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On strongly starlike and strongly convex functions with bounded radius and bounded boundary rotation

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Abstract

In this paper, we introduce and investigate new classes of normalized analytic functions in an open unit disk with bounded radius and bounded boundary rotation by using the subordination. We discuss inclusion results, co-efficient bounds, growth and distortion theorems of the classes. Moreover, we compute the radii of strong starlikeness, convexity and starlikeness of the classes. It is interesting to mention that most of our findings are best possible as compared to the existing results in the literature.

MSC: 30C45; 30C50

Keywords: Subordination; Strongly convex function; Janowski function; Bounded boundary rotation

1 Introduction and preliminaries

We represent \mathcal{A} as the class of functions $f(z)$, which is analytic in $E = \{z \in \mathbb{C} : |z| < 1\}$. Then, for each $z \in E$, $f(z)$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Suppose that $g(z)$ and $G(z)$ are analytic functions in E . Then $g(z)$ is said to be subordinate to $G(z)$ written as $g(z) \prec G(z)$ if and only if there exists $w(z)$, which is analytic in E with $w(0) = 0$ and $|w(z)| < 1$. Therefore, $g(z) = G(w(z))$ belongs to E . $g(z)$ is univalent in E follows that $g \prec G$ is equivalent to $g(0) = G(0)$ and $g(E) \subset G(E)$.

The function $F(A, B, z) = \left(\frac{1+Az}{1+Bz}\right)$ is the conformal mapping of the unit disk to circle, which is symmetric with respect to the real axis having center $\frac{1-AB}{1-B^2}$ and radius $\frac{A-B}{1-B^2}$ for each A, B such that $-1 \leq B < A \leq 1$. A function $p(z)$ (with $p(0) = 1$) is analytic in E and belongs to the class $P[A, B]$ if $p(z)$ is subordinate to $F(A, B, z)$. In [5], Janowski introduced and investigated the class $P[A, B]$. Later on, Noor and Arif [8] investigated the class $P_m[A, B]$. They investigated that a function $p(z)$ (with $p(0) = 1$) is analytic in E and belongs to the class

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$P_m[A, B]$ if and only if there exist $p_1(z), p_2(z) \in P[A, B]$ such that

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)p_2(z) \tag{1.2}$$

for $m \geq 2, -1 \leq B < A \leq 1$, and $z \in E$.

In [14], Rajapat et al. used the known family of fractional integral operators (with the Gauss hypergeometric function in the kernel) and defined new subclasses of strongly star-like and strongly convex functions of order β and type α in the open unit disk U . Moreover, they established several inclusion relationships and interesting results associated with the fractional integral operators.

Shiraishi et al. [16] investigated some new sufficient conditions for the class of strong Caratheodory functions in the open unit disk U . Example and several corollaries of the main results were presented.

We consider a function $F_\beta(A, B, z) = \left(\frac{1+Az}{1+Bz}\right)^\beta$ for $-1 \leq B \leq 1, -1 \leq A \leq 1 (A \neq B)$, and $0 < \beta \leq 1$, which is analytic and univalent in E . If $p(z)$ (with $p(0) = 1$) is subordinate to $F_\beta(A, B, z)$, then the function $p(z)$ is analytic in E and belongs to the class $\tilde{P}^\beta[A, B]$.

Lemma 1.1 ([10]) *Suppose $f \in \mathcal{V}_m(\rho)$ for $m \geq 2$ and $0 \leq \rho < 1$. Then $f(z) = (f_1(z))^{(1-\rho)}$ for $f_1(z) \in \mathcal{V}_m$.*

Lemma 1.2 ([6]) *Suppose $\xi = \xi_1 + i\xi_2, \zeta = \zeta_1 + i\zeta_2$ and $\Phi : D \subset C^2 \rightarrow C$ is a complex-valued function, which satisfies the following conditions:*

- (i) $\Psi(\xi, \zeta)$ is continuous in a domain $D \subset C^2$,
- (ii) $(1, 0) \in D$ and $\Phi(1, 0) > 0$,
- (iii) $\text{Re } \Phi(i\xi_2, \zeta_1) \leq 0$, whenever $(i\xi_2, \zeta_1) \in D$ and $\zeta_1 \leq -\frac{1}{2}(1 + \xi_2^2)$.

