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Hermite–Hadamard-type inequalities involving ψ-Riemann–Liouville fractional integrals via s-convex functions

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Abstract

In this paper, we establish some new Hermite–Hadamard-type inequalities involving ψ-Riemann–Liouville fractional integrals via s-convex functions in the second sense. Meanwhile, we present many useful estimates on these types of new Hermite–Hadamard-type inequalities. Finally, we give some applications to special means of real numbers.

Keywords: Hermite–Hadamard inequalities; ψ-Riemann–Liouville fractional integrals; s-convex functions

1 Introduction

The classical Hermite–Hadamard inequality is as follows:

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(s)\,ds \leq \frac{g(a)+g(b)}{2} \tag{1}$$

for convex functions $g : [a,b] \subset R \to R$ (see [1]).

In the past decade, fractional calculus has been regarded as one of the best tools to describe long-memory processes. Many researchers are interested in such a model. The most important of these models are described by differential equations with fractional derivatives. Their evolution is much more complex than the classical integer-order case, and the corresponding theory is also more difficult in the integer-order case. The theory of fractional integral inequalities plays an important role in mathematics.

The Hermite–Hadamard integral inequality for convex functions is one of the most famous inequalities. Ten recently published papers [2–11] are focused on the generalizations and variants for the convexity and Hermite–Hadamard inequality. Many mathematicians devoted to the promotion and expansion of (1). For more information, refer to [1, 12–18] and closely related references.

With a wide application of fractional integration and Hermite–Hadamard inequality, many researchers extended their research to the Hermite–Hadamard inequality, including fractional integration rather than ordinary integration; see [19–27]. Sarikaya et al. [19]

derived an interesting Hermite–Hadamard-type inequality, which contains the fractional integral instead of the ordinary one. The study attracted many researchers to consider the problem. So far, some new integral inequalities have been obtained by using fractional calculus. Sousa et al. [28] introduced fractional integral operators with ψ-Riemann–Liouville kernel and proved similar inequalities.

In addition to the classical convex functions, Hudzik and Maligranda [29] introduced the definition of s-convex functions in the second sense.

Definition 1.1 (see [30, Definition 1.4]) A function $g : I \subseteq R_+ \to R_+$ is said to be s-convex in the second sense on I if inequality $g(\lambda x + (1-\lambda)y) \le \lambda^s g(x) + (1-\lambda)^s g(y)$ for all $x, y \in I$ and $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

Definition 1.2 (see [28, Definition 4]) Let (a,b) $(-\infty \le a < b \le \infty)$ be a finite or infinite interval of the real line R, and let $\alpha > 0$. Also, let $\psi(x)$ be an increasing positive function on $(a,b]$ with continuous derivative $\psi'(x)$ on (a,b). Then the left- and right-sided ψ-Riemann–Liouville fractional integrals of a function f with respect to the function ψ on $[a,b]$ are defined by

$$I_{a^+}^{\alpha:\psi} g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)\big(\psi(x) - \psi(t)\big)^{\alpha-1} g(t)\,dt,$$

$$I_{b^-}^{\alpha:\psi} g(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)\big(\psi(t) - \psi(x)\big)^{\alpha-1} g(t)\,dt,$$

respectively, where Γ is the gamma function.

Lemma 1.3 *Let $h : [b,c] \to R$ be a differentiable mapping on (b,c) with $b < c$. Also, let $h \in L[b,c]$. Then we have the following equality for fractional integrals*:

$$\begin{aligned} &\frac{h(b)+h(c)}{2} - \frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\big[I^{\alpha:\psi}_{\psi^{-1}(b)^+}\big(h^\circ\psi\big)\big(\psi^{-1}(c)\big) + I^{\alpha:\psi}_{\psi^{-1}(c)^-}\big(h^\circ\psi\big)\big(\psi^{-1}(b)\big)\big] \\ &\quad = \frac{c-b}{2}\int_0^1 \big((1-t)^\alpha - t^\alpha\big)h'\big(tb + (1-t)c\big)\,dt. \end{aligned} \tag{2}$$

Proof From [31] we have

$$\begin{aligned} &\frac{h(b)+h(c)}{2} - \frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\big[I^{\alpha:\psi}_{\psi^{-1}(b)^+}\big(h^\circ\psi\big)\big(\psi^{-1}(c)\big) + I^{\alpha:\psi}_{\psi^{-1}(c)^-}\big(h^\circ\psi\big)\big(\psi^{-1}(b)\big)\big] \\ &\quad = \frac{1}{2(c-b)^\alpha}\int_{\psi^{-1}(b)}^{\psi^{-1}(c)} \big[\big(\psi(\nu) - b\big)^\alpha - \big(c - \psi(\nu)\big)^\alpha\big]\big(h'^\circ\psi\big)(\nu)\psi'(\nu)\,d\nu \\ &\quad = \frac{1}{2}\int_{\psi^{-1}(b)}^{\psi^{-1}(c)} \left[\left(\frac{\psi(\nu)-b}{c-b}\right)^\alpha - \left(\frac{c-\psi(\nu)}{c-b}\right)^\alpha\right]\big(h'^\circ\psi\big)(\nu)\psi'(\nu)\,d\nu \\ &\quad \left(\text{let } t = \frac{c-\psi(\nu)}{c-b}\right) \\ &\quad = \frac{c-b}{2}\int_0^1 \big((1-t)^\alpha - t^\alpha\big)h'\big(tb + (1-t)c\big)\,dt. \end{aligned}$$

