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New inequalities between the inverse hyperbolic tangent and the analogue for corresponding functions

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Abstract

In this paper, we present new inequalities which reveal further relationship for both the inverse tangent function arctan(*x*) and the inverse hyperbolic function arctanh(*x*). At the same time, we give the analogue for inverse hyperbolic tangent and other corresponding functions.

Keywords: Inequalities; Inverse tangent function; Inverse hyperbolic sine function; Inverse hyperbolic tangent function; Inverse sine function

1 Introduction

Masjed-Jamei [1] obtained the following inequality for $|x| < 1$:

$$(\arctan x)^2 \le \frac{x \ln(x + \sqrt{1 + x^2})}{\sqrt{1 + x^2}}. \tag{1}$$

Many similar or relative inequalities are discussed in references [2–14]. Recently, Zhu and Malesevic [15] affirmed inequality (1) for the large interval $(-\infty, \infty)$, pointed out that $\sinh^{-1}(x) = \ln(x + \sqrt{1 + x^2})$, and provided the following Theorems 1–6, which (or relative results) can be also found in [11, 12].

Theorem 1 ([15]) *The inequality*

$$(\arctan x)^2 \le \frac{x \ln(x + \sqrt{1 + x^2})}{\sqrt{1 + x^2}} \tag{2}$$

holds for all $x \in (-\infty, \infty)$, *and the power number* 2 *is the best in* (2).

Theorem 2 ([15]) *Let* $0 < r < \infty$, $\lambda = 1$, *and* $\mu = r \ln(r + \sqrt{r^2 + 1})/(\sqrt{r^2 + 1}(\arctan r)^2)$. *Then the double inequality*

$$\lambda(\arctan x)^2 \le \frac{x \ln(x + \sqrt{1 + x^2})}{\sqrt{1 + x^2}} \le \mu(\arctan x)^2 \tag{3}$$

holds for all $x \in (- - r, r)$, *where* λ *and* μ *are the best constants in* (3).

Theorem 3 ([15]) *We have*

$$-\frac{1}{45}x^6 \le (\arctan x)^2 - \frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \le -\frac{1}{45}x^6 + \frac{4}{105}x^8, \tag{4}$$

$$-\frac{1}{45}x^6 + \frac{4}{105}x^8 - \frac{11}{225}x^{10} \le (\arctan x)^2 - \frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \le -\frac{1}{45}x^6 \le +\frac{4}{105}x^8 - \frac{11}{225}x^{10} + \frac{586}{10{,}395}x^{12}. \tag{5}$$

Theorem 4 ([15]) *The inequality*

$$(\operatorname{arctanh} x)^2 \le \frac{x\arcsin x}{\sqrt{1-x^2}} \tag{6}$$

holds for all $x \in (-1,1)$, *and the power number* 2 *is the best in* (6).

Theorem 5 ([15]) *Let* $0 < r < 1$, $\alpha_1 = 1$, *and* $\beta_1 = r(\arcsin r)/(\sqrt{1--r^2}(\operatorname{arctanh} r)^2)$. *Then the double inequality*

$$\alpha_1(\operatorname{arctanh} x)^2 \le \frac{x\arcsin x}{\sqrt{1-x^2}} \le \beta_1(\operatorname{arctanh} x)^2 \tag{7}$$

holds for all $x \in (-r,r)$, *where* α_1 *and* β_1 *are the best constants in* (7).

Recently, Chen and Malešević [14] proposed the following results:

$$\frac{x\operatorname{arcsinh} x}{\sqrt{1+x^2+\alpha_2x^4}} \le (\arctan x)^2 \le \frac{x\operatorname{arcsinh} x}{\sqrt{1+x^2+\beta_2x^4}}, \quad x>0, \tag{8}$$

$$\frac{x\arcsin x}{1-\alpha_3x^2} < (\operatorname{arctanh} x)^2, \quad 0<x<1, \tag{9}$$

where $\alpha_2 = \frac{2}{45}$, $\beta_2 = 0$, and $\alpha_3 = \frac{1}{2}$ are the best possible constants.

In 2020, Zhu and Malešević [13] proposed natural approximation of Masjed-Jamei's inequality and provided two-sided bounds in a polynomial form of $(\arctan x)^2 - \frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$, which consists of explicit formulae of different degrees.

