Pak *Journal of Inequalities and Applications* (2020) 2020:130
https://doi.org/10.1186/s13660-020-02395-9

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the John–Nirenberg inequality

Hee Chul Pak[1*]

*Correspondence:
hpak@dankook.ac.kr
[1]Department of Mathematics, Dankook University, Cheonan-si, Republic of Korea

Abstract

We present a version of the John–Nirenberg inequality for a sub-class of BMO by estimating the corresponding mean oscillating distribution function via dyadic decomposition. The dominating functions are of the form of decreasing step functions which are finer than the classical exponential functions and might be much more efficient for some sophisticated analysis. We also prove that the modified BMO-norm is equivalent to the classical BMO-norm under the convexity assumption.

MSC: 30H35; 42B35; 46E30; 49K20

Keywords: BMO; John–Nirenberg inequality; Bounded mean oscillation; Dyadic decomposition

1 Introduction

The space of functions of bounded mean oscillation(BMO) first appeared in the work of John and Nirenberg [1] in the context of nonlinear partial differential equations that arise in the study of minimal surfaces. The space BMO can be characterized by the form of the John–Nirenberg inequality: it says that for $f \in \mathrm{BMO}$, one has

$$\frac{1}{|Q|}\left|\left\{x \in Q : \left|f(x) - f_Q\right| > t\right\}\right| \le e \cdot \exp\left(-\frac{1}{2^d e}\frac{t}{\|f\|_{\mathrm{BMO}}}\right) \tag{1.1}$$

for every cube $Q \subset \mathbb{R}^d$ and every $t > 0$. Here $f_Q := \frac{1}{|Q|}\int_Q f(x)\,dx$. One of the important applications of the John–Nirenberg inequality (1.1) is the L^p characterization of BMO-norm; that is, for $1 < p < \infty$,

$$\|f\|_{\mathrm{BMO}_p} := \left(\sup_Q \frac{1}{|Q|}\int_Q \left|f(x) - f_Q\right|^p dx\right)^{\frac{1}{p}} \tag{1.2}$$

is an equivalent to the classical BMO-norm.

This paper refines the John–Nirenberg inequality in two ways. We present a dominating function F that is finer than the exponential functions at the right-hand side of estimate (1.1) (Theorem 4). The function F has a form of decreasing step functions which possesses more merits than the exponential function (see an application of Theorem 4 at the end of

Springer

Sect. 3). Secondly, we introduce an appropriate base function α such that the functional

$$\|f\|_{\mathrm{BMO}_\alpha} := \alpha^{-1}\left(\sup_Q \frac{1}{|Q|}\int_Q \alpha\left(\left|f(x) - f_Q\right|\right)dx\right) \tag{1.3}$$

sets the levels for those functions located in the space BMO and forms a norm of the space BMO, which is equivalent to the original norm under the convexity assumption.

The base functions α that we have developed include the base functions of the form $\alpha(x) = x^p$ that guarantee the classical result (1.2). We, in fact, designate the base functions α to achieve the Hölder inequality which always incubates the Minkowski type triangle inequality.

In this paper, $(X, \mathfrak{M}, \mu)$ represents an abstract measure space and $\bar{\mathbb{R}}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

2 Hölder functions

Notions of Hölder functions have been developed to find appropriate base functions that permit the Hölder inequality. In this section, we briefly introduce the fundamentals of the Hölder functions; the details can be found in [2–4]. The notions presented here are modified versions without essential differences.

1. A *pre-Hölder function* $\alpha : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ is a strictly increasing absolutely continuous function. For example, the exponential function $\alpha(x) = e^x$ and $\alpha(x) = x^p$ $(0 < p < \infty)$ are in this category. If there exists a pre-Hölder function β satisfying

$$\alpha^{-1}(x)\beta^{-1}(x) = x \tag{2.1}$$

for all $x \in \bar{\mathbb{R}}_+$, then β is called the *conjugate* (*pre-Hölder*) *function* of α. In relation (2.1), the notations α^{-1}, β^{-1} are the inverse functions of α, β, respectively. Some examples of pre-Hölder pairs are: $(\alpha(x), \beta(x)) = (x^p, x^q)$ for $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$(\alpha, \beta) := (\lambda \circ A, \lambda \circ \tilde{A}), \tag{2.2}$$

where we set $\lambda(x) = A^{-1}(x)\tilde{A}^{-1}(x)$ for any Orlicz N-function A together with its complementary N-function $\tilde{A}$.