The proof is completed. □

Lemma 1.4 *Let $h : [b,c] \to R$ be a differentiable mapping on (b,c) with $b < c$. If $h \in L[b,c]$, then we have the following equality for fractional integrals*:

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\big[I^{\alpha:\psi}_{\psi^{-1}(b)^+}(h^\circ\psi)\big(\psi^{-1}(c)\big)+I^{\alpha:\psi}_{\psi^{-1}(c)^-}(h^\circ\psi)\big(\psi^{-1}(b)\big)\big]-h\left(\frac{b+c}{2}\right)\\
&\quad=\frac{c-b}{2}\int_0^1\big(k+t^\alpha-(1-t)^\alpha\big)h'\big(tb+(1-t)c\big)\,dt,
\end{aligned}\tag{3}$$

where

$$k=\begin{cases}1, & 0\le t<\frac{1}{2},\\ -1, & \frac{1}{2}\le t<1.\end{cases}$$

Proof Note that

$$\begin{aligned}
&\frac{c-b}{2}\int_0^1 kh'\big(tb+(1-t)c\big)\,dt\\
&\quad=\frac{c-b}{2}\int_0^{\frac{1}{2}} h'\big(tb+(1-t)c\big)\,dt-\frac{c-b}{2}\int_{\frac{1}{2}}^1 h'\big(tb+(1-t)c\big)\,dt\\
&\quad=\frac{h(c)-h(\frac{b+c}{2})}{2}+\frac{h(b)-h(\frac{b+c}{2})}{2}\\
&\quad=\frac{h(b)+h(c)}{2}-h\left(\frac{b+c}{2}\right).
\end{aligned}$$

By Lemma 1.3 we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\big[I^{\alpha:\psi}_{\psi^{-1}(b)^+}(h^\circ\psi)\big(\psi^{-1}(c)\big)+I^{\alpha:\psi}_{\psi^{-1}(c)^-}(h^\circ\psi)\big(\psi^{-1}(b)\big)\big]-h\left(\frac{b+c}{2}\right)\\
&\quad=\left[\frac{h(b)+h(c)}{2}-h\left(\frac{b+c}{2}\right)\right]\\
&\qquad-\left\{\frac{h(b)+h(c)}{2}-\frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\big[I^{\alpha:\psi}_{\psi^{-1}(b)^+}(h^\circ\psi)\big(\psi^{-1}(c)\big)+I^{\alpha:\psi}_{\psi^{-1}(c)^-}(h^\circ\psi)\big(\psi^{-1}(b)\big)\big]\right\}\\
&\quad=\frac{c-b}{2}\int_0^1 kh'\big(tb+(1-t)c\big)\,dt-\frac{c-b}{2}\int_0^1\big((1-t)^\alpha-t^\alpha\big)h'\big(tb+(1-t)c\big)\,dt\\
&\quad=\frac{c-b}{2}\int_0^1\big(k+t^\alpha-(1-t)^\alpha\big)h'\big(tb+(1-t)c\big)\,dt.
\end{aligned}$$

The proof is completed. □

Lemma 1.5 (see [32, Definition 1.1]) *Let (Ω,Λ,μ) be a measure space with $0<\mu(\Omega)<1$, and let $\phi : I \to R$ be a convex function defined on an open interval I in R. If $f : \Omega \to I$ is such that $f, \phi^\circ f \in L(\Omega,\Lambda,\mu)$, then*

$$\phi\left(\frac{1}{\mu(\Omega)}\int_\Omega f\,d\mu\right)\le\frac{1}{\mu(\Omega)}\int_\Omega\phi(f)\,d\mu.\tag{4}$$

In the case where Ω is strictly convex on I, we have equality in (4) if and only if f is constant almost everywhere on Ω.

Remark 1.6 Inequality (4) is reversed if ϕ is, that is,

$$\phi\left(\frac{1}{\mu(\Omega)}\int_{\Omega} f\,d\mu\right) \geq \frac{1}{\mu(\Omega)}\int_{\Omega}\phi(f)\,d\mu. \tag{5}$$

The main purpose of this paper is to introduce some new Hermite–Hadamard-type inequalities involving ψ-Riemann–Liouville fractional integrals via s-convex functions in the second sense. For these functions, we establish some results related to the left end of new inequalities similar to inequality (1). We give some applications to special mean of a positive real number.

2 Main results

We now in a position to establish some inequalities of Hermite–Hadamard type involving ψ-Riemann–Liouville fractional integrals (with $\alpha \in (0,1)$) via s-convex functions.

Theorem 2.1 *Let $\alpha \in (0,1)$, let $h : [b,c] \to R$ be a positive function with $0 \leq b < c$ and $h \in L[b,c]$, and let ψ be an increasing positive function on $[b,c]$ having a continuous derivative ψ' on (b,c). If h is an s-convex function on $[b,c]$, then we have the following inequality for fractional integrals*:

$$\begin{aligned} 2^{s-1}h\left(\frac{b+c}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(c-b)^{\alpha}}\left[I^{\alpha:\psi}_{\psi^{-1}(b)^{+}}\left(h^{\circ}\psi\right)\left(\psi^{-1}(c)\right) + I^{\alpha:\psi}_{\psi^{-1}(c)^{-}}\left(h^{\circ}\psi\right)\left(\psi^{-1}(b)\right)\right] \\ &\leq \left[\frac{3\alpha}{\alpha+s} - \frac{\alpha}{(\alpha+s)2^{\alpha+s}}\right]\frac{h(b)+h(c)}{2}. \end{aligned} \tag{6}$$