The values of μ in Theorem 2 and β_1 in Theorem 5 tend to be $+\infty$ for r tends to be $\pm\infty$ and ±1, respectively. In this paper, we obtain the following new inequalities, which improve the approximation effect of the inequalities in [15]. The main results are as follows.

Theorem 6 *The inequality*

$$(\arctan x)^2 \ge \frac{3(8+9x^2-8\sqrt{1+x^2})}{(4+11\sqrt{1+x^2})\sqrt{1+x^2}} \triangleq F(x) \tag{10}$$

holds for all $x \in (-\infty,\infty)$.

Theorem 7 *Let* $\kappa_1 = \frac{108}{11\pi^2} \approx 0.9947$ *and* $\kappa_2 = 1$. *The inequality*

$$\kappa_1(\arctan x)^2 \le F(x) \le \kappa_2(\arctan x)^2 \tag{11}$$

holds for all $x \in (-\infty, \infty)$, *where* κ_1 *and* κ_2 *are the best constants in* (11).

Theorem 8 *The inequality*

$$\frac{23}{75{,}600}x^8 \ge (\arctan x)^2 - F(x) \ge \frac{23}{75{,}600}x^8 - \frac{899}{1{,}134{,}000}x^{10} \tag{12}$$

holds for all $x \in (-\infty, \infty)$.

Theorem 9 *The inequality*

$$G_1(x) \triangleq (\operatorname{arctanh} x)^2 \le \left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2 \triangleq G_2(x) \tag{13}$$

holds for all $x \in (-1, 1)$.

Theorem 10 *Let* $\kappa_3 = 1$ *and* $\kappa_4 = \frac{16}{\pi^2} \approx 1.6211$. *The inequality*

$$\kappa_3(\operatorname{arctanh} x)^2 \le \left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2 \le \kappa_4(\operatorname{arctanh} x)^2 \tag{14}$$

holds for all $x \in (-1, 1)$, *where* κ_3 *and* κ_4 *are the best constants in* (14).

2 Proofs of Theorems 6–10

Let $\arctan x = t$, then one has that $x = \tan(t)$ and $\sqrt{1+x^2} = \sec(t)$, where $x \in (-\infty, \infty)$ and $t \in (-\pi/2, \pi/2)$. It can be verified that

$$\begin{aligned}
&(\arctan x)^2 = t^2, \\
&F(x) = -\frac{3}{4}\cos(t) - \frac{63}{16} + \frac{1125}{16(4\cos(t)+11)} = f_1(t), \\
&(\arctan x)^2 - F(x) = \bigl(t^2 - f_1(t)\bigr) = \delta_1(t), \\
&\delta_1'''(t) = \frac{(12(16\cos(t)^2 + 208\cos(t) + 1501))(\cos(t)-1)^2\sin(t)}{(4\cos(t)+11)^4}.
\end{aligned} \tag{15}$$

2.1 Proof of Theorem 6

From Eq. (15), one has that

$$\delta_1'''(t) > 0, \quad t \in (0, \pi/2), \qquad \delta_1''(0) = \delta_1'(0) = \delta_1(0) = 0, \tag{16}$$

which leads to

$$\delta_1''(t) > 0, \qquad \delta_1'(t) > 0, \quad t \in (0, \pi/2), \qquad \delta_1(t) \ge \delta_1(0) = 0, \quad t \in [0, \pi/2). \tag{17}$$

Note that $\delta_1(t) = \delta_1(-t)$, combining Eq. (15) with Eq. (17), one has that

$$\delta_1(t) \geq 0, \quad t \in (-\pi/2, \pi/2), \quad \text{and} \quad (\arctan x)^2 - F(x) \geq 0, \quad x \in (-\infty, \infty). \tag{18}$$

And we complete the proof.

