


RESEARCH

Open Access



# Endpoint boundedness for commutators of singular integral operators on weighted generalized Morrey spaces

Jinyun Qi<sup>1,2</sup>, Xuefang Yan<sup>1</sup> and Wenming Li<sup>1\*</sup> 

\*Correspondence:  
[lwmingg@sina.com](mailto:lwmingg@sina.com)

<sup>1</sup>School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, China  
Full list of author information is available at the end of the article

## Abstract

In this paper, we obtain the endpoint boundedness for the commutators of singular integral operators with *BMO* functions and the associated maximal operators on weighted generalized Morrey spaces. We also get similar results for the commutators of fractional integral operators with *BMO* functions and the associated maximal operators.

**MSC:** 42B20; 47G10

**Keywords:** Singular integral operator; Fractional integral operator; Commutator; Maximal operator; Weighted generalized Morrey space

## 1 Introduction and main results

The Morrey spaces were introduced by Morrey in [11] to investigate the local behavior of solutions to second order elliptic partial differential equations. Chiarenza and Frasca [2] showed the boundedness of the Hardy–Littlewood maximal operator, singular integral operators, and fractional integral operators on the Morrey spaces.

Let  $f$  be a measurable function on  $\mathbb{R}^n$ . The Hardy–Littlewood maximal function is defined by

$$M(f)(x) = \sup_B \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls  $B$  containing  $x$ .

We say that  $T$  is a singular integral operator if there exists a function  $K$  which satisfies the following conditions:

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y) dy,$$
$$|K(x)| \leq \frac{C}{|x|^n}, \quad |\nabla K(x)| \leq \frac{C}{|x|^{n+1}}, \quad x \neq 0.$$

© The Author(s) 2020. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

The  $BMO(\mathbb{R}^n)$  space is defined by

$$BMO(\mathbb{R}^n) = \left\{ b \in L_{loc}(\mathbb{R}^n) : \|b\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx < \infty \right\},$$

where  $b_B = \frac{1}{|B|} \int_B b(y) dy$ .

For the singular integral operator  $T$  and  $b \in BMO$ , the commutator  $[b, T]$  is defined by

$$[b, T]f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))K(x - y)f(y) dy.$$

For  $1 < p < \infty$ , we say a weight  $w \in A_p$  if

$$[w]_p = \sup_B \left( \frac{1}{|B|} \int_B w(y) dy \right) \left( \frac{1}{|B|} \int_B w(y)^{-p'/p} dy \right)^{p/p'} < \infty.$$

For  $p = 1$ , we write  $w \in A_1$  if  $Mw(y) \leq Cw(y)$ , a.e.  $y \in \mathbb{R}^n$ .

It is a classical result that the operators  $T$  are bounded on  $L^p(w)$  whenever  $1 < p < \infty$  and  $w \in A_p$ , and for  $p = 1$  and  $w \in A_1$ , we have the weak type result which can be found in [9]. Komori and Shirai extended them to the weighted Morrey spaces in [10].

Let  $f$  be a measurable function on  $\mathbb{R}^n$  and  $1 \leq p < \infty$ ,  $0 \leq \kappa < 1$ . For two weights  $w$  and  $u$ , the weighted Morrey space is defined by

$$L^{p,\kappa}(w, u) = \{f \in L_{loc}^p(w) : \|f\|_{L^{p,\kappa}(w,u)} < \infty\},$$

where

$$\|f\|_{L^{p,\kappa}(w,u)} = \sup_B \left( \frac{1}{u(B)^\kappa} \int_B |f(x)|^p w(x) dx \right)^{\frac{1}{p}},$$

and the supremum is taken over all balls  $B$  in  $\mathbb{R}^n$ . When  $w = u$ , we write  $L^{p,\kappa}(w, u)$  as  $L^{p,\kappa}(w)$ . Komori and Shirai in [10] proved that, for  $1 < p < \infty$  and  $w \in A_p$ ,  $T$  and  $[b, T]$  are bounded on  $L^{p,\kappa}(w)$ , and if  $p = 1$  and  $w \in A_1$ , then for all  $t > 0$  and any ball  $B$ ,

$$w(\{x \in B : |Tf(x)| > t\}) \leq \frac{C}{t} \|f\|_{L^{1,\kappa}(w)} w(B)^\kappa.$$

Qi et al. [14] obtained the weighted endpoint estimates for the commutators of the singular integral operators with  $BMO$  functions and associated maximal operators on the weighted Morrey space  $L^{1,\kappa}(w)$ . They also gave similar results for the commutators of the fractional integral operators with  $BMO$  functions and associated maximal operators.

Let  $w$  and  $u$  be two weights and  $1 \leq q \leq \beta \leq p \leq \infty$ . We define the generalized two-weight Morrey space  $(L^q(w), L^p(u))^\beta := (L^q(w), L^p(u))^\beta(\mathbb{R}^n)$  as the space of all measurable functions  $f$  satisfying  $\|f\|_{(L^q(w), L^p(u))^\beta} < \infty$ , where

$$\|f\|_{(L^q(w), L^p(u))^\beta} := \sup_{r>0} r \|f\|_{(L^q(w), L^p(u))^\beta},$$

with

$$r \|f\|_{(L^q(w), L^p(u))^\beta} = \left( \int_{\mathbb{R}^n} (u(B(y, r)))^{\frac{1}{\beta} - \frac{1}{q} - \frac{1}{p}} \|f\|_{L^q(w)}^p dy \right)^{\frac{1}{\beta}},$$

for any  $r > 0$ , with the usual modification when  $p = \infty$ . In the case  $w = u$ , the spaces  $(L^q(w), L^p(u))^\beta$  are the spaces  $(L^q(w), L^p)^\beta$  defined by Feuto in [7]. In the case  $w = u \equiv 1$ , the spaces  $(L^q(w), L^p(u))^\beta$  are the spaces  $(L^q, L^p)^\beta$  defined in [8] by Fofana. For  $q < \beta$  and  $p = \infty$ , the space  $(L^q(w), L^p)^\beta$  is the weighted Morrey space  $L^{q,\kappa}(w)$  with  $\kappa = \frac{1}{q} - \frac{1}{\beta}$ .

Feuto [7] proved that the singular integral operators, the commutators of the singular integral operators with *BMO* functions, and other operators were bounded on these generalized weighted Morrey spaces  $(L^q(w), L^p)^\beta$  for  $q > 1$ . Here we consider the boundedness of the commutators of the singular integral operators with *BMO* functions on the endpoint generalized weighted Morrey space  $(L^1(w), L^p)^\beta$ . The weighted endpoint estimates for the commutators of the singular integral operators with *BMO* functions have many applications in partial differential equations. The *BMO* functions and the associated maximal operators can be applied in optimization problems, see [5, 6].