2. The *spectrum* (or *exponent function*) p_α of a pre-Hölder function α is defined as

$$p_\alpha(x) := x\frac{\alpha'(x)}{\alpha(x)} \tag{2.3}$$

for almost every $x > 0$. For example, the Lebesgue base function $\alpha(x) = x^p$ has a point spectrum: $p_\alpha(x) = p$ with $1 \leq p < \infty$.

A pre-Hölder function α permits a conjugate function if and only if the bijection α satisfies the limit conditions

$$\lim_{x\to 0+} \frac{\alpha(x)}{x} = 0, \qquad \lim_{x\to\infty} \frac{\alpha(x)}{x} = \infty \tag{2.4}$$

together with the spectrum condition

$$p_\alpha(x) > 1 \tag{2.5}$$

for almost every $x > 0$. Also, for a pre-Hölder pair (α, β), we have

$$\frac{1}{p_\alpha}(s) + \frac{1}{p_\beta}(t) = 1 \quad \text{when } \alpha(s) = \beta(t).$$

For details, we refer to [4].

3. Let Φ be a two-variable function on $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$ defined by

$$\Phi(x,y) := \alpha^{-1}(x)\beta^{-1}(y).$$

Then we observe that the equation of the tangent plane T of Φ at a point $(\alpha(a), \beta(b))$ is represented by

$$T(x,y) = \frac{1}{p_\alpha}\frac{ab}{\alpha(a)}x + \frac{1}{p_\beta}\frac{ab}{\beta(b)}y + ab\theta_f \tag{2.6}$$

with $\theta_f = 1 - \frac{1}{p_\alpha} - \frac{1}{p_\beta}$. This motivates us to define the Hölder functions as follows.

Definition 1 Let $\hbar > 0$ be given. A pre-Hölder function α with the conjugate function β is said to be a *Hölder function* if, for any positive constants $a, b > 0$, there exist constants θ_1, θ_2, and θ_f (depending on a and b) such that

$$\theta_1 + \theta_2 + \theta_f \le \hbar$$

and that a dominating condition

$$\Phi(x,y) = \alpha^{-1}(x)\beta^{-1}(y) \le \theta_1\frac{ab}{\alpha(a)}x + \theta_2\frac{ab}{\beta(b)}y + ab\theta_f \tag{2.7}$$

holds for all $(x,y) \in \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$. A Hölder function α is called an *s-Hölder function* if we can choose $\theta_f = 0$ in (2.7).

For example, any (convex) function satisfying

$$\alpha(x) := \begin{cases} x^p & \text{for } 0 \le x \le 1, \\ x^q & \text{for sufficiently large } x \end{cases} \tag{2.8}$$

$(1 < p, q < \infty)$ is a Hölder function, and so are many variants of (2.8).

4. Let α be a Hölder function and β be the corresponding Hölder conjugate function. For the case when $(X, \mathfrak{M}, \mu)$ is an infinite measure space, we define

$$\|f\|_\alpha := \alpha^{-1}\left(\sup_{\mu(K)<\infty}\frac{1}{\mu(K)}\int_K \alpha\big(|f(x)|\big)\,d\mu\right), \tag{2.9}$$

where the supremum is taken over all measurable subsets K (of X) of finite measure. Hereafter, $⨍_K f(x)\,d\mu := \frac{1}{\mu(K)}\int_K f(x)\,d\mu$. Even though we define $\|\cdot\|_\alpha$ on an abstract measure space $(X, \mathfrak{M}, \mu)$, we restrict our attention to the Euclidean space $\mathbb{R}^d$ equipped with Lebesgue measure, and the measurable sets K at (2.9) are restricted to cubes Q of $\mathbb{R}^d$ in the next section.