If $q(z) = 1 + \sum_{n=1}^\infty c_n z^n$ is an analytic function in E with $(q(z), zq'(z)) \in D$ and $\text{Re } \Phi(q(z), zq'(z))$ for $z \in E$, then $\text{Re}(q(z)) > 0$, for $z \in E$.

Lemma 1.3 ([3]) *Suppose $r_a \leq \text{Re}(a) \sin(\frac{\alpha\pi}{2}) - \text{Re}(a) \cos(\frac{\alpha\pi}{2}), \text{Im}(a) \geq 0$. Then the disk $|w - a| \leq r_a$ is contained in the sector $|\arg(w)| \leq \frac{\alpha\pi}{2}$ for $0 \leq \alpha < 1$.*

2 Main results

In this section, we discuss the coefficient problem, analytic property, inclusion results, and radius problem. It is interesting to mention that our obtained results are sharp as compared to the existing results in the literature.

First we define the following.

Definition 2.1 A function $q(z)$ (with $q(0) = 1$) is analytic in E and belongs to the class $\tilde{P}_m^\beta[A, B]$ (for $m \geq 2$) if and only if there exist $p_1(z), p_2(z) \in \tilde{P}^\beta[A, B]$ such that

$$q(z) = \left(\frac{m}{4} + \frac{1}{2}\right)q_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)q_2(z) \quad \text{for } z \in E.$$

It is easy to see that the set $\tilde{P}_m^\beta[A, B]$ is convex. For special values of A, B , and β , we obtain several subclasses of analytic functions studied and investigated by several researchers [4, 7–9, 13–15].

Definition 2.2 An analytic function $f(z) \in \tilde{R}_m^\beta[A, B]$ if and only if

$$\frac{zf'(z)}{f(z)} \in \tilde{P}_m^\beta[A, B] \quad \text{for } z \in E. \tag{2.1}$$

Note that, for $\beta = 1$, the class $\tilde{R}_m^\beta[A, B]$ yields the class $R_m[A, B]$ [8] and $\tilde{R}_m^\beta[1, -1] = R_m(\beta)$. For $\beta = \frac{1}{2}$, $B = 0$, and $A = a$ with $a \in (0, 1]$, we obtain the class

$$\tilde{R}_m^{\frac{1}{2}}[a, 0] = q_1, q_2 < \sqrt{1 + az} \quad \text{for } z \in E. \tag{2.2}$$

Definition 2.3 A function $f \in \mathcal{A}$ belongs to $\tilde{V}_m^\beta[A, B]$ if and only if f is locally univalent and

$$\frac{(zf'(z))'}{f'(z)} \in \tilde{P}_m^\beta[A, B] \quad \text{for } z \in E. \tag{2.3}$$

From (2.1) and (2.3), it is considered that

$$f \in \tilde{V}_m^\beta[A, B] \iff zf'(z) \in \tilde{R}_m^\beta[A, B]. \tag{2.4}$$

Note that, for particular cases of A, B , and β , several classes are obtained and investigated in [8]. For $\beta = \frac{1}{2}$, $m = 2$, $B = 0$ and $A = a, a \in (0, 1]$, Aouf [1] obtained the following class:

$$\tilde{V}_m^{\frac{1}{2}}[a, 0] = \left\{ f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} = \left(\frac{m}{4} + \frac{1}{2}\right)q_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)q_2(z), q_1, q_2 < \sqrt{1 + az} \right\}.$$

Now we prove the following lemmas, which represent the coefficient inequality of functions belonging to the class $\tilde{P}_m^\beta[A, B]$. By choosing special values of A, B , and β , we may relate our finding with the existing results. We shall assume $m \geq 2, -1 \leq B < A \leq 1, 0 < \beta \leq 1$, unless otherwise stated.