Let  $\Psi : [0, \infty) \rightarrow [0, \infty)$  be an increasing function. We define space  $L^{\Psi,\infty}(w)$  as the space of all measurable functions  $f$  satisfying  $\|f\|_{L^{\Psi,\infty}(w)} < \infty$ , where

$$\|f\|_{L^{\Psi,\infty}(w)} := \sup_{t>0} t\Psi(w\{x \in \mathbb{R}^n : |f(x)| > t\}).$$

When  $\Psi(t) = t^{1/p}$  with  $0 < p < \infty$ , then the space  $L^{\Psi,\infty}(w)$  is the weak weighted Lebesgue space  $L^{p,\infty}(w)$ .

Let  $w, u$  be two weights,  $\Psi : [0, \infty) \rightarrow [0, \infty)$  be an increasing function and  $1 \leq \beta \leq p \leq \infty$ . We define the generalized weak weighted Morrey space  $(L^{\Psi,\infty}(w), L^p(u))^\beta$  as the space of all measurable functions  $f$  satisfying  $\|f\|_{(L^{\Psi,\infty}(w), L^p(u))^\beta} < \infty$ , where

$$\|f\|_{(L^{\Psi,\infty}(w), L^p(u))^\beta} := \sup_{r>0} r \|f\|_{(L^{\Psi,\infty}(w), L^p(u))^\beta},$$

with

$$r \|f\|_{(L^{\Psi,\infty}(w), L^p(u))^\beta} = \left( \int_{\mathbb{R}^n} (u(B(y, r)))^{\frac{1}{\beta}-1-\frac{1}{p}} \|f\chi_{B(y, r)}\|_{L^{\Psi,\infty}(w)}^p dy \right)^{\frac{1}{p}}.$$

When  $\Psi(t) = t$ ,  $w = u$ , the space  $(L^{\Psi,\infty}(w), L^p(u))^\beta$  is the generalized weak weighted Morrey space  $(L^{1,\infty}(w), L^p)^\beta$  defined in [7]. Feuto proved for the singular integral operator  $T$ , if  $w \in A_1$ , then

$$\|Tf\|_{(L^{1,\infty}(w), L^p)^\beta} \leq C \|f\|_{(L^1(w), L^p)^\beta}.$$

In this paper, we extend the methods used in [14] and obtain the endpoint boundedness for the commutators of the singular integral operators with *BMO* functions and the associated maximal operators on the generalized weighted Morrey spaces  $(L^1(w), L^p)^\beta$ . The results are more general than [14] and have different forms. We also give similar results for the commutators of the fractional integral operators with *BMO* functions and the associated maximal operators.

In order to state our results, we need to recall some notations and facts about the Young functions and Orlicz spaces; for further information, see [1]. A function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  is a Young function if it is convex and increasing, and if  $\Phi(0) = 0$  and  $\Phi(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

Given a locally integrable function  $f$  and a Young function  $\Phi$ , define the mean Luxemburg norm of  $f$  on a ball  $B$  by

$$\|f\|_{\Phi, B} = \inf \left\{ \lambda > 0 : \frac{1}{|B|} \int_B \Phi \left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

For  $\alpha, 0 \leq \alpha < n$ , and a Young function  $\Phi$ , we define the Orlicz maximal operator

$$M_{\alpha, \Phi} f(x) = \sup_{B \ni x} |B|^{\frac{\alpha}{n}} \|f\|_{\Phi, B}.$$

If  $\alpha = 0$ , we write  $M_{\alpha, \Phi}$  simply as  $M_{\Phi}$ . If  $\alpha = 0$  and  $\Phi(t) = t$ ,  $M_{\alpha, \Phi}$  is the Hardy–Littlewood maximal operator  $M$ . If  $\Phi_{\varepsilon}(t) = t \log(e + t)^{\varepsilon}$ ,  $\varepsilon \geq 0$ , we write  $M_{\Phi_{\varepsilon}}$  simply as  $M_{L(\log L)^{\varepsilon}}$ .

If  $0 < \alpha < n$  and  $\Phi(t) = t$ ,  $M_{\alpha, \Phi}$  is a fractional maximal operator of order  $\alpha$ , and we write it as  $M_{\alpha}$ . If  $\Phi_{\varepsilon}(t) = t \log(e + t)^{\varepsilon}$ , we write  $M_{\alpha, \Phi}$  simply as  $M_{\alpha, L(\log L)^{\varepsilon}}$ .

Given  $\alpha, 0 < \alpha < n$ , for an appropriate function  $f$  on  $\mathbb{R}^n$ , the fractional integral operator (or the Riesz potential) of order  $\alpha$  is defined by

$$I_{\alpha} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy.$$

For  $b \in BMO(\mathbb{R}^n)$ , we define the commutators of the operator  $I_{\alpha}$  and  $b$  by

$$[b, I_{\alpha}] f(x) = \int_{\mathbb{R}^n} \frac{(b(x) - b(y))f(y)}{|x - y|^{n-\alpha}} dy.$$

A weight  $w$  is said to belong to the class  $A_{p, q}$  for  $1 < p, q < \infty$  if there exists a positive constant  $C$  such that, for any ball  $B$  in  $\mathbb{R}^n$ ,

$$\left( \frac{1}{|B|} \int_B w(x)^q dx \right)^{1/q} \left( \frac{1}{|B|} \int_B w(x)^{-p'} dx \right)^{1/p'} \leq C < \infty.$$

The following theorems are our main results.

**Theorem 1.1** *If  $1 < q \leq \beta < p < \infty$  and  $w \in A_q$ , then the Hardy–Littlewood maximal operator  $M$  and  $M_{L(\log L)}$  are bounded on  $(L^q(w), L^p)^{\beta}$ .*

*If  $q = 1 \leq \beta < p < \infty$  and  $w \in A_1$ , then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|M(f)\|_{(L^{1, \infty}(w), L^p)^{\beta}} \leq C \|f\|_{(L^1(w), L^p)^{\beta}}.$$

**Theorem 1.2** *Let  $1 \leq \beta < p \leq \infty$ ,  $w \in A_1$ ,  $\Phi(t) = t \log(e + t)$ , and  $\Psi(t) = \frac{1}{\Phi(1/t)} = \frac{t}{\log(e+t-1)}$ , then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|\Psi(M_{L(\log L)} f)\|_{(L^{1, \infty}(w), L^p)^{\beta}} \leq C \|\Phi(|f|)\|_{(L^1(w), L^p)^{\beta}}.$$

**Theorem 1.3** *Let  $T$  be any singular integral operator,  $w \in A_1$ ,  $\Phi(t) = t \log(e + t)$ ,  $\Psi(t) = \frac{t}{\log(e+t-1)}$ , and  $b \in BMO$ ,  $1 \leq \beta < p \leq \infty$ . Then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|\Psi(|[b, T]f|)\|_{(L^{1, \infty}(w), L^p)^{\beta}} \leq C \|\Phi(|f|)\|_{(L^1(w), L^p)^{\beta}}.$$

We also study similar estimates for the commutators of the fractional integral operators with *BMO* functions and the associated maximal operators and get the following results.