For any measurable set K of finite measure, we have a Hölder type inequality

$$\left| \fint_K f(x)g(x)\,d\mu \right| \le \hbar \|f\|_\alpha \|g\|_\beta, \tag{2.10}$$

if the right-hand side is finite. We name the Hölder functions after Hölder inequality (2.10). For this reason, we briefly sketch the idea of the proof that justifies Definition 1. Let $a := \|f\|_\alpha \neq 0$ and $b := \|g\|_\beta \neq 0$. Then there exist θ_1, θ_2, and θ_f such that $\theta_1 + \theta_2 + \theta_f \le \hbar$ and

$$\begin{aligned} \left|f(x)g(x)\right| &= \alpha^{-1}\left(\alpha\left(\left|f(x)\right|\right)\right)\beta^{-1}\left(\beta\left(\left|g(x)\right|\right)\right) \\ &\le \theta_1 \frac{ab}{\alpha(a)}\alpha\left(\left|f(x)\right|\right) + \theta_2 \frac{ab}{\beta(b)}\beta\left(\left|g(x)\right|\right) + ab\theta_f. \end{aligned} \tag{2.11}$$

Integrating over K and dividing both sides by $\mu(K)$ yield

$$\begin{aligned} \fint_K \left|f(x)g(x)\right| d\mu &\le \theta_1 \frac{ab}{\alpha(a)} \fint_K \alpha\left(\left|f(x)\right|\right) d\mu + \theta_2 \frac{ab}{\beta(b)} \fint_K \beta\left(\left|g(x)\right|\right) d\mu \\ &\quad + \theta_f ab \fint_K d\mu \\ &\le \hbar \|f\|_\alpha \|g\|_\beta. \end{aligned}$$

When $(X, \mathfrak{M}, \mu)$ is a finite measure space, we define

$$\|f\|_\alpha := \alpha^{-1}\left(\int_X \alpha\left(\left|f(x)\right|\right) d\mu\right)$$

to get a version of Hölder type inequality

$$\left| \int_X f(x)g(x)\,d\mu \right| \le \hbar \|f\|_\alpha \|g\|_\beta.$$

The Hölder inequality always incubates the Minkowski type inequality

$$\|f_1 + f_2\|_\alpha \le \hbar\left\{\|f_1\|_\alpha + \|f_2\|_\alpha\right\}. \tag{2.12}$$

Also, for any constant $k \ge 0$ and $\|f\|_\alpha < \infty$, we have

$$\frac{k}{\hbar}\|f\|_\alpha \le \|kf\|_\alpha \le k\hbar\|f\|_\alpha.$$

In particular, when $\hbar = 1$, we have the homogeneity

$$\|kf\|_\alpha = k\|f\|_\alpha.$$

Quasi-homogeneity of the norm is good enough to exploit estimates for the existence theory of nonlinear partial differential equations[2, 3] and to study singular integrals on these genealogical function spaces.

5. We close this section by introducing the boundedness of the spectra of the convex base functions.

Proposition 2 (Boundedness of the spectrum) *Let (α,β) be a convex Hölder pair, and let p_α be the spectrum of α. Then there exist constants c_1, c_2 satisfying*

$$1 < c_1 \le p_\alpha(t) \le c_2 < \infty \tag{2.13}$$

for almost every $t > 0$. The same result holds for the spectrum p_β of β.

Proof From the convexity of α, we get

$$t\alpha'(t) \le \int_t^{2t} \alpha'(s)\,ds \le \int_0^{2t} \alpha'(s)\,ds = \alpha(2t) \quad \text{for } t > 0.$$

Hence in order to prove the second inequality $p_\alpha(t) \le c_2$, it is enough to demonstrate that $\alpha(2t) \le c\alpha(t)$ for some constant $c > 1$.