Lemma 2.4 Suppose $q(z) = 1 + \sum_{n=1}^\infty a_n z^n$ belongs to $\tilde{P}_m^\beta[A, B]$. Then

$$|a_n| \leq \frac{\beta m}{2} |A - B| \quad \text{for all } n \geq 1. \tag{2.4*}$$

Proof Suppose $q(z)$ belongs to $\tilde{P}_m^\beta[A, B]$ with $q(z) = 1 + \sum_{n=1}^\infty a_n z^n$. Then

$$q(z) = \left(\frac{m}{4} + \frac{1}{2}\right)q_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)q_2(z), \quad q_1, q_2 \in \tilde{P}^\beta[A, B].$$

If $q_1(z) = 1 + \sum_{n=1}^\infty b_n z^n$ and $q_2(z) = 1 + \sum_{n=1}^\infty c_n z^n$, then

$$1 + \sum_{n=1}^\infty a_n z^n = \left(\frac{m}{4} + \frac{1}{2}\right)\left(1 + \sum_{n=1}^\infty b_n z^n\right) - \left(\frac{m}{4} - \frac{1}{2}\right)\left(1 + \sum_{n=1}^\infty c_n z^n\right).$$

By comparing the coefficients of z^n and using the triangle inequality, we get

$$|a_n| \leq \left(\frac{m}{4} + \frac{1}{2}\right)|b_n| + \left(\frac{m}{4} - \frac{1}{2}\right)|c_n|.$$

Since $q_i(z) \prec \left(\frac{1+Az}{1+Bz}\right)^\beta = 1 + \beta(A - B)z + \dots$ for $i = 1, 2$. Using the well-known result of Rogosinski [15] on subordination, we have $|b_n| \leq \beta|A - B|$ and $|c_n| \leq \beta|A - B|$ for all $n \geq 1$. This implies that

$$|a_n| \leq \left(\frac{m}{4} + \frac{1}{2}\right)\beta|A - B| + \left(\frac{m}{4} - \frac{1}{2}\right)\beta|A - B|,$$

which is as required. This completes the proof. □

The following lemma extends the inequality bound of functions in the class $P_m[A, B]$.

Lemma 2.5 *If $q(z)$ belongs to $\tilde{P}_m^\beta[A, B]$ and $z = re^{i\theta}$, then*

$$\frac{m(k_1 - k_2) + 2(k_1 + k_2)}{4(1 - B^2r^2)} \leq \operatorname{Re}\{q(z)\} \leq \frac{m}{2} \left(\frac{1 + Ar}{1 + Br}\right)^\beta, \tag{2.5}$$

where $k_1 = (1 - (A - B)r - AB r^2)^\beta$, $k_2 = (1 + (A - B)r - AB r^2)^\beta$.

Proof Suppose $q(z) \in \tilde{P}_m^\beta[A, B]$. Then

$$q(z) = \left(\frac{m}{4} + \frac{1}{2}\right)q_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)q_2(z), \quad \text{where } q_1(z), q_2(z) \text{ in } \tilde{P}^\beta[A, B]. \tag{2.6}$$

Using the triangle inequality, we have

$$\operatorname{Re}(q(z)) \leq |q(z)| \leq \left(\frac{m}{4} + \frac{1}{2}\right)|q_1(z)| + \left(\frac{m}{4} - \frac{1}{2}\right)|q_2(z)|. \tag{2.7}$$

Since $q_1(z), q_2(z) \in \tilde{P}^\beta[A, B]$, then

$$\operatorname{Re}\{q_i(z)\} \leq \left(\frac{1 + A|w|}{1 + B|w|}\right)^\beta < \left(\frac{1 + Ar}{1 + Br}\right)^\beta \quad \text{for } i = 1, 2, \tag{2.8}$$

where $w(z)$ is defined in E , which is an analytic function with $w(0) = 0$ and $|w(z)| < r$. Using (2.8) in (2.7), we get the right-hand side of (2.5).

Moreover, (2.6) can be written as

$$\operatorname{Re}(q(z)) \geq \left(\frac{m}{4} + \frac{1}{2}\right)\operatorname{Re}\{q_1(z)\} - \left(\frac{m}{4} - \frac{1}{2}\right)|q_2(z)|. \tag{2.9}$$

Since $q_1(z), q_2(z) \in \tilde{P}^\beta[A, B]$ and

$$\operatorname{Re}\{q_i(z)\} \geq \left(\frac{1 - A|w|}{1 - B|w|}\right)^\beta > \left(\frac{1 - Ar}{1 - Br}\right)^\beta \quad \text{for } i = 1, 2. \tag{2.10}$$

Using (2.10) in (2.9), we have

$$\{\operatorname{Re} q(z)\} \geq \left(\frac{m}{4} + \frac{1}{2}\right)\left(\frac{1 - Ar}{1 - Br}\right)^\beta - \left(\frac{m}{4} - \frac{1}{2}\right)\left(\frac{1 + Ar}{1 + Br}\right)^\beta. \tag{2.11}$$