**Theorem 1.4** *Let  $0 < \alpha < n$ ,  $w \in A_1$ ,  $1/q = 1 - \alpha/n$ ,  $1 \leq \beta < p \leq \infty$ , and  $0 < 1 + 1/p - 1/\beta < 1/q$ ,  $\Phi(t) = t \log(e + t)$ ,  $\Psi(t) = \frac{t}{\log(e+t-1)}$ ,  $\Gamma(t) = t^{1/q} \log(e + t)^{-1}$ , and  $\Theta(t) = t^{1/q} \log(e + t^{-1})$ . Then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|\Psi(M_{\alpha,L(\log L)}f)\|_{(L^{\Gamma,\infty(w),L^p})^\beta} \leq C \|\Phi(|f|)\|_{(L^1(\Theta(w)),L^p(w))^\beta}.$$

**Theorem 1.5** *Let  $0 < \alpha < n$ ,  $w \in A_1$ ,  $b \in BMO$ ,  $1/q = 1 - \alpha/n$ ,  $1 \leq \beta < p \leq \infty$ , and  $0 < 1 + 1/p - 1/\beta < 1/q$ ,  $\Phi(t) = t \log(e + t)$ ,  $\Psi(t) = \frac{t}{\log(e+t-1)}$ ,  $\Gamma(t) = t^{1/q} \log(e + t)^{-1}$ , and  $\Theta(t) = t^{1/q} \log(e + t^{-1})$ . Then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|\Psi([b, I_\alpha]f)\|_{(L^{\Gamma,\infty(w),L^p})^\beta} \leq C \|\Phi(|f|)\|_{(L^1(\Theta(w)),L^p(w))^\beta}.$$

From these results, we see that the commutators of the fractional integral operators with the *BMO* functions and the associated maximal operators map the weighted Morrey spaces to some weighted Orlicz–Morrey spaces. Hence we can further consider the boundedness for these integral operators on general weighted Orlicz–Morrey spaces.

**2 Proof of Theorem 1.1, Theorem 1.2, and Theorem 1.3**

**Lemma 2.1** ([9]) *Let  $w \in A_\infty$ , then there exists a constant  $C > 0$  such that, for any cube  $Q$ ,  $w(2Q) \leq Cw(Q)$ .*

**Lemma 2.2** ([9]) *Let  $1 < p < \infty$  and  $w \in A_p$ . Then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|M(f)\|_{L^p(w)} \leq C\|f\|_{L^p(w)}.$$

*Let  $w \in A_1$ . Then there exists a constant  $C > 0$  independent of  $f$  such that*

$$\|M(f)\|_{L^1,\infty(w)} \leq C\|f\|_{L^1(w)}.$$

**Lemma 2.3** ([15]) *There exists a constant  $C > 0$  such that, for any ball  $B$  and all  $x \in B$ ,*

$$M(f \chi_{(2B)^c})(x) \leq C \sum_{i=1}^\infty \frac{1}{|2^{i+1}B|} \int_{2^{i+1}B} |f(y)| dy$$

*for every locally integrable function  $f$ .*

**Lemma 2.4** ([12]) *Let  $\Phi(t) = t \log(e + t)$ , then there exists a positive constant  $C$  such that, for any weight  $w$  and all  $t > 0$ ,*

$$w(\{x \in \mathbb{R}^n : M_{L(\log L)}f(x) > t\}) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) Mw(x) dx$$

*for every locally integrable function  $f$ .*

**Lemma 2.5** ([9]) *Let  $w \in A_1$ , then there exists a constant  $C > 0$  and  $\eta > 0$  such that, for any ball  $B$  and a measurable subset  $E \subset B$ ,*

$$\frac{w(E)}{w(B)} \leq C \left( \frac{|E|}{|B|} \right)^\eta.$$

**Lemma 2.6** ([7]) *Let  $1 \leq s \leq q < \infty$ ,  $w \in A_{q/s}$ , and  $T : L^q_{\text{loc}}(w) \rightarrow L^q_{\text{loc}}(w)$  a sublinear operator which satisfies the following property: for all balls  $B \subset \mathbb{R}^n$ ,  $x \in B$ ,*

$$T(f \chi_{(2B)^c})(x) \leq C \sum_{i=1}^{\infty} i \left( \frac{1}{|2^{i+1}B|} \int_{2^{i+1}B} |f(y)|^s dy \right)^{1/s}.$$

Then

- (1) *if  $q > 1$  and  $T$  is bounded on  $L^q(w)$ , then it is also bounded on  $(L^q(w), L^p)^\beta$  for  $q \leq \beta < p \leq \infty$ ,*
- (2) *if for all  $\lambda > 0$ ,*

$$w(\{x \in \mathbb{R}^n : |T(f)(x)| > \lambda\}) \leq C \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(y)| dy w(y) dy,$$

*then for  $1 \leq \beta < p \leq \infty$ ,  $T$  is bounded on  $(L^1(w), L^p)^\beta$  to  $(L^{1,\infty}(w), L^p)^\beta$ .*

*Proof of Theorem 1.1* By Lemma 2.2, Lemma 2.3, and Lemma 2.6, we obtain that the Hardy–Littlewood maximal operator  $M$  is bounded on  $(L^q(w), L^p)^\beta$  for  $w \in A_q$ , and for  $w \in A_1$ , then there exists a constant  $C > 0$  independent of  $f$  such that

$$\|M(f)\|_{(L^{1,\infty}(w), L^p)^\beta} \leq C \|f\|_{(L^1(w), L^p)^\beta}.$$

Because  $M_{L(\log L)} \approx M^2$ , which was obtained by Perez in [12], we have  $M_{L(\log L)}$  is bounded on  $(L^q(w), L^p)^\beta$ . This ends the proof. □

*Proof of Theorem 1.2* Fix  $y \in \mathbb{R}^n$  and  $r > 0$ , let  $B = B(y, r)$  be a ball centered at  $y$  with radius  $r$ . By Lemma 2.4, we have

$$\begin{aligned} w(\{x \in B : M_{L(\log L)} f(x) > t\}) &= \int_{\{x \in \mathbb{R}^n : M_{L(\log L)} f(x) > t\}} \chi_B(x) w(x) dx \\ &\leq C \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{t} \right) M(\chi_B w)(x) dx \\ &\leq C \left( \int_{3B} + \int_{(3B)^c} \right) \Phi \left( \frac{|f(x)|}{t} \right) M(\chi_B w)(x) dx \\ &\leq \text{I} + \text{II}. \end{aligned}$$

To estimate the term I, since  $w \in A_1$ , we have

$$\begin{aligned} I &\leq C \int_{3B} \Phi\left(\frac{|f(x)|}{t}\right) w(x) dx \\ &\leq C \Phi\left(\frac{1}{t}\right) \int_{3B} \Phi(|f(x)|) w(x) dx \\ &\leq C \Phi\left(\frac{1}{t}\right) \|\Phi(|f|) \chi_{3B}\|_{L^1(w)}. \end{aligned}$$

