Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A study on Copson operator and its associated sequence space

Hadi Roopaei[1*]

Dedicated to Prof. Maryam Mirzakhani who, in spite of a short lifetime, left a long standing impact on mathematics

*Correspondence:
h.roopaei@gmail.com
[1]Young Researchers and Elite Club, Marvdasht Branch, Islamic Azad University, Marvdasht, Iran

Abstract

In this research, we investigate two types of Copson matrices, the generalized Copson matrix and the Copson matrix of order *n*, and their associated sequence spaces generated by these matrices. We also investigate the topological properties, inclusions, and dual spaces of these new Banach spaces as well as compute the norm of Copson operators on the well-known matrix domains such as Hilbert and difference sequence spaces. Moreover, in a reverse manner, we investigate the norm of well-known operators on the Copson matrix domains generated with Copson matrices. Through this study we introduce several new inequalities, inclusions, and factorizations for well-known operators.

MSC: 26D15; 40C05; 40G05; 47B37

Keywords: Norm; Copson matrix; Hilbert matrix; Difference sequence space

1 Introduction

Let $p \geq 1$ and ω denote the set of all real-valued sequences. The space ℓ_p is the set of all real sequences $x = (x_k) \in \omega$ such that

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p} < \infty.$$

There are two different types of Copson matrices, the generalized Copson matrix and the Copson matrix of order n, we indicate them by the notations C^N and C^n, respectively. In the sequel, we introduce these matrices and their differences as well as their associated matrix domains.

We say that $A = (a_{n,k})$ is a quasi-summability matrix if it is an upper-triangular matrix, i.e., $a_{n,k} = 0$ for $n < k$ and $\sum_{n=0}^{k} a_{n,k} = 1$ for all k.

Copson matrix. The Copson matrix is a quasi-summability matrix which is defined by

$$c_{j,k} = \begin{cases} \frac{1}{k+1}, & 0 \leq j \leq k, \\ 0 & \text{otherwise} \end{cases}$$

for all $j,k \in \mathbb{N}_0$, that is,

$$C = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 0 & 1/2 & 1/3 & \cdots \\ 0 & 0 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and has the ℓ_p-norm $\|C\|_{\ell_p} = p$. This matrix is the transpose of the well-known Cesàro matrix.

Generalized Copson matrix. Suppose that $N \geq 1$ is a real number. The generalized Copson matrix $C^N = (c^N_{j,k})$ is defined by

$$c^N_{j,k} = \begin{cases} \frac{1}{k+N}, & 0 \leq j \leq k, \\ 0 & \text{otherwise} \end{cases}$$

for all $j,k \in \mathbb{N}_0$, has the ℓ_p-norm $\|C^N\|_{\ell_p} = p$ ([8], Lemma 2.3), and the matrix representation

$$c^N = \begin{pmatrix} \frac{1}{N} & \frac{1}{1+N} & \frac{1}{2+N} & \cdots \\ 0 & \frac{1}{1+N} & \frac{1}{2+N} & \cdots \\ 0 & 0 & \frac{1}{2+N} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that C^1 is the Copson matrix C. For more examples,

$$C^2 = \begin{pmatrix} 1/2 & 1/3 & 1/4 & \cdots \\ 0 & 1/3 & 1/4 & \cdots \\ 0 & 0 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} 1/3 & 1/4 & 1/5 & \cdots \\ 0 & 1/4 & 1/5 & \cdots \\ 0 & 0 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Also note that, for $N > 1$, the generalized Copson matrix is not a quasi-summability matrix.

The matrix domain of an infinite matrix T in a sequence space X is defined as

$$X_T = \{x \in \omega : Tx \in X\},$$

which is also a sequence space. The matrix domain associated with this matrix is the set $\{x = (x_k) \in \omega : C^N x \in \ell_p\}$, or

$$C^N_p = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=j}^{\infty} \frac{x_k}{k+N} \right|^p < \infty \right\},$$

which has the following norm:

$$\|x\|_{C^N_p} = \left(\sum_{j=0}^{\infty} \left| \sum_{k=j}^{\infty} \frac{x_k}{k+N} \right|^p \right)^{\frac{1}{p}}.$$

By using matrix domains of special triangle matrices in classical spaces, many authors have introduced and studied new Banach spaces. For the relevant literature, we refer to the papers [2–5, 7, 9, 10, 12, 13, 16, 17, 21, 23–25] and textbooks [2] and [20].