This implies that

$$\begin{aligned} \{\operatorname{Re} q(z)\} &\geq \left(\frac{m}{4} + \frac{1}{2}\right) \left(\frac{1 - (A - B)r - AB r^2}{1 - B^2 r^2}\right)^\beta \\ &\quad - \left(\frac{m}{4} - \frac{1}{2}\right) \left(\frac{1 + (A - B)r - AB r^2}{1 - B^2 r^2}\right)^\beta. \end{aligned} \tag{2.11*}$$

By simple calculations, we get the left-hand side of (2.5). Hence the proof. \square

Lemma 2.6 *If a function $q(z) \in \mathcal{A}$ belongs to $\tilde{P}_m^\beta[A, B]$, then $q(z) \in P_m(\gamma)$, where $\gamma = (\frac{1-A}{1-B})^\beta$ for $z \in E$.*

Proof Let $q(z)$ belong to $\tilde{P}_m^\beta[A, B]$. From (2.6) and (2.8), we have

$$\operatorname{Re}\{q_i(z)\} \geq \left(\frac{1 - A|w|}{1 - B|w|}\right)^\beta > \left(\frac{1 - Ar}{1 - Br}\right)^\beta, \quad i = 1, 2, z \in E. \tag{2.12}$$

From this it follows that each $q_i(z) \in P(\gamma)$ for $i = 1, 2$, where $\gamma = (\frac{1-A}{1-B})^\beta$ and $q(z) \in P_m(\gamma)$ for $\gamma = (\frac{1-A}{1-B})^\beta$. This implies that $\tilde{P}_m^\beta[A, B] \subseteq P_m(\gamma)$. Hence the proof. \square

In the following theorems, we discuss and investigate the coefficient problem, analytic property, inclusion results, and radius problem.

Theorem 2.7 *If $f \in \tilde{V}_m^\beta[A, B]$, then there exist $\sigma_1, \sigma_2 \in \tilde{S}(\gamma)$ with $\gamma = (\frac{1-A}{1-B})^\beta$ such that*

$$f'(z) = \frac{\left(\frac{\sigma_1(z)}{z}\right)^{\left(\frac{m}{4} + \frac{1}{2}\right)}}{\left(\frac{\sigma_2(z)}{z}\right)^{\left(\frac{m}{4} - \frac{1}{2}\right)}}, \quad z \in E. \tag{2.13}$$

Proof Let $f(z)$ belong to $\tilde{V}_m^\beta[A, B]$. Then $q(z) = \frac{(zf'(z))^\gamma}{f'(z)} \in \tilde{P}_m^\beta[A, B]$. By Lemma 2.6, $\tilde{P}_m^\beta[A, B] \subseteq P_m(\gamma)$. This implies that $\frac{(zf'(z))^\gamma}{f'(z)} \in P_m(\gamma)$ with $\gamma = (\frac{1-A}{1-B})^\beta$. It follows that $f \in V_m(\gamma)$. By Lemma 1.1, it has the form

$$f'(z) = (g(z))^{1-\lambda}, \quad \text{where } \lambda = \left(\frac{1 - A}{1 - B}\right)^\beta \text{ and } g(z) \in V_m. \tag{2.14}$$

Brannan [2] showed that each $g(z) \in V_m$ has the representation of the form

$$g'(z) = \frac{\left(\frac{s_1(z)}{z}\right)^{\left(\frac{m}{4} + \frac{1}{2}\right)}}{\left(\frac{s_2(z)}{z}\right)^{\left(\frac{m}{4} - \frac{1}{2}\right)}}, \quad s_1, s_2 \in \tilde{S}. \tag{2.15}$$

It has been proved in [12] that each $\sigma_i \in \tilde{S}(\gamma)$ is

$$\sigma_i(z) = z \left(\frac{S_i(z)}{z}\right)^{1-\gamma}, \quad S_i(z) \in \tilde{S}, \quad i = 1, 2. \tag{2.16}$$

Using (2.14), (2.15), and (2.16), we get our required result (2.13). This completes the proof. \square