Suppose that there exists a sequence $\{t_j\}_{j=1}^\infty$ of positive numbers such that

$$\alpha(2t_j) \ge 2^j \alpha(t_j), \quad j = 1, 2, \ldots. \tag{2.14}$$

We choose a sequence of mutually disjoint measurable subsets $\{X_j\}_{j=1}^\infty$ of a finite measurable subset X of $\mathbb{R}^d$ such that

$$\mu(X_j) = \frac{\alpha(1)\mu(X)}{2^j \alpha(t_j)}.$$

Define $f(x) = \sum_{j=1}^\infty t_j \chi_{X_j}(x)$. Then we have

$$\int_X \alpha\bigl(\bigl|f(x)\bigr|\bigr)\,d\mu = \sum_{j=1}^\infty \alpha(t_j)\mu(X_j) = \alpha(1)\mu(X).$$

However, from (2.14) one has

$$\int_X \alpha\bigl(2\bigl|f(x)\bigr|\bigr)\,d\mu \ge \sum_{j=1}^\infty 2^j \alpha(t_j)\mu(X_j) = \infty,$$

which violates the Minkowski type inequality (2.12). In all, we have shown that there is a positive real number $c_2 > 1$ with

$$p_\alpha(t) \le c_2$$

for almost every $t > 0$.

The same argument on the conjugate function β of α delivers the existence of a positive constant $c_0 > 1$ for which

$$p_\beta(t) \le c_0 \quad \text{for almost every } t > 0, \tag{2.15}$$

that is,

$$\beta'(t) \le c_0 \frac{\beta(t)}{t}. \tag{2.16}$$

From identity (2.1), we notice that

$$x = \beta\left(\frac{x}{\alpha^{-1}(x)}\right) \quad \text{or} \quad \alpha(x) = \beta\left(\frac{\alpha(x)}{x}\right). \tag{2.17}$$

Hence the spectrum condition (2.16) is equivalent to saying

$$\beta'\left(\frac{\alpha(t)}{t}\right) \le c_0 t. \tag{2.18}$$

On the other hand, differentiate both sides of identity (2.1) to have

$$\frac{\beta^{-1}(x)}{\alpha'(\alpha^{-1}(x))} + \frac{\alpha^{-1}(x)}{\beta'(\beta^{-1}(x))} = 1,$$

which is equivalent to

$$\frac{\beta^{-1}(\alpha(t))}{\alpha'(t)} + \frac{t}{\beta'(\alpha(t))} = 1. \tag{2.19}$$

Then from the second identity of (2.17), we obtain $\beta^{-1} \circ \alpha(x) = \frac{\alpha(x)}{x}$. Therefore we find that identity (2.19) is equivalent to

$$\frac{y}{\alpha'(x)} + \frac{x}{\beta'(y)} = 1 \quad \text{for } y := \frac{\alpha(x)}{x}, \tag{2.20}$$

or

$$\alpha'(t) = s + t\frac{\alpha'(t)}{\beta'(s)}, \qquad s = \frac{\alpha(t)}{t}. \tag{2.21}$$

Reflecting (2.18) to identity (2.21), we have

$$\alpha'(t) \ge \frac{c_0}{c_0 - 1} \frac{\alpha(t)}{t},$$

which implies that

$$p_\alpha(t) \ge c_1$$

with $c_1 = \frac{c_0}{c_0 - 1}$. □

As a direct application of Proposition 2, we have the following.

Corollary 3 *Let (α, β) be a convex Hölder pair. Then there exist some constants c_1, c_2 satisfying*

$$t^{c_1} \le \alpha(t) \le t^{c_2} \tag{2.22}$$

for almost every $t > 0$.

