}{(u^{\lambda_1}(x) + v^{\lambda_2}(y))^{\lambda}} f(x)\, dx, \quad y \in R_+^n,$$

is a bounded operator in $L^p(R_+^n)$ *if and only if* $\frac{n\lambda_1}{p} + \frac{n\lambda_2}{q} = \lambda\lambda_1\lambda_2$.

(ii) *When* T *is a bounded operator in* $L^p(R_+^n)$, *the operator norm of* T *is*

$$\|T\| = \frac{\Gamma^n(\frac{1}{\rho})}{\rho^{n-1} \lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}} \Gamma(\lambda) \Gamma(\frac{n}{\rho})} \Gamma\left(\frac{n}{\lambda_1 q}\right) \Gamma\left(\frac{n}{\lambda_2 p}\right).$$

Proof The corollary follows from Theorems 2 and 3. □

Theorem 4 *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n \geq 1$, $\rho > 0$, $\lambda > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\alpha < n(p-1)$, $\beta < n(q-1)$, $u(x) = (\sum_{i=1}^{n} x_i^{\rho})^{1/\rho}$, *and* $v(y) = (\sum_{i=1}^{n} y_i^{\rho})^{1/\rho}$. *Then*

(i) *There exists* M *such that*

$$\int_{R_+^n} \int_{R_+^n} \frac{1}{(\max\{u^{\lambda_1}(x), v^{\lambda_2}(y)\})^{\lambda}} f(x) g(y)\, dx\, dy \leq M \|f\|_{p,u^{\alpha}(x)} \|g\|_{q,v^{\beta}(y)} \tag{3.2}$$

if and only if $\frac{n\lambda_1 - \lambda_2\alpha}{p} + \frac{n\lambda_2 - \lambda_1\beta}{q} = \lambda\lambda_1\lambda_2$, *where* $f \in L^p_{u^{\alpha}(x)}(R_+^n)$ *and* $g \in L^q_{v^{\beta}(y)}(R_+^n)$.

(ii) *If inequality* (3.2) *holds, then its best constant factor is*

$$\inf M = \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}} \rho^{n-1} \Gamma(\frac{n}{\rho})} \left[\left(\frac{1}{\lambda_1}\left(\frac{n}{q} - \frac{\alpha}{p}\right)\right)^{-1} + \left(\frac{1}{\lambda_2}\left(\frac{n}{p} - \frac{\beta}{q}\right)\right)^{-1}\right].$$

Proof Set $K(u(x),v(y)) = \frac{1}{(\max\{u^{\lambda_1}(x),v^{\lambda_2}(y)\})^{\lambda}}$. Then $K(u(x),v(y))$ is a generalized homogeneous function for parameters $(\lambda,-\lambda_1,-\lambda_2)$. By Lemma 2 we get

$$\begin{aligned} W_1 &= \int_{R_+^n} K\bigl(1,v(t)\bigr)\bigl[v(t)\bigr]^{-\frac{\beta+n}{q}}\,dt \\ &= \int_{v(t)\le 1}\bigl[v(t)\bigr]^{-\frac{\beta+n}{q}}\,dt + \int_{v(t)>1}\bigl[v(t)\bigr]^{-\lambda\lambda_2-\frac{\beta+n}{q}}\,dt \\ &= \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_2\rho^{n-1}\Gamma(\frac{n}{\rho})}\left(\frac{1}{\lambda_2}\left(\frac{n}{p}-\frac{\beta}{q}\right)\right)^{-1} + \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_2\rho^{n-1}\Gamma(\frac{n}{\rho})}\left(\frac{1}{\lambda_1}\left(\frac{n}{q}-\frac{\alpha}{p}\right)\right)^{-1} \\ &= \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_2\rho^{n-1}\Gamma(\frac{n}{\rho})}\left[\left(\frac{1}{\lambda_2}\left(\frac{n}{p}-\frac{\beta}{q}\right)\right)^{-1} + \left(\frac{1}{\lambda_1}\left(\frac{n}{q}-\frac{\alpha}{p}\right)\right)^{-1}\right]. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} W_2 &= \int_{R_+^n} K\bigl(u(t),1\bigr)\bigl[u(t)\bigr]^{-\frac{\alpha+n}{p}}\,dt \\ &= \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1\rho^{n-1}\Gamma(\frac{n}{\rho})}\left[\left(\frac{1}{\lambda_1}\left(\frac{n}{q}-\frac{\alpha}{p}\right)\right)^{-1} + \left(\frac{1}{\lambda_2}\left(\frac{n}{p}-\frac{\beta}{q}\right)\right)^{-1}\right]. \end{aligned}$$

Then we have

$$W_1^{\frac{1}{p}}W_2^{\frac{1}{q}} = \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}\rho^{n-1}\Gamma(\frac{n}{\rho})}\left[\left(\frac{1}{\lambda_1}\left(\frac{n}{q}-\frac{\alpha}{p}\right)\right)^{-1} + \left(\frac{1}{\lambda_2}\left(\frac{n}{p}-\frac{\beta}{q}\right)\right)^{-1}\right].$$

In summary, Theorem 4 holds by Theorem 1. □

Corollary 2 *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $n\ge 1$, $\rho>0$, $\lambda>0$, $\lambda_1>0$, $\lambda_2>0$, $u(x)=(\sum_{i=1}^n x_i^{\rho})^{\frac{1}{\rho}}$, *and* $v(y)=(\sum_{i=1}^n y_i^{\rho})^{\frac{1}{\rho}}$. *Then*

(i) *The operator* T *defined by*

$$T(f)(y) = \int_{R_+^n}\frac{1}{\max\{u^{\lambda_1}(x),v^{\lambda_2}(y)\})^{\lambda}}f(x)\,dx, y\in R_+^n,$$

is a bounded operator in $L^p(R_+^n)$ *if and only if* $\frac{n\lambda_1}{p}+\frac{n\lambda_2}{q}=\lambda\lambda_1\lambda_2$.

(ii) *When* T *is a bounded operator in* $L^p(R_+^n)$, *the operator norm of* T *is*

$$\|T\| = \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}\rho^{n-1}\Gamma(\frac{n}{\rho})}\left(\frac{\lambda_1 q}{n}+\frac{\lambda_2 p}{n}\right).$$

Proof The corollary follows from Theorems 2 and 4. □

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The authors declare that they have no competing interests.

Authors' contributions
YH and JL carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. BY and QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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