For the term II, observe that for  $x \in (3B)^c$ ,  $x \in B'$ ,  $B'$  is a ball and  $B' \cap B \neq \emptyset$ . We have

$$\begin{aligned} \frac{1}{|B'|} \int_{B'} \chi_B(z) w(z) dz &= \frac{1}{|B'|} \int_{B' \cap B} w(z) dz \\ &\leq \frac{C}{|x-y|^n} \int_B w(z) dz = \frac{C}{|x-y|^n} w(B). \end{aligned}$$

Therefore we obtain

$$M(\chi_B w)(x) \leq C|x-y|^{-n} w(B).$$

Since  $w \in A_1$ , we get

$$\begin{aligned} II &\leq C \int_{(3B)^c} \Phi\left(\frac{|f(x)|}{t}\right) |x-y|^{-n} w(B) dx \\ &\leq C \sum_{j=1}^{\infty} \int_{3^{j+1}B \setminus 3^jB} \Phi\left(\frac{|f(x)|}{t}\right) \frac{w(B)}{|3^{j+1}B|} dx \\ &\leq C \sum_{j=1}^{\infty} \frac{w(B)}{w(3^{j+1}B)} \int_{3^{j+1}B} \Phi\left(\frac{|f(x)|}{t}\right) w(x) dx \\ &\leq C \Phi\left(\frac{1}{t}\right) \sum_{j=1}^{\infty} \frac{w(B)}{w(3^{j+1}B)} \|\Phi(|f|) \chi_{3^{j+1}B}\|_{L^1(w)}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \|\Psi(M_{L(\log L)} f) \chi_B\|_{L^{1,\infty}(w)} &= \sup_{t>0} t w\{x \in B : \Psi(M_{L(\log L)} f)(x) > t\} \\ &= \sup_{t>0} t w\{x \in B : M_{L(\log L)} f(x) > \Psi^{-1}(t)\} \\ &= \sup_{t>0} \Psi(t) w\{x \in B : M_{L(\log L)} f(x) > t\} \\ &\leq C \left( \|\Phi(|f|) \chi_{3B}\|_{L^1(w)} \right. \\ &\quad \left. + \sum_{j=1}^{\infty} \frac{w(B)}{w(3^{j+1}B)} \|\Phi(|f|) \chi_{3^{j+1}B}\|_{L^1(w)} \right). \end{aligned}$$

Thus, for any  $r > 0$ , by Lemma 2.1 and Lemma 2.5, we have

$$\begin{aligned}
 & r \left\| \Psi(M_L(\log L)f) \right\|_{(L^1(w), L^p)^\beta} \\
 &= \left( \int_{\mathbb{R}^n} (w(B(y, r)))^{\frac{1}{\beta}-1-\frac{1}{p}} \left\| \Psi(M_L(\log L)f) \chi_{B(y, r)} \right\|_{L^1(w)} \right)^p dy \Bigg)^{\frac{1}{p}} \\
 &\leq C \left( \int_{\mathbb{R}^n} (w(B(y, r)))^{\frac{1}{\beta}-1-\frac{1}{p}} \left\| \Phi(|f|) \chi_{B(y, 3r)} \right\|_{L^1(w)} \right)^p dy \Bigg)^{\frac{1}{p}} \\
 &\quad + C \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} \frac{w(B(y, r))}{w(B(y, 3^{j+1}r))} w(B(y, r))^{\frac{1}{\beta}-1-\frac{1}{p}} \left\| \Phi(|f|) \chi_{B(y, 3^{j+1}r)} \right\|_{L^1(w)} \right)^p dy \right)^{\frac{1}{p}} \\
 &\leq C \left( \int_{\mathbb{R}^n} (w(B(y, 3r)))^{\frac{1}{\beta}-1-\frac{1}{p}} \left\| \Phi(|f|) \chi_{B(y, 3r)} \right\|_{L^1(w)} \right)^p dy \Bigg)^{\frac{1}{p}} \\
 &\quad + C \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} \left( \frac{w(B(y, r))}{w(B(y, 3^{j+1}r))} \right)^{\frac{1}{\beta}-\frac{1}{p}} w(B(y, 3^{j+1}r))^{\frac{1}{\beta}-1-\frac{1}{p}} \right. \right. \\
 &\quad \left. \left. \times \left\| \Phi(|f|) \chi_{B(y, 3^{j+1}r)} \right\|_{L^1(w)} \right)^p dy \right)^{\frac{1}{p}} \\
 &\leq C \left\| \Phi(|f|) \right\|_{(L^1(w), L^p)^\beta} \left( 1 + \sum_{j=1}^{\infty} \frac{1}{3^{j\eta(\frac{1}{\beta}-\frac{1}{p})}} \right) \\
 &\leq C \left\| \Phi(|f|) \right\|_{(L^1(w), L^p)^\beta}.
 \end{aligned}$$

This ends the proof. □

**Lemma 2.7** ([13]) *Let  $T$  be any Calderón–Zygmund singular integral operator,  $\Phi(t) = t \log(e + t)$ ,  $\varepsilon > 0$ , and  $b \in BMO$ . Then there exists a positive constant  $C$  such that, for all weights  $w$ ,*

$$w(\{x \in \mathbb{R}^n : |[b, T]f(x)| > t\}) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) M_{L(\log L)^{1+\varepsilon}} w(x) dx.$$

**Lemma 2.8** ([9]) *Let  $w \in A_1$ , then there exist a constant  $C > 0$  and  $\theta > 0$  such that, for any ball  $B$ ,*

$$\left( \frac{1}{|B|} \int_B w(y)^{1+\theta} dy \right)^{\frac{1}{1+\theta}} \leq C \frac{1}{|B|} \int_B w(y) dy.$$

*Proof of Theorem 1.3* Fix  $y \in \mathbb{R}^n$  and  $r > 0$ , let  $B = B(y, r)$ . By Lemma 2.7, we have

$$\begin{aligned}
 w(\{x \in B : |[b, T]f(x)| > t\}) &= \int_{\{x \in \mathbb{R}^n : |[b, T]f(x)| > t\}} w(x) \chi_B(x) dx \\
 &\leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) M_{L(\log L)^{1+\varepsilon}}(w \chi_B)(x) dx \\
 &\leq C \left( \int_{3B} + \int_{(3B)^c} \right) \Phi\left(\frac{|f(x)|}{t}\right) M_{L(\log L)^{1+\varepsilon}}(w \chi_B)(x) dx \\
 &\leq \text{I} + \text{II}.
 \end{aligned}$$



To estimate the term I, since  $w \in A_1$ , it is easy to prove that  $M_{L(\log L)^{1+\varepsilon}}(w\chi_B)(x) \leq Cw(x)$ ,  $x \in 3B$ , we have

$$I \leq C \int_{3B} \Phi\left(\frac{|f(x)|}{t}\right) w(x) dx \leq C\Phi\left(\frac{1}{t}\right) \|\Phi(|f|)\chi_{3B}\|_{L^1(w)}.$$