Copson matrix of order n. Consider the Hausdorff matrix $H^\mu = (h_{j,k})_{j,k=0}^\infty$, with entries of the form:

$$h_{j,k} = \begin{cases} \int_0^1 \binom{j}{k}\theta^k(1-\theta)^{j-k}\,d\mu(\theta), & j \geq k, \\ 0, & j < k, \end{cases}$$

where μ is a probability measure on $[0,1]$.

Hardy's formula ([11], Theorem 216) states that the Hausdorff matrix is a bounded operator on ℓ_p if and only if $\int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta) < \infty$ and

$$\|H^\mu\|_{\ell_p} = \int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta). \tag{1.1}$$

In order to define and know the Copson matrix details, we need the following theorem also known as Hellinger–Toeplitz theorem.

Theorem 1.1 ([6], Proposition 7.2) *Suppose that $1 < p, q < \infty$. A matrix A maps ℓ_p into ℓ_q if and only if the transposed matrix A^t maps ℓ_{q^*} into ℓ_{p^*}. We then have $\|A\|_{\ell_p,\ell_q} = \|A^t\|_{\ell_{q^*},\ell_{p^*}}$, where p^* is the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$.*

For nonnegative real number n, and by choosing $d\mu(\theta) = n(1-\theta)^{n-1}\,d\theta$ in the definition of Hausdorff matrix, we gain the Cesàro matrix of order n, which, according to Hardy's formula, has the ℓ_p-norm

$$\frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}. \tag{1.2}$$

Now, the Copson matrix of order n, $C^n = (c^n_{j,k})$, which is defined as the transpose of Cesàro matrix of order n, has the entries

$$c^n_{j,k} = \begin{cases} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}}, & 0 \leq j \leq k, \\ 0 & \text{otherwise}, \end{cases} \tag{1.3}$$

and, according to Hellinger–Toeplitz theorem, the ℓ_p-norm

$$\|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)}. \tag{1.4}$$

Note that $C^0 = I$, where I is the identity matrix and $C^1 = C$ is the well-known Copson matrix. For more examples,

$$C^2 = \begin{pmatrix} 1 & 2/3 & 3/6 & \cdots \\ 0 & 1/3 & 2/6 & \cdots \\ 0 & 0 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} 1 & 3/4 & 6/10 & \cdots \\ 0 & 1/4 & 3/10 & \cdots \\ 0 & 0 & 1/10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We emphasize that the Copson matrix of order n is a quasi-summability matrix, while the generalized Copson matrix is not.

The Copson matrix domain C_p^n is the set of all sequences whose C^n-transforms are in the space ℓ_p; that is,

$$C_p^n = \left\{ x = (x_j) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} x_k \right|^p < \infty \right\},$$

which is a Banach space with the norm

$$\|x\|_{C_p^n} = \left(\sum_{j=0}^{\infty} \left| \sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} x_k \right|^p \right)^{1/p}.$$

Throughout this research, we use the notations $\|\cdot\|_{A_p,\ell_p}$, $\|\cdot\|_{\ell_p,A_p}$ and $\|\cdot\|_{A_p,B_p}$ for the norm of operators from the matrix domain A_p into sequence space ℓ_p, from ℓ_p into the matrix domain A_p, and from the matrix domain A_p into the matrix domain B_p, respectively.

Motivation. Although a lot of papers have been published on Cesàro matrix, Cesàro sequence spaces, and Cesàro function spaces and many mathematicians, like the pioneers Jagers, Bennett, Luxemburg, and Zaanen [1, 6, 14, 19], worked on that and later the work was continued by several mathematicians, there exist limited studies on the transpose of this matrix. Also, while we can extract many results of Copson matrix just by transposing the Cesàro matrix and applying some theorems like Hellinger–Toeplitz theorem, there are some special areas that work only for Copson matrices. We state one of these differences in this paper by computing the norm of Copson operators between difference sequence spaces, while we cannot do it for the Cesàro operator. Other examples introduce some topological properties, inequalities, and inclusions which are only applicable on Copson matrices.