Theorem 2.8 *Let $f(z)$ belong to $\tilde{R}_m^\beta[A, B]$ with $f(z) = z + \sum_{n=2}^\infty a_n z^n$ for $z \in E$. Then*

$$|a_n| \leq \frac{(\frac{\beta m}{2}|A - B|)_{n-1}}{(n - 1)!} \quad \text{for all } n \geq 2. \tag{2.17}$$

Proof Let $f(z)$ belong to $\tilde{R}_m^\beta[A, B]$. Then $q(z) = \frac{zf'(z)}{f(z)} \in \tilde{P}_m^\beta[A, B]$. Suppose $q(z) = \frac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^\infty c_n z^n$, where $q(z)$ is defined in E and is analytic with $q(0) = 1$. This implies

$$zf'(z) = f(z)q(z). \tag{2.18}$$

The series representation of (2.18) is

$$z + \sum_{n=2}^\infty n a_n z^n = \left(z + \sum_{n=2}^\infty a_n z^n \right) \left(1 + \sum_{n=1}^\infty c_n z^n \right) = z + \sum_{n=2}^\infty \sum_{i=0}^{n-1} c_i a_{n-i} z^n.$$

Comparing the coefficient of z^n , we obtain $(n - 1)a_n = \sum_{i=1}^{n-1} c_i a_{n-i}$, $c_0 = 1$ and hence

$$|a_n| = \frac{1}{n - 1} \sum_{i=1}^{n-1} |c_i| |a_{n-i}| \quad \text{for } n \geq 2. \tag{2.19}$$

Since $q(z) \in \tilde{P}_m^\beta[A, B]$ and using Lemma 2.4, we get $|c_i| \leq \frac{\beta m}{2}(A - B)$ for all $n \geq 1$. From (2.19), we have $|a_n| \leq \frac{\beta m(A - B)}{2(n - 1)} \sum_{i=1}^{n-1} |a_i|$. In particular,

$$\begin{aligned} |a_2| &\leq \frac{\beta m}{2}(A - B), \\ |a_3| &\leq \frac{\beta m(A - B)}{2} \left(\frac{\beta m}{2}(A - B) + 1 \right) = \frac{(\frac{\beta m}{2}(A - B))_2}{(2)!}. \end{aligned}$$

For $n = j$,

$$\begin{aligned} |a_j| &\leq \frac{\beta m(A - B)}{j - 1} \sum_{i=1}^{j-1} |a_i| = \frac{\beta m(A - B)}{j - 1} \prod_{i=1}^{j-2} \left(\frac{\beta m(A - B)}{i} + 1 \right) \\ &= \frac{(\frac{\beta m}{2}(A - B))_{j-1}}{(j - 1)!} \quad \text{for } j \geq 3, \end{aligned} \tag{2.20}$$

where we use the Pochhammer notation $(\beta)_m$ introduced in [11]:

$$(\beta)_m = \begin{cases} 1, & m = 0, \beta \in \mathbb{C} \setminus \{0\}, \\ \beta(\beta + 1) \cdot (\beta + m - 1), & m \in \mathbb{N}, \beta \in \mathbb{C}. \end{cases}$$

Consider

$$|a_{m+1}| \leq \frac{\beta m(A - B)}{m} \sum_{i=1}^{m-1} |a_i| + \frac{\beta m(A - B)}{m} |a_m|.$$

Using (2.20), we obtain the following:

$$\begin{aligned}
 |a_{j+1}| &\leq \frac{\beta \frac{m}{2}(A-B)}{j} \prod_{i=1}^{j-2} \left(\frac{\beta \frac{m}{2}(A-B)}{i} + 1 \right) + \frac{\frac{m}{2}(\beta(A-B))^2}{j(j-1)} \prod_{i=1}^{j-2} \left(\frac{\beta(A-B)}{i} + 1 \right) \\
 &= \frac{\beta \frac{m}{2}(A-B)}{j} \prod_{i=1}^{j-1} \left(\frac{\beta \frac{m}{2}(A-B)}{i} + 1 \right) = \frac{(\beta \frac{m}{2}(A-B))_m}{(j)!}.
 \end{aligned}$$

By induction, we get (2.17). Hence the proof. □

Theorem 2.9 *Let $f(z)$ belong to $\tilde{V}_m^\beta[A, B]$ with $f(z) = z + \sum_{n=2}^\infty a_n z^n, z \in E$. Then*

$$|a_n| \leq \frac{(\beta \frac{m}{2}(A-B))_{n-1}}{(n)!} \quad \text{for } n \geq 2.$$