For the term II, observe that for  $x \in (3B)^c$ ,  $x \in B'$ ,  $B'$  is a ball and  $B' \cap B \neq \emptyset$ , by Lemma 2.8, for any  $\delta : 0 < \delta \leq \theta$ , we have

$$\begin{aligned} \left(\frac{1}{|B'|} \int_{B'} (w(z)\chi_B(z))^{1+\delta} dz\right)^{\frac{1}{1+\delta}} &\leq \left(\frac{1}{|B'|} \int_B w(z)^{1+\delta} dz\right)^{\frac{1}{1+\delta}} \\ &= \left(\frac{|B|}{|B'|}\right)^{\frac{1}{1+\delta}} \left(\frac{1}{|B|} \int_B w(z)^{1+\delta} dz\right)^{\frac{1}{1+\delta}} \\ &\leq C\left(\frac{|B|}{|B'|}\right)^{\frac{1}{1+\delta}} \left(\frac{1}{|B|} \int_B w(z) dz\right) \\ &\leq C\left(\frac{|B|}{|B'|}\right)^{\frac{1}{1+\delta}} \frac{w(B)}{|B|}. \end{aligned}$$

Noticing the definition of the maximal function  $M$ , we obtain

$$\begin{aligned} M_{L(\log L)^{1+\varepsilon}}(w\chi_B)(x) &\leq (M(w^{1+\delta}\chi_B)(x))^{\frac{1}{1+\delta}} \\ &\leq C\left(\frac{|B|}{|x-y|^n}\right)^{\frac{1}{1+\delta}} \frac{w(B)}{|B|} \end{aligned}$$

and

$$\begin{aligned} II &\leq C \int_{(3B)^c} \Phi\left(\frac{|f(x)|}{t}\right) \left(\frac{|B|}{|x-y|^n}\right)^{\frac{1}{1+\delta}} \frac{w(B)}{|B|} dx \\ &\leq C \sum_{j=1}^{\infty} \int_{3^{j+1}B \setminus 3^jB} \Phi\left(\frac{|f(x)|}{t}\right) \left(\frac{|B|}{|3^{j+1}B|}\right)^{\frac{1}{1+\delta}} \frac{w(B)}{|B|} dx \\ &\leq C \sum_{j=1}^{\infty} \left(\frac{|B|}{|3^{j+1}B|}\right)^{\frac{1}{1+\delta}} \frac{w(B)}{|B|} \frac{|3^{j+1}B|}{w(3^{j+1}B)} \int_{3^{j+1}B} \Phi\left(\frac{|f(x)|}{t}\right) w(x) dx \\ &\leq C\Phi\left(\frac{1}{t}\right) \sum_{j=1}^{\infty} \left(\frac{|B|}{|3^{j+1}B|}\right)^{\frac{-\delta}{1+\delta}} \frac{w(B)}{w(3^{j+1}B)} \|\Phi(|f|)\chi_{3^{j+1}B}\|_{L^1(w)}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \|\Psi(|[b, T]f|)\chi_B\|_{L^{1,\infty}(w)} &= \sup_{t>0} tw\{x \in B : \Psi(|[b, T]f|)(x) > t\} \\ &= \sup_{t>0} \Psi(t)w\{x \in B : |[b, T]f(x)| > t\} \end{aligned}$$

$$\leq C \left( \|\Phi(|f|)\chi_{3B}\|_{L^1(w)} + \sum_{j=1}^{\infty} \left( \frac{|B|}{|3^{j+1}B|} \right)^{\frac{\delta}{1+\delta}} \frac{w(B)}{w(3^{j+1}B)} \|\Phi(|f|)\chi_{3^{j+1}B}\|_{L^1(w)} \right).$$

Thus, for any  $r > 0$ , by Lemma 2.1 and Lemma 2.5, we have

$$\begin{aligned} & r \|\Psi(|[b, T]f|)\|_{(L^1(w), L^p)^\beta} \\ &= \left( \int_{\mathbb{R}^n} (w(B(y, r)))^{\frac{1}{\beta}-1-\frac{1}{p}} \|\Psi(|[b, T]f|)\chi_{B(y, r)}\|_{L^1(w)}^p dy \right)^{\frac{1}{p}} \\ &\leq C \left( \int_{\mathbb{R}^n} (w(B(y, r)))^{\frac{1}{\beta}-1-\frac{1}{p}} \|\Phi(|f|)\chi_{B(y, 3r)}\|_{L^1(w)}^p dy \right)^{\frac{1}{p}} \\ &\quad + C \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} \left( \frac{|B|}{|3^{j+1}B|} \right)^{\frac{\delta}{1+\delta}} \frac{w(B(y, r))}{w(B(y, 3^{j+1}r))} \|\Phi(|f|)\chi_{B(y, 3^{j+1}r)}\|_{L^1(w)} \right)^p dy \right)^{\frac{1}{p}} \\ &\leq C \left( \int_{\mathbb{R}^n} (w(B(y, 3r)))^{\frac{1}{\beta}-1-\frac{1}{p}} \|\Phi(|f|)\chi_{B(y, 3r)}\|_{L^1(w)}^p dy \right)^{\frac{1}{p}} \\ &\quad + C \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} \left( \frac{|B|}{|3^{j+1}B|} \right)^{\eta(\frac{1}{\beta}-\frac{1}{p})-\frac{\delta}{1+\delta}} w(B(y, 3^{j+1}r))^{\frac{1}{\beta}-1-\frac{1}{p}} \right. \right. \\ &\quad \left. \left. \times \|\Phi(|f|)\chi_{B(y, 3^{j+1}r)}\|_{L^1(w)} \right)^p dy \right)^{\frac{1}{p}} \\ &\leq C \|\Phi(|f|)\|_{(L^1(w), L^p)^\beta} \left( 1 + \sum_{j=1}^{\infty} \left( \frac{1}{3^{(j+1)n}} \right)^{\eta(\frac{1}{\beta}-\frac{1}{p})-\frac{\delta}{1+\delta}} \right) \\ &\leq C \|\Phi(|f|)\|_{(L^1(w), L^p)^\beta}, \end{aligned}$$

in which we take  $\delta > 0$  small enough such that  $\eta(\frac{1}{\beta} - \frac{1}{p}) - \frac{\delta}{1+\delta} > 0$ . This ends the proof.  $\square$

### 3 Proof of Theorem 1.4 and Theorem 1.5

Given an increasing function  $\varphi : [0, \infty) \rightarrow [0, \infty)$ , as in [3], we define the function  $h_\varphi$  by

$$h_\varphi(s) = \sup_{t>0} \frac{\varphi(st)}{\varphi(t)}, \quad 0 \leq s < \infty.$$

If  $\varphi$  is submultiplicative, then  $h_\varphi \approx \varphi$ . Also, for all  $s, t > 0$ ,  $\varphi(st) \leq h_\varphi(s)\varphi(t)$ .