Proof Since h is an s-convex function on $[b,c]$, for every $x,y \in [b,c]$ with $\lambda = \frac{1}{2}$, we have

$$h\left(\frac{x+y}{2}\right) \leq \frac{1}{2^s}h(x) + \frac{1}{2^s}h(y),$$

that is, with $x = tb + (1-t)c, y = (1-t)b + tc$,

$$2^s h\left(\frac{b+c}{2}\right) \leq h\big(tb + (1-t)c\big) + h\big((1-t)b + tc\big). \tag{7}$$

Multiplying both sides of (7) by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 t^{\alpha-1}h\big(tb+(1-t)c\big)\,dt + \int_0^1 t^{\alpha-1}h\big((1-t)b+tc\big)\,dt \\ &\quad\geq \int_0^1 t^{\alpha+s-1}h\left(\frac{b+c}{2}\right)dt \\ &\quad\geq \frac{2^s}{\alpha}h\left(\frac{b+c}{2}\right). \end{aligned}$$

Next,

$$\frac{\Gamma(\alpha+1)}{2(c-b)^{\alpha}}\left[I^{\alpha:\psi}_{\psi^{-1}(b)^{+}}\left(h^{\circ}\psi\right)\left(\psi^{-1}(c)\right) + I^{\alpha:\psi}_{\psi^{-1}(c)^{-}}\left(h^{\circ}\psi\right)\left(\psi^{-1}(b)\right)\right]$$

$$= \frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\biggl[\frac{1}{\Gamma(\alpha)}\int_{\psi^{-1}(b)}^{\psi^{-1}(c)} \psi'(t)\bigl(\psi\bigl(\psi^{-1}(c)\bigr)-\psi(t)\bigr)^{\alpha-1}\bigl(h^\circ\psi\bigr)(t)\,dt$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{\psi^{-1}(b)}^{\psi^{-1}(c)} \psi'(t)\bigl(\psi(t)-\psi\bigl(\psi^{-1}(b)\bigr)\bigr)^{\alpha-1}\bigl(h^\circ\psi\bigr)(t)\,dt\biggr]$$

$$= \frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\times\frac{1}{\Gamma(\alpha)}\biggl[\int_{\psi^{-1}(b)}^{\psi^{-1}(c)} \psi'(t)\bigl(c-\psi(t)\bigr)^{\alpha-1}h\bigl(\psi(t)\bigr)\,dt$$

$$+ \int_{\psi^{-1}(b)}^{\psi^{-1}(c)} \psi'(t)\bigl(\psi(t)-b\bigr)^{\alpha-1}h\bigl(\psi(t)\bigr)\,dt\biggr]$$

$$\bigl(\text{let } m=\psi(t)\bigr)$$

$$= \frac{\alpha}{2(c-b)}\biggl[\int_b^c\biggl(\frac{c-m}{c-b}\biggr)^{\alpha-1}h(m)\,dm+\int_b^c\biggl(\frac{m-b}{c-b}\biggr)^{\alpha-1}h(m)\,dm\biggr]$$

$$\biggl(\text{let } u=\frac{c-m}{c-b}, v=\frac{m-b}{c-b}, \text{ then let } t=u \text{ and } t=v\biggr)$$

$$= \frac{\alpha}{2}\biggl[\int_0^1 t^{\alpha-1}h\bigl(tb+(1-t)c\bigr)\,dt+\int_0^1 t^{\alpha-1}h\bigl((1-t)b+tc\bigr)\,dt\biggr]$$

$$\geq \frac{\alpha}{2}\times\frac{2^s}{\alpha}h\biggl(\frac{b+c}{2}\biggr)$$

$$= 2^{s-1}h\biggl(\frac{b+c}{2}\biggr),$$

so the left-hand side inequality in (6) is proved.

To prove the right-hand side inequality in (6), since h is an s-convex function, for $t\in[0,1]$, we have

$$h\bigl(tb+(1-t)c\bigr)\leq t^sh(b)+(1-t)^sh(c)$$

and

$$h\bigl((1-t)b+tc\bigr)\leq (1-t)^sh(b)+t^sh(c),$$

and then

$$h\bigl(tb+(1-t)c\bigr)+h\bigl((1-t)b+tc\bigr)\leq \bigl(t^s+(1-t)^s\bigr)\bigl(h(b)+h(c)\bigr). \tag{8}$$

Multiplying both sides of (8) by $t^{\alpha-1}$ and then integrating, we obtain

$$\int_0^1 t^{\alpha-1}h\bigl(tb+(1-t)c\bigr)\,dt+\int_0^1 t^{\alpha-1}h\bigl((1-t)b+tc\bigr)\,dt$$

$$\leq \int_0^1 t^{\alpha-1}\bigl(t^s+(1-t)^s\bigr)\bigl(h(b)+h(c)\bigr)\,dt$$

$$= \biggl[\int_0^1 t^{\alpha+s-1}\,dt+\int_0^1 t^{\alpha-1}(1-t)^s\,dt\biggr]\bigl(h(b)+h(c)\bigr)$$

$$= \biggl[\frac{1}{\alpha+s}+\int_0^{\frac{1}{2}} t^{\alpha-1}(1-t)^s\,dt+\int_{\frac{1}{2}}^1 t^{\alpha-1}(1-t)^s\,dt\biggr]\bigl(h(b)+h(c)\bigr)$$

$$\leq \left[\frac{1}{\alpha+s} + \int_0^{\frac{1}{2}} (1-t)^{\alpha+s-1}\,dt + \int_{\frac{1}{2}}^1 t^{\alpha+s-1}\,dt\right]\big(h(b)+h(c)\big)$$
$$\leq \left[\frac{3}{\alpha+s} + \frac{1}{(\alpha+s)2^{\alpha+s}}\right]\big(h(b)+h(c)\big).$$