Proof Let $f(z)$ belong to $\tilde{V}_m^\beta[A, B]$ with $f(z) = z + \sum_{n=2}^\infty a_n z^n$. Then $zf'(z) \in \tilde{R}_m^\beta[A, B]$. By using Theorem 2.8, we get

$$|a_n| \leq \frac{(\beta \frac{m}{2}(A-B))_{n-1}}{(n)!} \quad \text{for } n \geq 2. \quad \square$$

Corollary 2.10 *For $A = 1, B = -1, f \in \tilde{V}_m^\beta[1, -1] \equiv \tilde{V}_m(\beta)$ and using Theorem 2.9, we obtain*

$$|a_n| \leq \frac{(2m\beta)_{n-1}}{(n)!} \quad \text{for } n \geq 2.$$

Theorem 2.11 *If $f \in \tilde{V}_m^\beta[A, B]$, then $f \in \tilde{R}_m^\beta(\rho)$, where*

$$\rho = \frac{(2\gamma - 1) + \sqrt{(1 - 2\gamma)^2 + 8}}{4}, \quad \gamma = \left(\frac{1 - A}{1 - B} \right)^\beta. \tag{2.21}$$

Proof Suppose that

$$\begin{aligned}
 \frac{zf'(z)}{f(z)} &= (1 - \rho)q(z) + \rho \\
 &= (1 - \rho) \left\{ \left(\frac{m}{4} + \frac{1}{2} \right) q_1(z) - \left(\frac{m}{4} - \frac{1}{2} \right) q_2(z) \right\} + \rho,
 \end{aligned} \tag{2.22}$$

where $q(z)$ is analytic in E with $q(0) = 1$. From (2.21), we have

$$\frac{(zf'(z))'}{f'(z)} = (1 - \rho) \left[q(z) + \frac{zq'(z)}{(1 - \rho)q(z) + \rho} \right] + \rho. \tag{2.23}$$

$f \in \tilde{V}_m^\beta[A, B]$ implies that $\frac{(zf'(z))'}{f'(z)} \in \tilde{P}_m^\beta[A, B]$ and $\tilde{P}_m^\beta[A, B] \subseteq P_m(\gamma)$, where $\gamma = \left(\frac{1-A}{1-B} \right)^\beta$. From this it follows that

$$\frac{1}{1 - \gamma} \left(\frac{(zf'(z))'}{f'(z)} - \gamma \right) \in P_m. \tag{2.24}$$

Using (2.23) and (2.24), we get

$$\frac{(1-\rho)}{1-\gamma} \left[q(z) + \frac{\frac{1}{1-\rho} z Q'(z)}{q(z) + \frac{\rho}{1-\rho}} \right] + \frac{(\rho-\gamma)}{1-\gamma} \in P_m. \tag{2.25}$$

Now we define

$$\Phi_{\eta_1, \eta_2}(z) = \frac{1}{\eta_2 + 1} \frac{z}{(1-z)^{\eta_1}} + \frac{\eta_2}{\eta_2 + 1} \frac{z}{(1-z)^{\eta_1+1}},$$

with $\eta_1 = \frac{1}{1-\rho}$, $\eta_2 = \frac{\rho}{1-\rho}$. Using (2.22) with the convolution techniques given by Noor [7], we have

$$q(z) + \frac{\eta_1 z Q(z)}{q(z) + \eta_2} = (1-\rho) \left[\left(\frac{m}{4} + \frac{1}{2} \right) \left\{ q_1(z) + \frac{\eta_1 z Q'_1(z)}{q_1(z) + \eta_2} \right\} - \left(\frac{m}{4} - \frac{1}{2} \right) \left\{ q_2(z) + \frac{\eta_1 z Q'_2(z)}{q_2(z) + \eta_2} \right\} \right] + \rho. \tag{2.26}$$

Thus, using (2.25) and (2.26), we have

$$\frac{(1-\rho)}{1-\gamma} \left\{ q_i(z) + \frac{\eta_1 z Q'_i(z)}{q_i(z) + \eta_2} \right\} + \frac{(\rho-\gamma)}{1-\gamma} \in P \quad \text{for } i = 1, 2 \text{ and } z \in E. \tag{2.27}$$