Proof The result follows from solving the two separable ordinary differential inequalities

$$c_1 \le x\frac{\alpha'(x)}{\alpha(x)} \le c_2. \qquad \square$$

3 Refined John–Nirenberg inequality

In this section Q stands for a cube whose sides are parallel to the axes, and $|A|$ is the Lebesgue measure of the set A in $\mathbb{R}^d$, $d \ge 1$. For a locally integrable function f on $\mathbb{R}^d$, we let

$$f_Q = \fint_Q f(x)\,dx$$

and define distribution function μ_Q with respect to $f - f_Q$ by

$$\mu_Q(t) := \left|\left\{x \in Q : \left|f(x) - f_Q\right| > t\right\}\right|. \tag{3.1}$$

Let α be a pre-Hölder function. We denote a class of functions by

$$\mathrm{BMO}_\alpha\left(\mathbb{R}^d\right) = \left\{f \in L^1_{loc}\left(\mathbb{R}^d\right) : \|f\|_{\mathrm{BMO}_\alpha} < \infty\right\},$$

where we set

$$\|f\|_{\mathrm{BMO}_\alpha} = \alpha^{-1}\left(\sup_Q \fint_Q \alpha\left(\left|f(x) - f_Q\right|\right)dx\right).$$

When α is the identity function, we write $\mathrm{BMO}_\alpha(\mathbb{R}^d) := \mathrm{BMO}(\mathbb{R}^d)$.

We now state a refined version of the John–Nirenberg inequality on the sub-classes of BMO equipped with the huge class of (convex pre-)Hölder base functions α.

Theorem 4 (Refinement of the John–Nirenberg inequality) *Let α be a convex pre-Hölder function or Hölder function and $f \in \mathrm{BMO}_\alpha$. For every $\lambda > 1$, define step functions F_λ by*

$$F_\lambda(t) = \sum_{k=0}^{\infty}\left(\frac{1}{\lambda}\right)^k \chi_{(0,T]}(t - kT) \quad (t > 0),$$

where $T := 2^d\alpha^{-1}(\lambda\alpha(\|f\|_{\mathrm{BMO}_\alpha}))$. Then one has

$$\frac{1}{|Q|}\mu_Q(t) \le F_\lambda(t) \quad (t > 0) \tag{3.2}$$

for every cube $Q \subset \mathbb{R}^d$ and every $\lambda > 1$. The upper bound function F_λ is finer than the exponential function, that is to say, for every $\lambda > 1$, we have

$$\frac{1}{|Q|}\mu_Q(t) \le F_\lambda(t) \le \lambda\exp\left(-\frac{t\ln\lambda}{2^d\alpha^{-1}(\lambda\alpha(\|f\|_{\mathrm{BMO}_\alpha}))}\right). \tag{3.3}$$

Proof We define

$$E_Q(t) := \left\{x \in Q : \left|f(x) - f_Q\right| > t\right\},$$

$$\Theta(t) := \sup_Q \frac{\mu_Q(t)}{|Q|}.$$

We want to show that, for any $\lambda > 1$,

$$\Theta(t) \le \sum_{k=0}^{\infty} \frac{1}{\lambda^k} \chi_{(0,T]}(t - kT) \lesssim e^{b-\frac{t}{c}}$$

for some positive constants b and c. First, we observe that $\Theta(t) \le 1$ for all $t > 0$. So, obviously, we have

$$\Theta(t) \le \chi_{(0,T]}(t) \le e^{b-b\frac{t}{T}} \quad \text{for } 0 < t \le T, \tag{3.4}$$

where $b > 0$ and $T > 0$ are constants to be chosen later. Now take a number s so large that $s > 1$ and

$$\|f\|_{\mathrm{BMO}_\alpha} < s.$$

Then we have

$$⨍_Q \alpha\bigl(|f(x) - f_Q|\bigr)\,dx < \alpha(s) \tag{3.5}$$

for any cube Q.