2 The Copson Banach spaces C_p^n and C_∞^n

In this section, the sequence spaces C_p^n $(1 \le p < \infty)$ and C_∞^n are introduced by using the Copson matrix of order n, and the inclusions, basis, and duals of this matrix domain will investigated.

Lemma 2.1 *The Copson matrix of order n, C^n, is invertible and its inverse $C^{-n} = (c_{j,k}^{-n})$ is defined by*

$$c_{j,k}^{-n} = \begin{cases} (-1)^{(k-j)} \binom{n}{k-j} \binom{n+j}{j}, & j \le k \le j+n, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof Let us recall the forward difference matrix of order n, $\Delta^n = (\delta_{j,k}^n)$, which is a lower triangle matrix with entries

$$\delta_{j,k}^n = \begin{cases} (-1)^{(k-j)} \binom{n}{k-j}, & j \le k \le j+n, \\ 0 & \text{otherwise}. \end{cases}$$

This matrix has the inverse $\Delta^{-n} = (\delta^{-n}_{j,k})$ with the following entries:

$$\delta^{-n}_{j,k} = \begin{cases} \binom{n+k-j-1}{k-j}, & j \le k, \\ 0 & \text{otherwise.} \end{cases}$$

From relation (1.3), one can see that the Copson matrix of order n and its inverse can be rewritten based on the forward difference operator and its inverse. For $j \le k$, we have

$$c^n_{j,k} = \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} = \frac{\delta^{-n}_{j,k}}{\binom{n+k}{k}} \quad \text{and} \quad c^{-n}_{j,k} = \delta^n_{j,k}\binom{n+j}{j}.$$

Now, by a simple calculation, we deduce that

$$\left(C^{-n}C^n\right)_{i,k} = \frac{\binom{n+i}{i}}{\binom{n+k}{k}}\sum_j \delta^n_{i,j}\delta^{-n}_{j,k} = \frac{\binom{n+i}{i}}{\binom{n+k}{k}}\left(\Delta^n\Delta^{-n}\right)_{i,k} = \frac{\binom{n+i}{i}}{\binom{n+k}{k}}I_{i,k},$$

which completes the proof. □

Now, we introduce the sequence spaces C^n_p and C^n_∞ as the set of all sequences whose C^n-transforms are in the spaces ℓ_p and ℓ_∞, respectively; that is,

$$C^n_p = \left\{x = (x_j) \in \omega : \sum_{j=0}^{\infty}\left|\sum_{k=j}^{\infty}\frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}}x_k\right|^p < \infty\right\}$$

and

$$C^n_\infty = \left\{x = (x_j) \in \omega : \sup_j\left|\sum_{k=j}^{\infty}\frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}}x_k\right| < \infty\right\}.$$

Theorem 2.2 *The spaces C^n_p and C^n_∞ are Banach spaces with the norms*

$$\|x\|_{C^n_p} = \left(\sum_{j=0}^{\infty}\left|\sum_{k=j}^{\infty}\frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}}x_k\right|^p\right)^{1/p}$$

and

$$\|x\|_{C^n_\infty} = \sup_{j\in\mathbb{N}_0}\left|\sum_{k=j}^{\infty}\frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}}x_k\right|,$$

respectively.

Proof We omit the proof which is a routine verification. □

Theorem 2.3 *The spaces C^n_p and C^n_∞ are linearly isomorphic to ℓ_p and ℓ_∞, respectively.*

Proof We only prove that C^n_p is linearly isomorphic to ℓ_p. Since C^n is invertible, hence the map $T : C^n_p \to \ell_p$ as $Tx = C^n x$ for any $x \in C^n_p$ is bijective, which proves the isomorphism. □

Theorem 2.4 *The inclusion $C_p^n \subset C_q^n$ is strict, where $1 \leq p < q < \infty$.*