Since the first two conditions of Lemma 1.2 are obviously satisfied, we satisfy condition (iii) as follows:

$$\begin{aligned} \operatorname{Re} \Phi(i\xi_2, \xi_1) &= \frac{\rho-\gamma}{1-\gamma} + \frac{1-\rho}{1-\gamma} \operatorname{Re} \left\{ i\xi_2 + \frac{\rho \xi_1}{(1-\rho)i\xi_2 + \rho} \right\} \\ &\leq \frac{\rho-\gamma}{1-\gamma} - \frac{1-\rho}{2(1-\gamma)} \left\{ \frac{\rho(1+\xi_2^2)}{(1-\rho)^2 \xi_2^2 + \rho^2} \right\}, \quad \text{for } \xi_1 \leq -\frac{1+\xi_2^2}{2} \\ &= \frac{A + B\xi_2^2}{D}, \end{aligned} \tag{2.28}$$

where

$$\begin{aligned} A &= 2(\rho-\gamma)\rho^2 - \rho(1-\rho), \\ B &= 2(\rho-\gamma)(1-\rho)^2 - \rho(1-\rho), \\ D &= 2(1-\gamma)((1-\rho)^2 u_2^2 + \rho^2). \end{aligned}$$

It is observed that (2.28) gives negative value for $A \leq 0$ and $B \leq 0$. Therefore, using $A \leq 0$, we have

$$\rho = \frac{(2\gamma-1) + \sqrt{(1-2\gamma)^2 + 8}}{4}, \quad \gamma = \left(\frac{1-A}{1-B} \right)^\beta, \tag{2.29}$$

and from $B \leq 0$, we obtain $0 \leq \rho < 1$. Hence all three conditions of Lemma 1.2 are satisfied. From this it follows that $q_i(z) \in P$ for $i = 1, 2$ and $z \in E$. Therefore $q(z) \in P_m$ and $f \in R_m(\rho)$, where ρ is given in (2.29). This completes the proof. \square

Corollary 2.12 *If $\beta = \frac{1}{2}, B = 0, A = a, a \in (0, 1]$ and $f \in \tilde{V}_m^{\frac{1}{2}}[a, 0]$. Then, using Theorem 2.11, we have $f \in \tilde{R}_m^{\frac{1}{2}}(\rho)$, where*

$$\rho = \frac{(2\gamma - 1) + \sqrt{(1 - 2\gamma)^2 + 8}}{4}, \quad \gamma = \sqrt{1 - a}. \tag{2.30}$$

Theorem 2.13 *If $f(z)$ belongs to $\tilde{R}_m^\beta[A, B]$, then $f(z) \in R_m(\gamma)$, $\gamma = (\frac{1-A}{1-B})^\beta$ for $z \in E$.*

Proof $f(z)$ belongs to $\tilde{R}_m^\beta[A, B]$ implies $q(z) = \frac{zf'(z)}{f(z)}$ for some $q \in \tilde{P}_m^\beta[A, B]$. Using Lemma 2.6, we get $\frac{zf'(z)}{f(z)} \in P_m(\gamma)$, where $\gamma = (\frac{1-A}{1-B})^\beta$. Thus $f(z) \in R_m(\gamma)$ for $z \in E$. This completes the proof. □

Corollary 2.14 *If $\beta = 1, m = 2$ and $f \in S^*[A, B]$. Then, using Theorem 2.13, we have $f(z) \in S^*(\gamma)$, where $\gamma = (\frac{1-A}{1-B})$.*

3 Conclusion

We introduced some new classes of analytic functions with bounded radius and bounded boundary rotation by using the subordination. We discussed inclusion results, coefficient bounds, growth and distortion theorems of the classes. The radii of starlikeness and strong starlikeness of the classes have been computed. It is observed that most of the results are best possible.

Acknowledgements

Authors are thankful to the Editor-in-Chief Prof. Dr. Gradimir V. Milovanovic and all anonymous reviewers for their valuable comments towards the improvement of the paper. The second author is grateful to Air Marshal Javed Ahmed, HI(M) (Retd), Vice-Chancellor, Air University, Islamabad, Pakistan, for providing the excellent research facilities.

Funding

We have no funding for this project of research.

Availability of data and materials

Please contact the authors for data requests.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 14 April 2019 Accepted: 23 April 2020 Published online: 07 May 2020

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