2.2 Proof of Theorem 7

From Theorem 6, one has that

$$F(x) \leq \kappa_2 (\arctan x)^2.$$

Now we prove that $\kappa_1 (\arctan x)^2 \leq F(x)$. From Eq. (15), one has that

$$\begin{aligned}
&\kappa_1 (\arctan x)^2 - F(x) = \kappa_1 t^2 - f_1(t) = \delta_2(t), \\
&\delta_2'''(t) = -f_1'''(t) = \delta_1'''(t) > 0, \quad t \in (0, \pi/2), \\
&\delta_2''(0) = \frac{216 - 22\pi^2}{11\pi^2} \approx -0.01 < 0, \qquad \delta_2''(\pi/2) = \frac{26136 - 2250\pi^2}{1331\pi^2} \approx 0.2 > 0.
\end{aligned} \tag{19}$$

From Eq. (19), there exists a unique root $t_1 \in (0, \pi/2)$ such that

$$\begin{aligned}
&\delta_2''(t) < 0, \quad t \in (0, t_1), \qquad \delta_2'(0) = 0, \\
&\delta_2''(t) > 0, \quad t \in (t_1, \pi/2), \qquad \delta_2'(\pi/2) = \frac{1188 - 372\pi}{121\pi} \approx 0.05 > 0.
\end{aligned} \tag{20}$$

From Eq. (19), there exists a unique root $t_2 \in (t_1, \pi/2)$ such that

$$\begin{aligned}
&\delta_2'(t) < 0, \quad t \in (0, t_2), \qquad \delta_2(0) = 0, \\
&\delta_2'(t) > 0, \quad t \in (t_2, \pi/2), \qquad \delta_2(\pi/2) = 0.
\end{aligned} \tag{21}$$

From Eq. (21), one has that

$$\delta_2(t) \leq 0, \quad t \in [0, t_2] \cup [t_2, \pi/2) = [0, \pi/2). \tag{22}$$

Note that $\delta_2(t) = \delta_2(-t)$, combining Eq. (19) with Eq. (22), one has that

$$\delta_2(t) \leq 0, \quad t \in (-\pi/2, \pi/2), \quad \text{and} \quad \kappa_1 (\arctan x)^2 \leq F(x), \quad x \in (-\infty, \infty). \tag{23}$$

Note that

$$\lim_{x \to \infty} \frac{F(x)}{(\arctan x)^2} = \kappa_1, \qquad \lim_{x \to 0} \frac{F(x)}{(\arctan x)^2} = \kappa_2,$$

both κ_1 and κ_2 are the best constants. And the proof is completed.

2.3 Proof of Theorem 8

Let $f_2(t) = \frac{23}{75{,}600}(\tan t)^8$ and $f_3(t) = \frac{23}{75{,}600}(\tan t)^8 - \frac{899}{1{,}134{,}000}(\tan t)^{10}$. Equation (12) in Theorem 8 is equivalent to

$$\delta_3(t) = \delta_1(t) - f_2(t) \le 0, \qquad \delta_4(t) = \delta_1(t) - f_3(t) \ge 0, \quad t \in (-\pi/2, \pi/2). \tag{24}$$

It can be verified that

$$\begin{aligned} f_2'''(t) &= \frac{23\sin(t)^5(2\cos(t)^4 - 26\cos(t)^2 + 45)}{4725(\cos t)^{11}}, \\ f_3'''(t) &= \frac{\sin(t)^5(1175\cos(t)^6 - 18871\cos(t)^4 + 50{,}261\cos(t)^2 - 29{,}667)}{28{,}350(\cos t)^{13}}. \end{aligned} \tag{25}$$

Let $\phi_1(t) = 907{,}200\cos(t)^{12} + 12{,}700{,}800\cos(t)^{11} + 97{,}807{,}500\cos(t)^{10} + 97{,}795{,}724\cos(t)^9 + 97{,}642{,}636\cos(t)^8 + 96{,}990{,}540\cos(t)^7 + 96{,}802{,}860\cos(t)^6 + 103{,}838{,}238\cos(t)^5 + 126{,}378{,}882\cos(t)^4 + 148{,}760{,}458\cos(t)^3 + 130{,}005{,}062\cos(t)^2 + 67{,}501{,}665\cos(t) + 15{,}153{,}435$ and $\phi_2(t) = 5{,}443{,}200\cos(t)^{13} + 81{,}648{,}000\cos(t)^{12} + 668{,}493{,}000\cos(t)^{11} + 1{,}255{,}037{,}200\cos(t)^{10} + 1{,}837{,}671{,}000\cos(t)^9 + 2{,}404{,}568{,}576\cos(t)^8 + 2{,}978{,}639{,}640\cos(t)^7 + 3{,}789{,}264{,}297\cos(t)^6 + 5{,}266{,}619{,}820\cos(t)^5 + 7{,}153{,}847{,}855\cos(t)^4 + 7{,}714{,}708{,}320\cos(t)^3 + 5{,}610{,}730{,}675\cos(t)^2 + 2{,}369{,}206{,}620\cos(t) + 434{,}354{,}547$. Combining Eq. (24) with Eq. (25), one has that