Proof Let $x \in C_p^n$. Then we have $C^n x \in \ell_p$. Since the inclusion $\ell_p \subset \ell_q$ holds for $1 \leq p < q < \infty$, we have $C^n x \in \ell_q$, which implies that $x \in C_q^n$. Hence, we conclude that the inclusion $C_p^n \subset C_q^n$ holds. Now, since the inclusion $\ell_p \subset \ell_q$ is strict, we can choose $y = (y_j) \in \ell_q \setminus \ell_p$. By defining $x = C^{-n} y$, we have $C^n x = y$, which results in $C^n x \in \ell_q \setminus \ell_p$. Hence, we conclude that $x \in C_q^n \setminus C_p^n$, and so the inclusion $C_p^n \subset C_q^n$ is strict. □

Theorem 2.5 *The inclusion $C_p^n \subset C_\infty^n$ is strict, where $1 \leq p < \infty$.*

Proof Choose any $x \in C_p^n$. Then we have $C^n x \in \ell_p$. Since the inclusion $\ell_p \subset \ell_\infty$ holds for $1 \leq p < \infty$, we have $C^n x \in \ell_\infty$. This implies that $x \in C_\infty^n$. Hence, we conclude that the inclusion $C_p^n \subset C_\infty^n$ holds. Similar to the proof of the previous theorem, we can choose x such that $C^n x = ((-1)^j) \in \ell_\infty \setminus \ell_p$, and consequently it results in $x \in C_\infty^n \setminus C_p^n$. Therefore, the inclusion $C_p^n \subset C_\infty^n$ is strict. □

It is known from Theorem 2.3 of Jarrah and Malkowsky [15] that if T is a triangle then the domain λ_T of T in a normed sequence space λ has a basis if and only if λ has a basis. As a direct consequence of this fact, we have the following.

Corollary 2.6 *Define the sequence $(h^{(k)}) = (h_j^{(k)})$ for each $k \in \mathbb{N}$ by*

$$\left(h^{(k)}\right)_j = \begin{cases} (-1)^{k-j}\binom{n+j}{j}\binom{n}{k-j}, & k \geq j, \\ 0, & k < j \end{cases} \qquad (j \in \mathbb{N}_0). \tag{2.1}$$

Then the sequence $(h^{(k)})$ is a basis for the space C_p^n and every sequence $x \in C_p^n$ has a unique representation of the form $x = \sum_k (C^n x)_k h^{(k)}$.

The following lemma is essential to determine the dual spaces. Throughout the paper, $\mathcal{N}$ is the collection of all finite subsets of $\mathbb{N}$.

Lemma 2.7 ([26]) *The following statements hold*:

(i) $T = (t_{j,k}) \in (\ell_1, \ell_1)$ *if and only if*

$$\sup_k \sum_{j=0}^\infty |t_{j,k}| < \infty.$$

(ii) $T = (t_{j,k}) \in (\ell_p, \ell_1)$ *if and only if*

$$\sum_{k=0}^\infty \left(\sum_{j=0}^\infty |t_{j,k}| \right)^{p*} < \infty,$$

where $1 < p < \infty$.

(iii) $T = (t_{j,k}) \in (\ell_\infty, \ell_1)$ *if and only if*

$$\sup_{K \in \mathcal{N}} \sum_{j=0}^\infty \left| \sum_{k \in K} t_{j,k} \right| < \infty.$$

(iv) $T = (t_{j,k}) \in (\ell_1, c)$ *if and only if*

$$\lim_{j\to\infty} t_{j,k} \quad \text{exists for each } k \in \mathbb{N} \tag{2.2}$$

and

$$\sup_{j,k} |t_{j,k}| < \infty. \tag{2.3}$$

(v) $T = (t_{j,k}) \in (\ell_p, c)$ *if and only if* (2.2) *holds and*

$$\sup_j \sum_{k=0}^{\infty} |t_{j,k}|^{p*} < \infty, \tag{2.4}$$

where $1 < p < \infty$.