For a cube Q, we employ a dyadic decomposition of Q as follows: Sub-divide Q into 2^d equal closed sub-cubes of side length equal to half of the side length of Q by bisecting the sides, and select such sub-cubes $Q_1^1, Q_2^1, \dots, Q_{2^d}^1$. Repeat this process for any sub-cubes Q_j^1 of the second generation to produce the third generation $Q_1^2, Q_2^2, \dots, Q_{2^{2d}}^2$. Continuing this process, we obtain the collection of all dyadic cubes of Q, to say $\mathcal{D}(Q)$, and $\mathcal{F}(Q)$ is the collection of all the *first* cubes $Q' \in \mathcal{D}(Q)$ such that

$$⨍_{Q'} \alpha\bigl(|f(x) - f_Q|\bigr)\,dx > \alpha(s),$$

where the "first Q'" means that, for any cube $\tilde{Q} \in \mathcal{D}(Q)$ with $Q' \subset \tilde{Q}$, $\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \alpha(|f(x) - f_Q|)\,dx < \alpha(s)$. Then we note that

$$\left\{x \in Q : \sup_{x \in Q' \in \mathcal{D}(Q)} ⨍_{Q'} \alpha\bigl(|f(x) - f_Q|\bigr)\,dx > \alpha(s)\right\} = \bigcup_{Q' \in \mathcal{F}(Q)} Q'.$$

Also, by virtue of the Lebesgue differentiation theorem, for almost every $x \in E_Q(s)$, there exists $Q' \in \mathcal{D}(Q)$ (indeed, $Q' \in \mathcal{F}(Q)$) such that $x \in Q'$ and

$$\alpha(s) < ⨍_{Q'} \alpha\bigl(|f(y) - f_Q|\bigr)\,dy.$$

This means that the statement

$$\text{“}x \in E_Q(s) \quad \text{implies} \quad x \in \bigcup_{Q' \in \mathcal{F}(Q)} Q'\text{”} \tag{3.6}$$

holds for almost every x. For $Q' \in \mathcal{F}(Q)$, we take the parent $\tilde{Q}$ of Q', that is, the smallest cube $\tilde{Q} \in \mathcal{D}(Q) \setminus \mathcal{F}(Q)$ containing Q'. Then Jensen's inequality for the convex function α or the Hölder type inequality (2.10) for the Hölder function α leads to

$$\begin{aligned} \frac{1}{|Q'|} \int_{Q'} \left|f(x) - f_Q\right| dx &\le \left(\frac{|\tilde{Q}|}{|Q'|}\right) \alpha^{-1}\left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \alpha\left(\left|f(x) - f_Q\right|\right) dx\right) \\ &\le 2^d s. \end{aligned} \tag{3.7}$$

Hence from the fact that for $x \in E_Q(t)$ with $t > 2^d s$,

$$\begin{aligned} t < \left|f(x) - f_Q\right| &\le \left|f(x) - f_{Q'}\right| + |f_{Q'} - f_Q| \\ &\le \left|f(x) - f_{Q'}\right| + \frac{1}{|Q'|} \int_{Q'} \left|f(x) - f_Q\right| dx \\ &\le \left|f(x) - f_{Q'}\right| + 2^d s, \end{aligned}$$

we have

$$t - 2^d s < \left|f(x) - f_{Q'}\right|.$$

By (3.6), we obtain that for $t > 2^d s$,

$$\begin{aligned} \mu_Q(t) &= \left|E_Q(s) \cap E_Q(t)\right| \\ &\le \left|\bigcup_{Q' \in \mathcal{F}(Q)} Q' \cap E_Q(t)\right| \\ &\le \sum_{Q' \in \mathcal{F}(Q)} \left|Q' \cap E_Q(t)\right| \\ &\le \sum_{Q' \in \mathcal{F}(Q)} \frac{|\{x \in Q' : |f(x) - f_{Q'}| > t - 2^d s\}|}{|Q'|} \left|Q'\right| \\ &\le \Theta\left(t - 2^d s\right) \sum_{Q' \in \mathcal{F}(Q)} \left|Q'\right|. \end{aligned}$$