$$\begin{aligned} \delta_3'''(t) &= \frac{\sin(t)(\cos(t)-1)^3}{(4\cos(t)+11)^4(\cos t)^{11}}\phi_1(t) < 0, \quad \forall t \in (0, \pi/2), \\ \delta_4'''(t) &= \frac{\sin(t)(\cos(t)-1)^4}{28{,}350(4\cos(t)+11)^4(\cos t)^{13}}\phi_2(t) > 0, \quad \forall t \in (0, \pi/2), \\ \delta_3''(0) &= 0, \qquad \delta_4''(0) = 0. \end{aligned} \tag{26}$$

From Eq. (25), one has that

$$\delta_3''(t) < 0, \qquad \delta_4''(t) > 0, \quad \forall t \in (0, \pi/2), \qquad \delta_3'(0) = 0, \qquad \delta_4'(0) = 0. \tag{27}$$

From Eq. (27), one obtains that

$$\delta_3'(t) < 0, \qquad \delta_4'(t) > 0, \quad \forall t \in (0, \pi/2), \qquad \delta_3(0) = 0, \qquad \delta_4(0) = 0, \tag{28}$$

which leads to

$$\delta_3(t) \le 0, \qquad \delta_4(t) \ge 0, \quad \forall t \in [0, \pi/2). \tag{29}$$

Note that $\delta_i(t) = \delta_i(-t)$, $i = 3, 4$, combining with Eq. (29), both Eq. (24) and Theorem 8 are proved.

2.4 Proof of Theorem 9

Let $\arcsin(x) = s$, then one has that $x = \sin(s)$, where $x \in (-1, 1)$, $s \in (-\pi/2, \pi/2)$. It can be verified that

$$\begin{aligned} &(\operatorname{arctanh} x) = \frac{1}{2}\ln\left(\frac{1+\sin(s)}{1-\sin(s)}\right) > 0, \\ &\left(\frac{-\ln(1-x^2)}{\arcsin x}\right) = \frac{-\ln(1-(\sin s)^2)}{s} > 0, \quad s \in (0, \pi/2). \end{aligned} \tag{30}$$

Let

$$\begin{aligned} &(\operatorname{arctanh} x) - \left(\frac{-\ln(1-x^2)}{\arcsin x}\right) = \frac{1}{2}\ln\left(\frac{1+\sin(s)}{1-\sin(s)}\right) - \frac{-\ln(1-(\sin s)^2)}{s} = \delta_5(s), \\ &\delta_6(s) = \delta_5'(s)\cdot s^2, \qquad \phi_3(s) = -2 + \sin(s)s + 2\cos(s). \end{aligned} \tag{31}$$

It can be verified that

$$\phi_3''(s) = -\sin(s)s < 0, \quad s \in (0, \pi/2), \qquad \phi_3'(0) = \phi_3(0) = 0,$$

which leads to

$$\phi_3(s) \le 0, \qquad \delta_6'(s) = \frac{s}{(\cos s)^2}\phi_3(s) \le 0, \qquad \delta_6(0) = 0, \quad s \in [0, \pi/2). \tag{32}$$

Combining Eq. (31) with Eq. (32), one obtains that

$$\delta_6(s) \le 0, \qquad \delta_5'(s) \le 0, \qquad \delta_5(0) = 0, \quad s \in [0, \pi/2). \tag{33}$$

Combining Eq. (31) with Eq. (33), we have that

$$\delta_5(s) \le 0, \quad s \in [0, \pi/2), \qquad 0 \le (\operatorname{arctanh} x)^2 \le \left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2, \quad x \in [0, 1). \tag{34}$$

Note that $G_i(-x) = G_i(x)$, $i = 1, 2$, combining with Eq. (34), we have proved both Eq. (13) and Theorem 9.