(vi) $T = (t_{j,k}) \in (\ell_\infty, c)$ *if and only if* (2.2) *holds and*

$$\lim_{j\to\infty} \sum_{k=0}^{\infty} |t_{j,k}| = \sum_{k=0}^{\infty} \left| \lim_{j\to\infty} t_{j,k} \right|.$$

(vii) $T = (t_{j,k}) \in (\ell_1, \ell_\infty)$ *if and only if* (2.3) *holds.*

(viii) $T = (t_{j,k}) \in (\ell_p, \ell_\infty)$ *if and only if* (2.4) *holds, where* $1 < p < \infty$.

(ix) $T = (t_{j,k}) \in (\ell_\infty, \ell_\infty)$ *if and only if*

$$\sup_j \sum_{k=0}^{\infty} |t_{j,k}| < \infty.$$

Definition The α-, β-, and γ-duals of a sequence space X are defined by

$$X^\alpha = \left\{ a = (a_k) \in \omega : \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for all } x = (x_k) \in X \right\},$$

$$X^\beta = \left\{ a = (a_k) \in \omega : \left(\sum_{k=1}^{n} a_k x_k \right) \in c \text{ for all } x = (x_k) \in X \right\},$$

$$X^\gamma = \left\{ a = (a_k) \in \omega : \left(\sum_{k=1}^{n} a_k x_k \right) \in \ell_\infty \text{ for all } x = (x_k) \in X \right\},$$

respectively. In the following, we find the Köthe dual of the Copson sequence space.

Theorem 2.8 *The* α*-duals of the spaces* C_1^n, C_p^n $(1 < p < \infty)$, *and* C_∞^n *are as follows*:

$$\left(C_1^n\right)^\alpha = \left\{ b = (b_j) \in \omega : \sup_k \sum_{j=0}^{\infty} \left| (-1)^{k-j} \binom{n+j}{j} \binom{n}{k-j} b_j \right| < \infty \right\},$$

$$\left(C_p^n\right)^\alpha = \left\{ b = (b_j) \in \omega : \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} \left| (-1)^{k-j} \binom{n+j}{j} \binom{n}{k-j} b_j \right| \right)^{p*} < \infty \right\},$$

and

$$\left(C_\infty^n\right)^\alpha = \left\{b = (b_j) \in \omega : \sup_{K \in \mathcal{N}} \sum_{j=0}^{\infty} \left| \sum_{k \in K} (-1)^{k-j} \binom{n+j}{j} \binom{n}{k-j} b_j \right| < \infty \right\}.$$

Proof Let $b = (b_j) \in \omega$. Consider the matrix $A = (a_{j,k})$ defined by

$$a_{j,k} = \begin{cases} (-1)^{k-j}\binom{n+j}{j}\binom{n}{k-j} b_j, & 0 \le j \le k, \\ 0, & \text{otherwise.} \end{cases}$$

Given any $x = (x_j) \in C_p^n$ $(1 \le p \le \infty)$, we have $b_j x_j = (Ay)_j$ for all $j \in \mathbb{N}$. This implies that $bx \in \ell_1$ with $x \in C_p^n$ if and only if $Ay \in \ell_1$ with $y \in \ell_p$. Hence, we conclude that $b \in (C_p^n)^\alpha$ if and only if $A \in (\ell_p, \ell_1)$. This completes the proof by Lemma 2.7. □

Theorem 2.9 *Let us define the following sets*:

$$B_1 = \left\{b = (b_k) \in \omega : \lim_{j \to \infty} \sum_{k=i}^{j} (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \text{ exists for each } k \in \mathbb{N} \right\},$$

$$B_2 = \left\{b = (b_k) \in \omega : \sup_{j,k} \left| \sum_{k=i}^{j} (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \right| < \infty \right\},$$

$$B_3 = \left\{b = (b_k) \in \omega : \sup_{j} \sum_{k=0}^{\infty} \left| \sum_{k=i}^{j} (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \right|^{p*} < \infty \right\},$$

and

$$B_4 = \left\{b = (b_k) \in \omega : \lim_{j \to \infty} \sum_{k=0}^{\infty} \left| \sum_{k=i}^{j} (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \right| = \sum_{k=0}^{\infty} \left| \sum_{k=i}^{\infty} (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \right| \right\}.$$

Then we have $(C_1^n)^\beta = B_1 \cap B_2$, $(C_p^n)^\beta = B_1 \cap B_3$ $(1 < p < \infty)$, *and* $(C_\infty^n)^\beta = B_1 \cap B_4$.