In this section, we set  $\Phi(t) = t \log(e + t)$ , it is submultiplicative and so  $h_\Phi \approx \Phi$ . Let  $0 < \alpha < n$ , and  $q$  be a number  $1/q = 1 - \alpha/n$ . Denote

$$\Gamma(t) = \begin{cases} 0, & t = 0, \\ \frac{t}{\Phi(t^{\alpha/n})}, & t > 0. \end{cases}$$

So

$$\Gamma(t) \approx t^{1/q} \log(e + t)^{-1}.$$

The function  $\Gamma$  is invertible with

$$\Gamma^{-1}(t) \approx [t \log(e + t)]^q = \Phi(t)^q.$$

**Lemma 3.1** ([3]) *If  $\varphi(t)/t$  is decreasing, then for any positive sequence  $\{t_j\}$ ,*

$$\varphi\left(\sum_j t_j\right) \leq \sum_j \varphi(t_j).$$

**Lemma 3.2** ([14]) *Let  $0 < \alpha < n$ ,  $1/q = 1 - \alpha/n$ . Then there exists a constant  $C > 0$  such that, for any  $t > 0$ , for any weight  $w$ , we have*

$$\begin{aligned} &\Gamma(w(\{x \in \mathbb{R}^n : M_{\alpha,L \log L}(f)(x) > t\})) \\ &\leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(y)|}{t}\right) h_\psi(Mw(y)) dy. \end{aligned}$$

*Proof of Theorem 1.4* Fix  $y \in \mathbb{R}^n$  and  $r > 0$ , let  $B = B(y, r)$ . By Lemma 3.2, we have

$$\begin{aligned} \Gamma(w(\{x \in B : M_{\alpha,L(\log L)}f(x) > t\})) &= \Gamma\left(\int_{\{x \in \mathbb{R}^n : M_{\alpha,L(\log L)}f(x) > t\}} w(x)\chi_B(x) dx\right) \\ &\leq C \int_{\mathbb{R}^n} \Gamma\left(\frac{|f(x)|}{t}\right) h_\Gamma(M(w\chi_B))(x) dx \\ &\leq C \left(\int_{3B} + \int_{(3B)^c}\right) \Gamma\left(\frac{|f(x)|}{t}\right) h_\Gamma(M(w\chi_B))(x) dx \\ &\leq \text{I} + \text{II}. \end{aligned}$$

Now we estimate the term I. Noticing that, for  $s > 0$ , we have

$$h_\Gamma(s) = \sup_{t>0} \frac{\Gamma(st)}{\Gamma(t)} = s \sup_{t>0} \frac{\Phi(t^{\alpha/n})}{\Phi((st)^{\alpha/n})} \leq C\Theta(s).$$

Since  $w \in A_1$ , we get

$$\begin{aligned} \text{I} &\leq C \int_{3B} \Phi\left(\frac{|f(x)|}{t}\right) h_\Gamma(w(x)) dx \\ &\leq C \int_{3B} \Phi\left(\frac{|f(x)|}{t}\right) \Theta(w(x)) dx \\ &\leq C\Phi(1/t) \|\Phi(|f|\chi_{3B})\|_{L^1(\Theta(w))}. \end{aligned}$$

For the term II, observe that for  $x \in (3B)^c$ ,  $x \in B'$ ,  $B'$  is a ball and  $B' \cap B \neq \emptyset$ . As in the proof of Theorem 1.2, we have

$$M(\chi_B w)(x) \leq C|x - y|^{-n}w(B).$$

Since  $w \in A_1$ ,  $\Theta$  is submultiplicative, we get

$$\text{II} \leq C \int_{(3B)^c} \Phi\left(\frac{|f(x)|}{t}\right) h_\psi(|x - y|^{-n}w(B)) dx$$

$$\begin{aligned}
 &\leq C \sum_{j=1}^{\infty} \int_{3^{j+1}B \setminus 3^jB} \Phi\left(\frac{|f(x)|}{t}\right) \Theta\left(\frac{w(B)}{|3^{j+1}B|}\right) dx \\
 &\leq C \sum_{j=1}^{\infty} \int_{3^{j+1}B} \Phi\left(\frac{|f(x)|}{t}\right) \Theta\left(\frac{w(3^{j+1}B)}{|3^{j+1}B|} \frac{w(B)}{w(3^{j+1}B)}\right) dx \\
 &\leq C \sum_{j=1}^{\infty} \int_{3^{j+1}B} \Phi\left(\frac{|f(x)|}{t}\right) \Theta\left(w(x) \frac{w(B)}{w(3^{j+1}B)}\right) dx \\
 &\leq C \sum_{j=1}^{\infty} \Theta\left(\frac{w(B)}{w(3^{j+1}B)}\right) \int_{3^{j+1}B} \Phi\left(\frac{|f(x)|}{t}\right) \Theta(w(x)) dx \\
 &\leq C \Phi(1/t) \sum_{j=1}^{\infty} \Theta\left(\frac{w(B)}{w(3^{j+1}B)}\right) \|\Phi(|f|) \chi_{3^{j+1}B}\|_{L^1(\Theta(w))}.
 \end{aligned}$$

Hence, we obtain

$$\begin{aligned}
 &\|\Psi(M_{\alpha,L(\log L)f}) \chi_B\|_{L^{\Gamma,\infty}(w)} \\
 &= \sup_{t>0} t \Gamma(w\{x \in B : \Psi(M_{\alpha,L(\log L)f})(x) > t\}) \\
 &= \sup_{t>0} t \Gamma(w\{x \in B : M_{\alpha,L(\log L)f}(x) > \Psi^{-1}(t)\}) \\
 &= \sup_{t>0} \Psi(t) \Gamma(w\{x \in B : M_{\alpha,L(\log L)f}(x) > t\}) \\
 &\leq C \left( \|\Phi(|f|) \chi_{3B}\|_{L^1(\Theta(w))} \right. \\
 &\quad \left. + \sum_{j=1}^{\infty} \Theta\left(\frac{w(B)}{w(3^{j+1}B)}\right) \|\Phi(|f|) \chi_{3^{j+1}B}\|_{L^1(\Theta(w))} \right).
 \end{aligned}$$