So then

$$\frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\big[I^{\alpha:\psi}_{\psi^{-1}(b)^+}\big(h^\circ\psi\big)\big(\psi^{-1}(c)\big) + I^{\alpha:\psi}_{\psi^{-1}(c)^-}\big(h^\circ\psi\big)\big(\psi^{-1}(b)\big)\big]$$
$$= \frac{\alpha}{2}\left[\int_0^1 t^{\alpha-1}h\big(tb+(1-t)c\big)\,dt + \int_0^1 t^{\alpha-1}h\big((1-t)b+tc\big)\,dt\right]$$
$$\leq \frac{\alpha}{2}\times\left[\frac{3}{\alpha+s} - \frac{1}{(\alpha+s)2^{\alpha+s}}\right]\big(h(b)+h(c)\big)$$
$$= \left[\frac{3\alpha}{\alpha+s} - \frac{\alpha}{(\alpha+s)2^{\alpha+s}}\right]\frac{h(b)+h(c)}{2}.$$

The proof is completed. □

Theorem 2.2 *Let $h:[b,c]\to R$ be a positive function with $0\leq b<c$ such that $h\in L[b,c]$, and let ψ be an increasing positive function on $[b,c]$ having a continuous derivative ψ' on (b,c). If h' is an s-convex function on $[b,c]$ for some fixed $s\in(0,1]$, then we have the following inequality for fractional integrals*:

$$\left|\frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\big[I^{\alpha:\psi}_{\psi^{-1}(b)^+}\big(h^\circ\psi\big)\big(\psi^{-1}(c)\big) + I^{\alpha:\psi}_{\psi^{-1}(c)^-}\big(h^\circ\psi\big)\big(\psi^{-1}(b)\big)\big] - h\left(\frac{b+c}{2}\right)\right|$$
$$\leq \frac{c-b}{2(s+1)}\big(\big|h'(b)\big| + \big|h'(c)\big|\big). \tag{9}$$

Proof Using Lemma 1.4 and the s-convexity of h, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\big[I^{\alpha:\psi}_{\psi^{-1}(b)^+}\big(h^\circ\psi\big)\big(\psi^{-1}(c)\big) + I^{\alpha:\psi}_{\psi^{-1}(c)^-}\big(h^\circ\psi\big)\big(\psi^{-1}(b)\big)\big] - h\left(\frac{b+c}{2}\right)\right|$$
$$= \frac{c-b}{2}\left|\int_0^1 \big(k+t^\alpha-(1-t)^\alpha\big)h'\big(tb+(1-t)c\big)\,dt\right|$$
$$\leq \frac{c-b}{2}\left\{\int_0^{\frac{1}{2}} \big(1+t^\alpha-(1-t)^\alpha\big)\big[t^s\big|h'(b)\big| + (1-t)^s\big|h'(c)\big|\big]\,dt\right.$$
$$\left. + \int_{\frac{1}{2}}^1 \big((1-t)^\alpha+1-t^\alpha\big)\big[t^s\big|h'(b)\big| + (1-t)^s\big|h'(c)\big|\big]\,dt\right\}$$
$$= \frac{c-b}{2}\left\{\big|h'(b)\big|\int_0^{\frac{1}{2}}\big[t^s+t^{\alpha+s}-t^s(1-t)^\alpha\big]\,dt\right.$$
$$+ \big|h'(c)\big|\int_0^{\frac{1}{2}}\big[(1-t)^s+t^\alpha(1-t)^s-(1-t)^{\alpha+s}\big]\,dt$$
$$+ \big|h'(b)\big|\int_{\frac{1}{2}}^1\big[t^s(1-t)^\alpha+t^s-t^{\alpha+s}\big]\,dt$$
$$\left. + \big|h'(c)\big|\int_{\frac{1}{2}}^1\big[(1-t)^{\alpha+s}+(1-t)^s-t^\alpha(1-t)^s\big]\,dt\right\}$$

$$\leq \frac{c-b}{2}\left\{|h'(b)|\int_0^{\frac{1}{2}} t^s\,dt + |h'(c)|\int_0^{\frac{1}{2}}(1-t)^s\,dt + |h'(b)|\int_{\frac{1}{2}}^1 t^s\,dt + |h'(c)|\int_{\frac{1}{2}}^1 (1-t)^s\,dt\right\}$$

$$= \frac{c-b}{2}\left\{|h'(b)|\int_0^1 t^s\,dt + |h'(c)|\int_0^1 (1-t)^s\,dt\right\}$$

$$= \frac{c-b}{2(s+1)}\left(|h'(b)| + |h'(c)|\right).$$

The proof is completed. □

Theorem 2.3 *Let $h:[b,c]\to R$ be a positive function with $0\leq b<c$ such that $h\in L[b,c]$, and let ψ be an increasing positive function on $[b,c]$ having a continuous derivative ψ' on (b,c). If $|h'|^q$ $(q>1)$ is an s-convex function on $[b,c]$ for some fixed $s\in(0,1]$, then we have the following inequality for fractional integrals*:

$$\left|\frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\left[I^{\alpha:\psi}_{\psi^{-1}(b)^+}(h\circ\psi)(\psi^{-1}(c)) + I^{\alpha:\psi}_{\psi^{-1}(c)^-}(h\circ\psi)(\psi^{-1}(b))\right] - h\left(\frac{b+c}{2}\right)\right|$$
$$\leq (c-b)\left(\frac{1}{(\alpha p+1)2^{\alpha p+1}}\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}$$
$$\times\left[\left(|h'(b)|^q + (2^{s+1}-1)|h'(c)|^q\right)^{\frac{1}{q}} + \left((2^{s+1}-1)|h'(b)|^q + |h'(c)|^q\right)^{\frac{1}{q}}\right], \tag{10}$$

where $\frac{1}{p} = 1-\frac{1}{q}$.