Therefore, by (3.7) and (3.5), we get

$$\begin{aligned} \mu_Q(t) &\le \Theta\left(t - 2^d s\right) \sum_{Q' \in \mathcal{F}(Q)} \frac{1}{\alpha(s)} \int_{Q'} \alpha\left(\left|f(x) - f_Q\right|\right) dx \\ &= \frac{1}{\alpha(s)} \Theta\left(t - 2^d s\right) \int_{\cup Q'} \alpha\left(\left|f(x) - f_Q\right|\right) dx \\ &\le \frac{1}{\alpha(s)} \Theta\left(t - 2^d s\right) \cdot |Q| \cdot \alpha\left(\|f\|_{\mathrm{BMO}_\alpha}\right) \end{aligned}$$

for $t > 2^d s$. That is,

$$\frac{1}{|Q|} \mu_Q(t) \le \frac{\alpha(\|f\|_{\mathrm{BMO}_\alpha})}{\alpha(s)} \Theta\left(t - 2^d s\right).$$

We take the supremum over Q to have

$$\Theta(t) \le \frac{\Theta(t-T)}{\lambda} \quad (t > T), \tag{3.8}$$

where we put $\lambda = \frac{\alpha(s)}{\alpha(\|f\|_{\mathrm{BMO}_\alpha})} > 1$ and $T := 2^d s = 2^d \alpha^{-1}(\lambda\alpha(\|f\|_{\mathrm{BMO}_\alpha}))$.

We revisit (3.4) to have

$$\Theta(t) \le \chi_{(0,T]}(t) \quad \text{for } 0 < t \le T. \tag{3.9}$$

Applying (3.9) into (3.8), we have

$$\Theta(t) \le \frac{\Theta(t-T)}{\lambda} \le \frac{1}{\lambda}\chi_{(0,T]}(t-T) \quad (T < t \le 2T).$$

Continue this process to get, for $t > 0$,

$$\Theta(t) \le \sum_{k=1}^{\infty} \frac{1}{\lambda^{k-1}} \chi_{(0,T]}\bigl(t-(k-1)T\bigr).$$

In order to find a smooth dominating function for $F_\lambda(t)$, consider (3.4) to have

$$\chi_{(0,T]}(t) \le \exp\left(b - b\frac{t}{T}\right) \quad \text{for } 0 < t \le T = 2^d \alpha^{-1}\bigl(\lambda\alpha\bigl(\|f\|_{\mathrm{BMO}_\alpha}\bigr)\bigr). \tag{3.10}$$

Then we have that, for $kT < t \le (k+1)T$,

$$\frac{1}{\lambda^k}\chi_{(0,T]}(t-kT) \le \frac{1}{\lambda^k}\exp\left(b - b\frac{t-kT}{T}\right) = \frac{1}{\lambda^k}e^{kb}\exp\left(b - b\frac{t}{T}\right).$$

Choosing $b = \ln\lambda$, we obtain that, for $t > 0$,

$$F_\lambda(t) \le \lambda \exp\left(-\frac{t\ln\lambda}{2^d\alpha^{-1}(\lambda\alpha(\|f\|_{\mathrm{BMO}_\alpha}))}\right)$$

for every $\lambda > 0$. The proof is now completed. □

By Theorem 4, we can see that

$$\begin{aligned}
\|f\|_{\mathrm{BMO}_\alpha} &= \alpha^{-1}\left(\sup_Q \frac{1}{|Q|}\int_0^\infty \alpha'(t)\mu_Q(t)\,dt\right) \\
&\le \alpha^{-1}\left(\int_0^\infty \alpha'(t)F_\lambda(t)\,dt\right) \\
&= \alpha^{-1}\left(\sum_{k=0}^{\infty}\left(\frac{1}{\lambda}\right)^k \int_{kT}^{(k+1)T} \alpha'(t)\,dt\right) \\
&= \alpha^{-1}\left(\sum_{k=0}^{\infty} \frac{\alpha((k+1)T) - \alpha(kT)}{\lambda^k}\right).
\end{aligned} \tag{3.11}$$