Thus, for any  $r > 0$ , we have

$$\begin{aligned}
 &r \|\Psi(M_{\alpha,L(\log L)f})\|_{(L^{\Gamma,\infty}(w),L^p)^\beta} \\
 &= \left( \int_{\mathbb{R}^n} (w(B(y,r)))^{\frac{1}{\beta}-1-\frac{1}{p}} \|\Psi(M_{\alpha,L(\log L)f}) \chi_{B(y,r)}\|_{L^{\Gamma,\infty}(w)}^p dy \right)^{\frac{1}{\beta}} \\
 &\leq C \left( \int_{\mathbb{R}^n} (w(B(y,r)))^{\frac{1}{\beta}-1-\frac{1}{p}} \|\Phi(|f|) \chi_{B(y,3r)}\|_{L^1(\Theta(w))}^p dy \right)^{\frac{1}{\beta}} \\
 &\quad + C \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} \Theta\left(\frac{w(B(y,r))}{w(B(y,3^{j+1}r))}\right) w(B(y,r))^{\frac{1}{\beta}-1-\frac{1}{p}} \right. \right. \\
 &\quad \left. \left. \times \|\Phi(|f|) \chi_{B(y,3^{j+1}r)}\|_{L^1(\Theta(w))} \right)^p dy \right)^{\frac{1}{\beta}} \\
 &\leq C \left( \int_{\mathbb{R}^n} (w(B(y,3r)))^{\frac{1}{\beta}-1-\frac{1}{p}} \|\Phi(|f|) \chi_{B(y,3r)}\|_{L^1(\Theta(w))}^p dy \right)^{\frac{1}{\beta}} \\
 &\quad + C \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} \frac{\log(e+3^{jm})}{3^{jm(\frac{1}{\beta}-1-\frac{1}{p}+\frac{1}{q})}} w(B(y,3^{j+1}r))^{\frac{1}{\beta}-1-\frac{1}{p}} \right. \right. \\
 &\quad \left. \left. \times \|\Phi(|f|) \chi_{B(y,3^{j+1}r)}\|_{L^1(\Theta(w))} \right)^p dy \right)^{\frac{1}{\beta}}.
 \end{aligned}$$

$$\begin{aligned} & \times \left\| \Phi(|f|)\chi_{B(y,3^{j+1}r)} \right\|_{L^1(\Theta(w))}^p dy \Big)^{\frac{1}{p}} \\ & \leq C \left\| \Phi(|f|) \right\|_{(L^1(\Theta(w)),L^p(w))^\beta} \left( 1 + \sum_{j=1}^\infty \frac{\log(e + 3^{jm})}{3^{jm(\frac{1}{\beta}-1-\frac{1}{p}+\frac{1}{q})}} \right) \\ & \leq C \left\| \Phi(|f|) \right\|_{(L^1(\Theta(w)),L^p(w))^\beta}. \end{aligned}$$

This ends the proof. □

**Lemma 3.3** ([4]) *Let  $0 < \alpha < n$ ,  $1/q = 1 - \alpha/n$ ,  $w \in A_1$ , and  $b \in BMO$ . Then there exists a constant  $C > 0$  such that, for any  $t > 0$ ,*

$$\begin{aligned} & \Gamma(w(\{x \in \mathbb{R}^n : |[b, I_\alpha]f(x)| > t\})) \\ & \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(y)|}{t}\right) \Theta(w(y)) dy. \end{aligned}$$

**Lemma 3.4** ([9]) *Let  $f(x) \geq 0$ ,  $f \in L^1_{loc}(\mathbb{R}^n)$ , and  $0 < \delta < 1$ , then  $M(f)^\delta \in A_1$ .*

*Proof of Theorem 1.5* Fix  $y \in \mathbb{R}^n$  and  $r > 0$ , let  $B = B(y, r)$ . For any  $w \in A_1$  and  $\delta : 0 < \delta \leq \theta$ , by Lemma 3.4, we have  $M(w^{1+\delta}\chi_B)^{1/(1+\delta)} \in A_1$ . By Lemma 3.3, we obtain

$$\begin{aligned} & \Gamma(w(\{x \in B : |[b, I_\alpha]f(x)| > t\})) \\ & = \Gamma\left(\int_{\{x \in \mathbb{R}^n : |[b, I_\alpha]f(x)| > t\}} w(x)\chi_B(x) dx\right) \\ & \leq C \Gamma\left(\int_{\{x \in \mathbb{R}^n : |[b, I_\alpha]f(x)| > t\}} M(w\chi_B)(x) dx\right) \\ & \leq C \Gamma\left(\int_{\{x \in \mathbb{R}^n : |[b, I_\alpha]f(x)| > t\}} (M(w^{1+\delta}\chi_B)(x))^{1/(1+\delta)} dx\right) \\ & \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) \Theta((M(w^{1+\delta}\chi_B)(x))^{1/(1+\delta)}) dx \\ & \leq C \left(\int_{3B} + \int_{(3B)^c}\right) \Phi\left(\frac{|f(x)|}{t}\right) \Theta((M(w^{1+\delta}\chi_B)(x))^{1/(1+\delta)}) dx \\ & \leq \text{I} + \text{II}. \end{aligned}$$

Now we estimate the term I. Noticing that  $w \in A_1$ , Lemma 2.8, we have  $\Theta((M(w^{1+\delta}\chi_B)(x))^{1/(1+\delta)}) \leq C\Theta(Mw(x)) \leq C\Theta(w(x))$ . Then

$$\text{I} \leq C \int_{3B} \Phi\left(\frac{|f(x)|}{t}\right) \Theta(w(x)) dx \leq C\Phi(1/t) \left\| \Phi(|f|)\chi_{3B} \right\|_{L^1(\Theta(w))}.$$

For the term II, as the proof of Theorem 1.3, for  $x \in (3B)^c$ ,

$$(M(w^{1+\delta}\chi_B)(x))^{\frac{1}{1+\delta}} \leq C \left(\frac{|B|}{|x-y|^n}\right)^{\frac{1}{1+\delta}} \frac{w(B)}{|B|}.$$

By Lemma 2.2, we get

$$\begin{aligned} \text{II} &\leq C \int_{(3B)^c} \Phi\left(\frac{|f(x)|}{t}\right) \Theta\left(\left(\frac{|B|}{|x-y|^n}\right)^{\frac{1}{1+\delta}} \frac{w(B)}{|B|}\right) dx \\ &\leq C \sum_{j=1}^{\infty} \int_{3^{j+1}B \setminus 3^jB} \Phi\left(\frac{|f(x)|}{t}\right) \Theta\left(\left(\frac{|B|}{|3^{j+1}B|}\right)^{\eta - \frac{\delta}{1+\delta}} w(x)\right) dx \\ &\leq C \Phi(1/t) \sum_{j=1}^{\infty} \Theta\left(\frac{|B|}{|3^{j+1}B|}\right)^{\eta - \frac{\delta}{(1+\delta)}} \|\Phi(|f|)\chi_{3^{j+1}B}\|_{L^1(\Theta(w))}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &\|\Psi(|[b, I_\alpha]f|)\chi_B\|_{L^{\Gamma, \infty}(w)} \\ &= \sup_{t>0} t \Gamma(w\{x \in B : \Psi(|[b, I_\alpha]f|)(x) > t\}) \\ &= \sup_{t>0} t \Gamma(w\{x \in B : |[b, I_\alpha]f(x)| > \Psi^{-1}(t)\}) \\ &= \sup_{t>0} \Psi(t) \Gamma(w\{x \in B : |[b, I_\alpha]f(x)| > t\}) \\ &\leq C \left( \|\Phi(|f|)\chi_{3B}\|_{L^1(\Theta(w))} \right. \\ &\quad \left. + \sum_{j=1}^{\infty} \Theta\left(\frac{|B|}{|3^{j+1}B|}\right)^{\eta - \frac{\delta}{(1+\delta)}} \|\Phi(|f|)\chi_{3^{j+1}B}\|_{L^1(\Theta(w))} \right). \end{aligned}$$