2.5 Proof of Theorem 10

Directly from Theorem 9, we have proved the left-hand side in Eq. (14) in Theorem 10.

$$\kappa_3(\operatorname{arctanh} x)^2 \le \left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2.$$

Now, we will prove the right-hand side of Eq. (14). Combining with Eq. (30), let

$$\begin{aligned} &\frac{4}{\pi}(\operatorname{arctanh} x) - \left(\frac{-\ln(1-x^2)}{\arcsin x}\right) = \frac{4}{2\pi}\ln\left(\frac{1+\sin(s)}{1-\sin(s)}\right) - \frac{-\ln(1-(\sin s)^2)}{s} \triangleq \delta_7(s), \\ &\delta_8(s) = \delta_7'(s)\cdot s^2, \qquad \phi_4(s) = \frac{2(2\sin(s)s + 4\cos(s) - \pi)}{\pi}. \end{aligned} \tag{35}$$

It can be verified that

$$\phi_4''(s) = \frac{-4\sin(s)s}{\pi} < 0, \quad s \in (0, \pi/2), \qquad \phi_4'(0) = \phi_4(\pi/2) = 0,$$

which leads to

$$\phi_4(s) \geq 0, \qquad \delta_8'(s) = \frac{s}{(\cos s)^2}\phi_4(s) \geq 0, \qquad \delta_8(0) = 0, \quad s \in [0, \pi/2). \tag{36}$$

Combining Eq. (35) with Eq. (36), one obtains that

$$\delta_8(s) \geq 0, \qquad \delta_7'(s) \geq 0, \qquad \delta_7(0) = 0, \quad s \in [0, \pi/2). \tag{37}$$

Combining Eq. (35) with Eq. (37), we have that

$$\begin{aligned} &\delta_7(s) \geq 0, \quad s \in [0, \pi/2), \\ &0 \leq \left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2 \leq \left(\frac{4}{\pi}\operatorname{arctanh} x\right)^2, \quad x \in [0, 1). \end{aligned} \tag{38}$$

Note that $G_i(-x) = G_i(x)$, $i = 1, 2$, combining with Eq. (38), one obtains that

$$\left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2 \leq \kappa_2(\operatorname{arctanh} x)^2, \quad x \in (-1, 1). \tag{39}$$

Combining Theorem 9 with Eq. (39), we have completed the proofs of both Eq. (14) and Theorem 10.

3 Discussions and conclusions

The values of μ in Theorem 2 and β_1 in Theorem 5 tend to be $+\infty$ for r tends to be $\pm\infty$ and ± 1, respectively, while the values of κ_i in Theorems 7 and 10 are constant. The error plots of the bounds from Eq. (2) and Eq. (6) in [15], from Eq. (8) and Eq. (9) in [14], and from Eq. (6) and Eq. (13) are plotted in Fig. 1. It shows that the results of Eq. (11) and

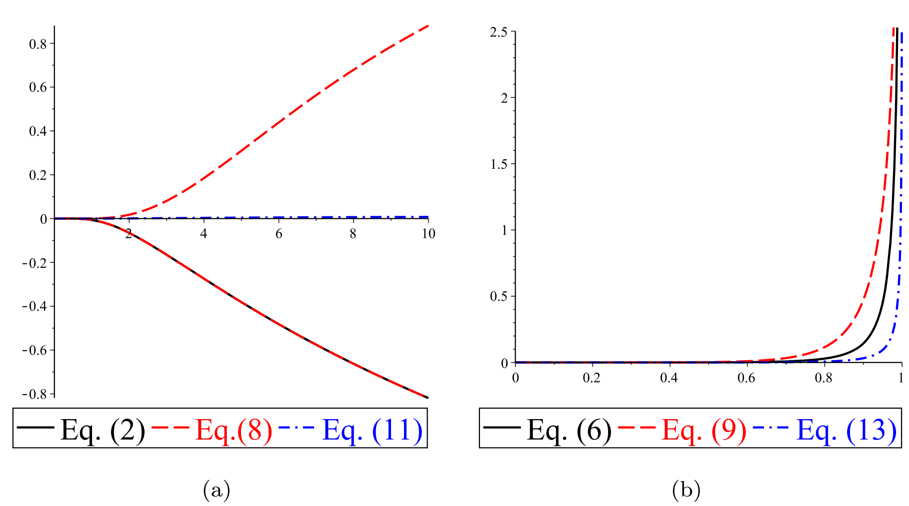

Figure 1 Error plots of bounds from (**a**) Eq. (2), (8), and (11); and (**b**) Eq. (6), (9), and (13)

Eq. (13) in this paper achieve better approximation effect than those of Eq. (2), Eq. (6), Eq. (8), and Eq. (9).

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Competing interests
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Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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