Proof Using Lemma 1.4 and the Hölder inequality via the s-convexity of $|h'|^q$ $(q>1)$, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\left[I^{\alpha:\psi}_{\psi^{-1}(b)^+}(h\circ\psi)(\psi^{-1}(c)) + I^{\alpha:\psi}_{\psi^{-1}(c)^-}(h\circ\psi)(\psi^{-1}(b))\right] - h\left(\frac{b+c}{2}\right)\right|$$

$$\leq \frac{c-b}{2}\left\{\int_0^{\frac{1}{2}}\left(1+t^\alpha-(1-t)^\alpha\right)\left|h'(tb+(1-t)c)\right|dt + \int_{\frac{1}{2}}^1\left((1-t)^\alpha+1-t^\alpha\right)\left|h'(tb+(1-t)c)\right|dt\right\}$$

$$\leq \frac{c-b}{2}\left\{\left(\int_0^{\frac{1}{2}}\left(1+t^\alpha-(1-t)^\alpha\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|h'(tb+(1-t)c)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1\left((1-t)^\alpha+1-t^\alpha\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|h'(tb+(1-t)c)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq \frac{c-b}{2}\left(\int_0^{\frac{1}{2}}\left(1+t^\alpha-(1-t)^\alpha\right)^p dt\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{2}}\left[t^s|h'(b)|^q+(1-t)^s|h'(c)|^q\right]dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1\left[t^s|h'(b)|^q+(1-t)^s|h'(c)|^q\right]dt\right)^{\frac{1}{q}}\right\}$$

$$\leq \frac{c-b}{2}\left(\int_0^{\frac{1}{2}}\left(1+t^\alpha-(1-t^\alpha)\right)^p dt\right)^{\frac{1}{p}}$$

$$\times \left\{ \left(\frac{1}{(s+1)2^{s+1}} \left| h'(b) \right|^q + \frac{1}{s+1} \left(1 - \frac{1}{2^{s+1}} \right) \left| h'(c) \right|^q \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{1}{s+1} \left(1 - \frac{1}{2^{s+1}} \right) \left| h'(b) \right|^q + \frac{1}{(s+1)2^{s+1}} \left| h'(c) \right|^q \right)^{\frac{1}{q}} \right\}$$

$$\le \frac{c-b}{2} \left(2^p \int_0^{\frac{1}{2}} t^{\alpha p}\, dt \right)^{\frac{1}{p}} \left(\frac{1}{(s+1)2^{s+1}} \right)^{\frac{1}{q}}$$

$$\times \left[\left(\left| h'(b) \right|^q + \left(2^{s+1} - 1 \right) \left| h'(c) \right|^q \right)^{\frac{1}{q}} + \left(\left(2^{s+1} - 1 \right) \left| h'(b) \right|^q + \left| h'(c) \right|^q \right)^{\frac{1}{q}} \right]$$

$$\le (c-b) \left(\frac{1}{(\alpha p+1)2^{\alpha p+1}} \right)^{\frac{1}{p}} \left(\frac{1}{(s+1)2^{s+1}} \right)^{\frac{1}{q}}$$

$$\times \left[\left(\left| h'(b) \right|^q + \left(2^{s+1} - 1 \right) \left| h'(c) \right|^q \right)^{\frac{1}{q}} + \left(\left(2^{s+1} - 1 \right) \left| h'(b) \right|^q + \left| h'(c) \right|^q \right)^{\frac{1}{q}} \right].$$

The proof is completed. □

Corollary 2.4 *Let $h : [b,c] \to R$ be a positive function with $0 \le b < c$ such that $h \in L[b,c]$ $\psi(\cdot)$ is an increasing and positive monotone function on $[b,c]$, having a continuous derivative ψ' on (b,c). If $|h'|^q$ $(q > 1)$ is an s-convex function on $[b,c]$ for some fixed $s \in (0,1]$, then we have the following inequality for fractional integrals*:

$$\left| \frac{\Gamma(\alpha+1)}{2(c-b)^\alpha} \left[I^{\alpha:\psi}_{\psi^{-1}(b)^+} \left(h^\circ \psi \right) \left(\psi^{-1}(c) \right) + I^{\alpha:\psi}_{\psi^{-1}(c)^-} \left(h^\circ \psi \right) \left(\psi^{-1}(b) \right) \right] - h\left(\frac{b+c}{2} \right) \right|$$

$$\le (c-b) \left(\frac{1}{(\alpha p+1)2^{\alpha p+1}} \right)^{\frac{1}{p}} \left(\frac{1}{(s+1)2^{s+1}} \right)^{\frac{1}{q}}$$

$$\times \left(1 + \left(2^{s+1} - 1 \right)^{\frac{1}{q}} \right) \left(\left| h'(b) \right| + \left| h'(c) \right| \right), \tag{11}$$

where $\frac{1}{p} = 1 - \frac{1}{q}$.