When α is a convex pre-Hölder function or the Hölder function, Jensen's inequality or the Hölder type inequality (2.10), respectively, yields

$$\|f\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}_\alpha}. \tag{3.12}$$

We can argue that for a convex Hölder pair (α, β), the classical $\mathrm{BMO}(\mathbb{R}^d)$-norm and the modified $\mathrm{BMO}_\alpha(\mathbb{R}^d)$-norm are equivalent: For it, we will demonstrate that the identity function $\mathrm{id} : \mathrm{BMO} \to \mathrm{BMO}_\alpha$ is continuous at $\mathbf{0}$, which implies that

$$\|f\|_{\mathrm{BMO}_\alpha} \leq C\|f\|_{\mathrm{BMO}}$$

for some $C > 0$. In fact, for any sequence $\{f_n\}$ converging to $\mathbf{0}$ in $\mathrm{BMO}(\mathbb{R}^d)$, we have

$$\alpha\left(\|f_n\|_{\mathrm{BMO}_\alpha}\right) = \sup_Q \int_0^\infty \alpha'(t) \frac{\mu_Q(t)}{|Q|}\, dt.$$

We apply Theorem 4 for the case when the base function α is the identity function so that we have

$$\frac{1}{|Q|}\mu_Q(t) \leq F_2(t) = \sum_{k=0}^\infty \left(\frac{1}{2}\right)^k \chi_{(0,T]}(t - kT),$$

where $T := 2^{d+1}\|f\|_{\mathrm{BMO}}$. Then, as in (3.11), we obtain

$$\begin{aligned}\alpha\left(\|f_n\|_{\mathrm{BMO}_\alpha}\right) &\leq \sum_{k=0}^\infty \frac{\alpha((k+1)2^{d+1}\|f_n\|_{\mathrm{BMO}}) - \alpha(k2^{d+1}\|f_n\|_{\mathrm{BMO}})}{2^k} \\ &\leq \sum_{k=0}^\infty \frac{\alpha((k+1)2^{d+1}\|f_n\|_{\mathrm{BMO}})}{2^k}.\end{aligned}$$

Corollary 3 yields

$$\left(\|f_n\|_{\mathrm{BMO}_\alpha}\right)^{c_1} \leq \alpha\left(\|f_n\|_{\mathrm{BMO}_\alpha}\right) \leq \sum_{k=0}^\infty \frac{((k+1)2^{d+1}\|f_n\|_{\mathrm{BMO}})^{c_2}}{2^k} \lesssim \|f_n\|_{\mathrm{BMO}}^{c_2}$$

for some $1 < c_1 < c_2$. Therefore we obtain the equivalence between $\mathrm{BMO}(\mathbb{R}^d)$-norm and $\mathrm{BMO}_\alpha(\mathbb{R}^d)$-norm:

Corollary 5 *Let (α, β) be a convex Hölder pair. Then there exists a constant C_α such that*

$$\|f\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}_\alpha} \leq C_\alpha\|f\|_{\mathrm{BMO}}.$$

Acknowledgements
The author would like to thank the anonymous referees.

Funding
This research was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2019R1I1A3A01057195).

Availability of data and materials
Not applicable.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author alone read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 January 2020 Accepted: 24 April 2020 Published online: 07 May 2020

References

1. John, F., Nirenberg, L.: On functions of bounded mean oscillation. Commun. Pure Appl. Math. **14**, 415–426 (1961)
2. Pak, H.-C.: Existence of solutions for a nonlinear elliptic equation with general flux term. Fixed Point Theory Appl. (2011). https://doi.org/10.1155/2011/496417
3. Pak, H.-C., Park, Y.-J.: Trace operator and a nonlinear boundary value problem in a new space. Bound. Value Probl. (2014). https://doi.org/10.1186/s13661-014-0153-z
4. Pak, H.-C., Park, Y.-J.: Spectrum and singular integrals on a new weighted function space. Acta Math. Sin. Engl. Ser. **34**(11), 1692–1702 (2018)