Thus, for any  $r > 0$ , we have

$$\begin{aligned} &r \|\Psi(|[b, I_\alpha]f|)\chi_B\|_{(L^{\Gamma, \infty}(w), L^p)^\beta} \\ &= \left( \int_{\mathbb{R}^n} (w(B(y, r)))^{\frac{1}{\beta} - 1 - \frac{1}{p}} \|\Psi(|[b, I_\alpha]f|)\chi_{B(y, r)}\|_{L^{\Gamma, \infty}(w)}^p dy \right)^{\frac{1}{\beta}} \\ &\leq C \left( \int_{\mathbb{R}^n} (w(B(y, r)))^{\frac{1}{\beta} - 1 - \frac{1}{p}} \|\Phi(|f|)\chi_{B(y, 3r)}\|_{L^1(\Theta(w))}^p dy \right)^{\frac{1}{\beta}} \\ &\quad + C \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} \Theta\left(\frac{|B(y, r)|}{|B(y, 3^{j+1}r)|}\right)^{\eta - \frac{\delta}{(1+\delta)}} w(B(y, r))^{\frac{1}{\beta} - 1 - \frac{1}{p}} \right. \right. \\ &\quad \left. \left. \times \|\Phi(|f|)\chi_{B(y, 3^{j+1}r)}\|_{L^1(\Theta(w))} \right)^p dy \right)^{\frac{1}{\beta}} \\ &\leq C \left( \int_{\mathbb{R}^n} (w(B(y, 3r)))^{\frac{1}{\beta} - 1 - \frac{1}{p}} \|\Phi(|f|)\chi_{B(y, 3r)}\|_{L^1(\Theta(w))}^p dy \right)^{\frac{1}{\beta}} \\ &\quad + C \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} \frac{\log(e + 3^{jn(\eta - \frac{\delta}{1+\delta})})}{3^{jn(\frac{1}{\beta} - 1 - \frac{1}{p} + \frac{1}{q}) - \frac{\delta}{\delta(1+q)}} w(B(y, 3^{j+1}r))^{\frac{1}{\beta} - 1 - \frac{1}{p}} \right. \right. \\ &\quad \left. \left. \times \|\Phi(|f|)\chi_{B(y, 3^{j+1}r)}\|_{L^1(\Theta(w))} \right)^p dy \right)^{\frac{1}{\beta}} \end{aligned}$$

$$\begin{aligned} &\leq C \|\Phi(|f|)\|_{(L^1(\Theta(w)), LP(w))^\beta} \left( 1 + \sum_{j=1}^{\infty} \frac{\log(e + 3^{jn(\eta - \frac{\delta}{1+\delta})})}{3^{jn(\eta(\frac{1}{\beta} - 1 - \frac{1}{p} + \frac{1}{q}) - \frac{\delta}{\delta(1+q)})}} \right) \\ &\leq C \|\Phi(|f|)\|_{(L^1(\Theta(w)), LP(w))^\beta}, \end{aligned}$$

in which we take  $\delta > 0$  small enough such that  $\eta(\frac{1}{q} - 1 - \frac{1}{p} + \frac{1}{\beta}) - \frac{\delta}{q(1+\delta)} > 0$  and  $\eta - \frac{\delta}{1+\delta} > 0$ . This ends the proof. □

**Acknowledgements**

The authors are very grateful to the anonymous referees for their valuable suggestions and comments, which helped to improve the quality of the paper.

**Funding**

Jinyun Qi was supported by the fund of Langfang Normal University (No. LSLY201504) and the doctor’s fund of Hebei Normal University (No. L2018B32).

**Availability of data and materials**

Not applicable.

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors contributed equally to this work. All authors read and approved the final manuscript.

**Author details**

<sup>1</sup>School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, China. <sup>2</sup>College of Science, Langfang Normal University, Langfang, China.

**Publisher’s Note**

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 January 2020 Accepted: 24 April 2020 Published online: 07 May 2020

**References**

1. Bennett, C., Sharpley, R.: Interpolation of Operators. Academic Press, Boston (1988)
2. Chiarenza, F., Frasca, M.: Morrey spaces and Hardy–Littlewood maximal function. *Rend. Mat. Appl.* **7**(7), 273–279 (1987)
3. Cruz-Uribe, D., Fiorenza, A.: Endpoint estimates and weighted norm inequalities for commutators of fractional integrals. *Publ. Mat.* **47**(1), 103–131 (2003)
4. Cruz-Uribe, D., Fiorenza, A.: Weighted endpoint estimates for commutators of fractional integrals. *Czechoslov. Math. J.* **57**(1), 153–160 (2007)
5. Dai, Z.F., Chen, X.H., Wen, F.H.: A modified Perry’s conjugate gradient method-based derivative-free method for solving large-scale nonlinear monotone equations. *Appl. Math. Comput.* **270**, 378–386 (2015)
6. Dai, Z.F., Zhu, H.: A modified Hestenes–Stiefel-type derivative-free method for large-scale nonlinear monotone equations. *Mathematics* **8**(2), 168 (2020)
7. Feuto, J.: Norm inequalities in generalized Morrey spaces. *J. Fourier Anal. Appl.* **20**(4), 896–909 (2014)
8. Fofana, I.: Étude d’une classe d’espaces de fonctions contenant les espaces de Lorentz. *Afr. Math.* **2**(1), 29–50 (1988)
9. Grafakos, L.: Modern Fourier Analysis, 2nd edn. Graduate Texts in Mathematics, vol. 250. Springer, Berlin (2009)
10. Komori, Y., Shirai, S.: Weighted Morrey spaces and a singular integral operator. *Math. Nachr.* **282**(2), 219–231 (2009)
11. Morrey, C.B.: On the solutions of quasi-linear elliptic partial differential equations. *Trans. Am. Math. Soc.* **43**(1), 126–166 (1938)
12. Pérez, C.: Endpoint estimates for commutators of singular integral operators. *J. Funct. Anal.* **128**(1), 163–185 (1995)
13. Pérez, C., Pradolini, G.: Sharp weighted endpoint estimates for commutators of singular integral operators. *Mich. Math. J.* **49**, 23–37 (2003)
14. Qi, J.Y., Shi, H.X., Li, W.M.: Weighted endpoint estimates for commutators of singular integral operators on Orlicz–Morrey spaces. *J. Funct. Spaces* **2019**, Article ID 5458101 (2019)
15. Sawano, Y., Hakim, D.I., Gunawan, H.: Non-smooth atomic decomposition for generalized Orlicz–Morrey spaces. *Math. Nachr.* **288**(14–15), 1741–1775 (2015)