Proof We consider inequality (10), and we let $a_1 = |h'(b)|^q$, $b_1 = (2^{s+1} - 1)|h'(c)|^q$, $a_2 = (2^{s+1} - 1)|h'(b)|^q$, $b_2 = |h'(c)|^q$. Here $0 < \frac{1}{q} < 1$ for $q > 1$. Using the inequality $\sum_{i=1}^n (a_i + b_i)^r \le \sum_{i=1}^n a_i^r + \sum_{i=1}^n b_i^r$ for $0 < r < 1$, $a_i > 0$, $b_i > 0$, $i = 1, 2, \ldots, n$, we obtain the required result. This completes the proof. □

Theorem 2.5 *Let $h : [b,c] \to R$ be a positive function with $0 \le b < c$ such that $h \in L[b,c]$, and let ψ is an increasing positive function on $[b,c]$ having a continuous derivative ψ' on (b,c). If $|h'|^q$ $(q > 1)$ is an s-convex function on $[b,c]$ for some fixed $s \in (0,1]$, then we have the following inequality for fractional integrals*:

$$\left| \frac{\Gamma(\alpha+1)}{2(c-b)^\alpha} \left[I^{\alpha:\psi}_{\psi^{-1}(b)^+} \left(h^\circ \psi \right) \left(\psi^{-1}(c) \right) + I^{\alpha:\psi}_{\psi^{-1}(c)^-} \left(h^\circ \psi \right) \left(\psi^{-1}(b) \right) \right] - h\left(\frac{b+c}{2} \right) \right|$$

$$\le \frac{c-b}{2} \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\frac{\alpha-1}{2} + \frac{1}{2^\alpha} \right)^{1-\frac{1}{q}} \left(\frac{1}{(s+1)2^{s+1}} \right)^{\frac{1}{q}}$$

$$\times \left[\left(\left| h'(b) \right|^q + \left(2^{s+1} - 1 \right) \left| h'(c) \right|^q \right)^{\frac{1}{q}} + \left(\left(2^{s+1} - 1 \right) \left| h'(b) \right|^q + \left| h'(c) \right|^q \right)^{\frac{1}{q}} \right]. \tag{12}$$

Proof Using Lemma 1.4 and the power mean inequality via the s-convexity of $|h'|^q$ $(q>1)$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(c-b)^\alpha}\left[I^{\alpha:\psi}_{\psi^{-1}(b)^+}\bigl(h^\circ\psi\bigr)\bigl(\psi^{-1}(c)\bigr)+I^{\alpha:\psi}_{\psi^{-1}(c)^-}\bigl(h^\circ\psi\bigr)\bigl(\psi^{-1}(b)\bigr)\right]-h\left(\frac{b+c}{2}\right)\right| \\
&\quad\le\frac{c-b}{2}\left\{\int_0^{\frac{1}{2}}\bigl(1+t^\alpha-(1-t)^\alpha\bigr)\bigl|h'\bigl(tb+(1-t)c\bigr)\bigr|\,dt\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1\bigl((1-t)^\alpha+1-t^\alpha\bigr)\bigl|h'\bigl(tb+(1-t)c\bigr)\bigr|\,dt\right\} \\
&\quad\le\frac{c-b}{2}\left\{\left(\int_0^{\frac{1}{2}}\bigl(1+t^\alpha-(1-t)^\alpha\bigr)\,dt\right)^{1-\frac{1}{q}}\right. \\
&\qquad\times\left(\int_0^{\frac{1}{2}}\bigl(1+t^\alpha-(1-t)^\alpha\bigr)\bigl|h'\bigl(tb+(1-t)c\bigr)\bigr|^q\,dt\right)^{\frac{1}{q}} \\
&\qquad\left.+\left(\int_{\frac{1}{2}}^1\bigl((1-t)^\alpha+1-t^\alpha\bigr)\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\bigl((1-t)^\alpha+1-t^\alpha\bigr)\bigl|h'\bigl(tb+(1-t)c\bigr)\bigr|^q\,dt\right)^{\frac{1}{q}}\right\} \\
&\quad\le\frac{c-b}{2}\left(\int_0^{\frac{1}{2}}\bigl(1+t^\alpha-(1-t)^\alpha\bigr)\,dt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left\{\left(\int_0^{\frac{1}{2}}\bigl(1+t^\alpha-(1-t)^\alpha\bigr)\bigl[t^s\bigl|h'(b)\bigr|^q+(1-t)^s\bigl|h'(c)\bigr|^q\bigr]\,dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\int_{\frac{1}{2}}^1\bigl((1-t)^\alpha+1-t^\alpha\bigr)\bigl[t^s\bigl|h'(b)\bigr|^q+(1-t)^s\bigl|h'(c)\bigr|^q\bigr]\,dt\right)^{\frac{1}{q}}\right\} \\
&\quad\le\frac{c-b}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\alpha-1}{2}+\frac{1}{2^\alpha}\right)^{1-\frac{1}{q}} \\
&\qquad\times\left\{\left(\bigl|h'(b)\bigr|^q\int_0^{\frac{1}{2}}\bigl[t^s+t^{\alpha+s}-t^s(1-t)^\alpha\bigr]\,dt\right.\right. \\
&\qquad\left.+\bigl|h'(c)\bigr|^q\int_0^{\frac{1}{2}}\bigl[(1-t)^s+t^\alpha(1-t)^s-(1-t)^{\alpha+s}\bigr]\,dt\right)^{\frac{1}{q}} \\
&\qquad+\left(\bigl|h'(b)\bigr|^q\int_{\frac{1}{2}}^1\bigl[t^s(1-t)^\alpha+t^s-t^{\alpha+s}\bigr]\,dt\right. \\
&\qquad\left.\left.+\bigl|h'(c)\bigr|^q\int_{\frac{1}{2}}^1\bigl[(1-t)^{\alpha+s}+(1-t)^s-t^\alpha(1-t)^s\bigr]\,dt\right)^{\frac{1}{q}}\right\} \\
&\quad\le\frac{c-b}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\alpha-1}{2}+\frac{1}{2^\alpha}\right)^{1-\frac{1}{q}} \\
&\qquad\times\left\{\left(\bigl|h'(b)\bigr|^q\int_0^{\frac{1}{2}}t^s\,dt+\bigl|h'(c)\bigr|^q\int_0^{\frac{1}{2}}(1-t)^s\,dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\bigl|h'(b)\bigr|^q\int_{\frac{1}{2}}^1t^s\,dt+\bigl|h'(c)\bigr|^q\int_{\frac{1}{2}}^1(1-t)^s\,dt\right)^{\frac{1}{q}}\right\} \\
&\quad\le\frac{c-b}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\alpha-1}{2}+\frac{1}{2^\alpha}\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times \left\{ \left(\frac{1}{(s+1)2^{s+1}} \left| h'(b) \right|^q + \frac{1}{s+1} \left(1 - \frac{1}{2^{s+1}} \right) \left| h'(c) \right|^q \right)^{\frac{1}{q}} + \left(\frac{1}{s+1} \left(1 - \frac{1}{2^{s+1}} \right) \left| h'(b) \right|^q + \frac{1}{(s+1)2^{s+1}} \left| h'(c) \right|^q \right)^{\frac{1}{q}} \right\}$$

$$\leq \frac{c-b}{2} \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\frac{\alpha-1}{2} + \frac{1}{2^\alpha} \right)^{1-\frac{1}{q}} \left(\frac{1}{(s+1)2^{s+1}} \right)^{\frac{1}{q}} \times \left[\left(\left| h'(b) \right|^q + \left(2^{s+1} - 1 \right) \left| h'(c) \right|^q \right)^{\frac{1}{q}} + \left(\left(2^{s+1} - 1 \right) \left| h'(b) \right|^q + \left| h'(c) \right|^q \right)^{\frac{1}{q}} \right].$$

The proof is completed. □

Corollary 2.6 *Let $h : [b,c] \to R$ be a positive function with $0 \leq b < c$ such that $h \in L[b,c]$, and let ψ be an increasing positive function on $[b,c]$ having a continuous derivative ψ' on (b,c). If $|h'|^q$ $(q > 1)$ is an s-convex function on $[b,c]$ for some fixed $s \in (0,1]$, then we have the following inequality for fractional integrals*:

$$\left| \frac{\Gamma(\alpha+1)}{2(c-b)^\alpha} \left[I^{\alpha:\psi}_{\psi^{-1}(b)^+} \left(h^\circ \psi \right) \left(\psi^{-1}(c) \right) + I^{\alpha:\psi}_{\psi^{-1}(c)^-} \left(h^\circ \psi \right) \left(\psi^{-1}(b) \right) \right] - h\left(\frac{b+c}{2} \right) \right| \leq \frac{c-b}{2} \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\frac{\alpha-1}{2} + \frac{1}{2^\alpha} \right)^{1-\frac{1}{q}} \left(\frac{1}{(s+1)2^{s+1}} \right)^{\frac{1}{q}} \times \left(1 + \left(2^{s+1} - 1 \right)^{\frac{1}{q}} \right) \left(\left| h'(b) \right| + \left| h'(c) \right| \right). \tag{13}$$

Proof We can obtain the result using the technique in the proof of Corollary 2.4 by considering inequality (13). □

Theorem 2.7 *Let $h : [b,c] \to R$ be a positive function with $0 \leq b < c$ such that $h \in L[b,c]$, and let ψ be an increasing positive function on $[b,c]$ having a continuous derivative ψ' on (b,c). If $|h'|^q$ $(q > 1)$ is a concave function on $[b,c]$, then we have the following inequality for fractional integrals*:

$$\left| \frac{\Gamma(\alpha+1)}{2(c-b)^\alpha} \left[I^{\alpha:\psi}_{\psi^{-1}(b)^+} \left(h^\circ \psi \right) \left(\psi^{-1}(c) \right) + I^{\alpha:\psi}_{\psi^{-1}(c)^-} \left(h^\circ \psi \right) \left(\psi^{-1}(b) \right) \right] - h\left(\frac{b+c}{2} \right) \right| \leq (c-b) \left(\frac{1}{(\alpha p+1)2^{\alpha p+1}} \right)^{\frac{1}{p}} \left(\frac{1}{2} \right)^{\frac{1}{q}} \left(\left| h'\left(\frac{b+3c}{4} \right) \right| + \left| h'\left(\frac{3b+c}{4} \right) \right| \right), \tag{14}$$

where $\frac{1}{p} = 1 - \frac{1}{q}$.

Proof Using Lemma 1.4 and the Hölder inequality, we have

$$\left| \frac{\Gamma(\alpha+1)}{2(c-b)^\alpha} \left[I^{\alpha:\psi}_{\psi^{-1}(b)^+} \left(h^\circ \psi \right) \left(\psi^{-1}(c) \right) + I^{\alpha:\psi}_{\psi^{-1}(c)^-} \left(h^\circ \psi \right) \left(\psi^{-1}(b) \right) \right] - h\left(\frac{b+c}{2} \right) \right| \leq \frac{c-b}{2} \left(\int_0^{\frac{1}{2}} \left(1 + t^\alpha - (1-t)^\alpha \right)^p dt \right)^{\frac{1}{p}} \left\{ \left(\int_0^{\frac{1}{2}} \left| h'\left(tb + (1-t)c \right) \right|^q dt \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 \left| h'\left(tb + (1-t)c \right) \right|^q dt \right)^{\frac{1}{q}} \right\}$$

$$
\begin{aligned}
&= (c-b)\left(\frac{1}{(\alpha p+1)2^{\alpha p+1}}\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{2}}\left|h'\bigl(tb+(1-t)c\bigr)\right|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{1}{2}}^1\left|h'\bigl(tb+(1-t)c\bigr)\right|^q\,dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Noting that $|h'|^q$ $(q > 1)$ is concave on $[b,c]$ and using the Jensen integral inequality (5), we have

$$
\int_0^{\frac{1}{2}}\left|h'\bigl(tb+(1-t)c\bigr)\right|^q\,dt \le \left(\int_0^{\frac{1}{2}} t^*\,dt\right)\left|h'\left(\frac{\int_0^{\frac{1}{2}}(tb+(1-t)c)\,dt}{\int_0^{\frac{1}{2}} t^*\,dt}\right)\right|^q \le \frac{1}{2}\left|h'\left(\frac{b+3c}{4}\right)\right|^q.
$$

Similarly,

$$
\int_{\frac{1}{2}}^1\left|h'\bigl(tb+(1-t)c\bigr)\right|^q\,dt \le \left(\int_{\frac{1}{2}}^1 t^*\,dt\right)\left|h'\left(\frac{\int_{\frac{1}{2}}^1(tb+(1-t)c)\,dt}{\int_{\frac{1}{2}}^1 t^*\,dt}\right)\right|^q \le \frac{1}{2}\left|h'\left(\frac{3b+c}{4}\right)\right|^q.
$$

In this formula, t^* is an arbitrary constant independent of t. Combined with the previous inequality, we get the required results. The proof is completed. □

3 Applications to some special means

Bivariate means are with respect to two elements. Consider the following bivariate means (see [33]) for arbitrary $m,n \in R$, $m \neq n$:

the harmonic mean

$$
H(m,n) = \frac{2}{\frac{1}{m}+\frac{1}{n}}, \quad m,n \in R \setminus \{0\},
$$

the arithmetic mean

$$
A(m,n) = \frac{m+n}{2}, \quad m,n \in R,
$$

the logarithmic mean

$$
L(m,n) = \frac{n-m}{\ln|n| - \ln|m|}, \quad |m| \neq |n|, mn \neq 0,
$$

the r-logarithmic mean

$$
L_r(m,n) = \left[\frac{n^{r+1}-m^{r+1}}{(r+1)(n-m)}\right]^{\frac{1}{r}}, \quad r \in Z \setminus \{-1,0\}, m,n \in R, m \neq n.
$$

Now we give some applications to special means of a real number.

Proposition 3.1 *Let $m, n \in R_+$, $m < n$, $r \in Z$, $|r| \geq 2$, $s \in (0,1]$, and $q > 1$. Then*

$$\left|L_r^r(m,n) - A^r(m,n)\right| \leq \begin{cases} \frac{(n-m)|r|}{s+1} A(|m|^{r-1}, |n|^{r-1}), \\ 2(n-m)|r|(\frac{1}{(p+1)2^{p+1}})^{\frac{1}{p}}(\frac{1}{(s+1)2^{s+1}})^{\frac{1}{q}} \\ \quad \times (1 + (2^{s+1}-1)^{\frac{1}{q}}) A(|m|^{r-1}, |n|^{r-1}), \\ (n-m)|r|(\frac{1}{4})^{\frac{1}{p}}(\frac{1}{(s+1)2^{s+1}})^{\frac{1}{q}} \\ \quad \times (1 + (2^{s+1}-1)^{\frac{1}{q}}) A(|m|^{r-1}, |n|^{r-1}), \end{cases}$$

where $\frac{1}{p} = 1 - \frac{1}{q}$.

Proof Applying Theorem 2.2, Corollary 2.4, and Corollary 2.6, respectively, for $h(x) = x^r$, $\psi(x) = x$, and $\alpha = 1$, we immediately obtain the result. □

Proposition 3.2 *Let $m, n \in R_+$, $m < n$, $r \in Z$, $s \in (0,1]$, and $q > 1$. Then*

$$\left|L^{-1}(m,n) - H(m^{-1}, n^{-1})\right| \leq \begin{cases} \frac{n-m}{s+1} A(|m|^{-2}, |n|^{-2}), \\ 2(n-m)(\frac{1}{(p+1)2^{p+1}})^{\frac{1}{p}}(\frac{1}{(s+1)2^{s+1}})^{\frac{1}{q}} \\ \quad \times (1 + (2^{s+1}-1)^{\frac{1}{q}}) A(|m|^{-2}, |n|^{-2}), \\ (n-m)(\frac{1}{4})^{\frac{1}{p}}(\frac{1}{(s+1)2^{s+1}})^{\frac{1}{q}} \\ \quad \times (1 + (2^{s+1}-1)^{\frac{1}{q}}) A(|m|^{-2}, |n|^{-2}). \end{cases}$$

where $\frac{1}{p} = 1 - \frac{1}{q}$.

Proof Applying Theorem 2.2, Corollary 2.4, and Corollary 2.6 respectively, for $h(x) = \frac{1}{x}$, $\psi(x) = x$, and $\alpha = 1$, we immediately obtain the result. □

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All data generated or analyzed during this study are included in this published paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Using the second sense of *s*-convex function, some new Hermite–Hadamard-type inequalities are established, involving the fractional integration of ψ-Riemann–Liouville. At the same time the authors gave many useful estimates for these new Hermite–Hadamard-type inequalities. The main idea of this paper was proposed by YZ and HS. ZC and WX prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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