Proof $b = (b_k) \in (C_1^n)^\beta$ if and only if the series $\sum_{k=0}^{\infty} b_k x_k$ is convergent for all $x = (x_k) \in C_1^n$. The equality

$$\begin{aligned} \sum_{k=0}^{j} b_k x_k &= \sum_{k=0}^{j} b_k \left(\sum_{i=0}^{k} (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} y_i \right) \\ &= \sum_{k=0}^{j} \left(\sum_{i=k}^{j} (-1)^{k-i} \binom{n+i}{i} \binom{n}{k-i} b_i \right) y_k \end{aligned}$$

implies that $b = (b_k) \in (C_1^n)^\beta$ if and only if the matrix $B = (b_{j,k})$ is in the class (ℓ_1, c), where

$$b_{j,k} = \begin{cases} \sum_{i=k}^{j} (-1)^{k-i}\binom{n+i}{i}\binom{n}{k-i} b_i, & 0 \le j \le k, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we deduce from Lemma 2.7 that

$$\lim_{j\to\infty}\sum_{i=k}^{n}(-1)^{k-i}\binom{n+i}{i}\binom{n}{k-i}b_i \quad \text{exists for each } k\in\mathbb{N}$$

and

$$\sup_{j,k}\left|\sum_{i=k}^{j}(-1)^{k-i}\binom{n+i}{i}\binom{n}{k-i}b_i\right| < \infty,$$

which means $b=(b_k)\in B_1\cap B_2$, and so we have $(C_1^n)^\beta = B_1\cap B_2$. The other results can be proved similarly. □

Theorem 2.10 *The γ-duals of the spaces C_1^n, C_p^n $(1<p<\infty)$, and C_∞^n are as follows*:

$$(C_1^n)^\gamma = \left\{b=(b_k)\in\omega : \sup_{j,k}\left|\sum_{i=k}^{j}(-1)^{k-i}\binom{n+i}{i}\binom{n}{k-i}b_i\right| < \infty\right\},$$

$$(C_p^n)^\gamma = \left\{b=(b_k)\in\omega : \sup_{j}\sum_{k=0}^{\infty}\left|\sum_{i=k}^{j}(-1)^{k-i}\binom{n+i}{i}\binom{n}{k-i}b_i\right|^{p*} < \infty\right\},$$

and

$$(C_\infty^n)^\gamma = \left\{b=(b_k)\in\omega : \sup_{j}\sum_{k=0}^{\infty}\left|\sum_{i=k}^{j}(-1)^{k-i}\binom{n+i}{i}\binom{n}{k-i}b_i\right| < \infty\right\}.$$

Proof It follows with the same technique as that in the proof of Theorem 2.9. □

3 Norm of operators on Copson matrix domain

In this section we intend to compute the norm of well-known operators, such as Hilbert, Hausdorff, and Copson operators, on the Copson matrix domain. In so doing, we need the following lemma.

Lemma 3.1 *Let U be a bounded operator on ℓ_p and A_p and B_p be two matrix domains such that $A_p \simeq \ell_p$.*

(i) *If BT is a bounded operator on ℓ_p, then T is a bounded operator from ℓ_p into B_p and*

$$\|T\|_{\ell_p,B_p} = \|BT\|_{\ell_p}.$$

(ii) *If T has a factorization of the form $T=UA$, then T is a bounded operator from the matrix domain A_p into ℓ_p and*

$$\|T\|_{A_p,\ell_p} = \|U\|_{\ell_p}.$$

(iii) *If $BT=UA$, then T is a bounded operator from the matrix domain A_p into B_p and*

$$\|T\|_{A_p,B_p} = \|U\|_{\ell_p}.$$

In particular, if $AT = UA$, then T is a bounded operator from the matrix domain A_p into A_p and

$$\|T\|_{A_p} = \|U\|_{\ell_p}.$$

Also, if T and A commute, then $\|T\|_{A_p} = \|T\|_{\ell_p}$.

Proof (i) For every $x \in \ell_p$,

$$\|T\|_{\ell_p,B_p} = \sup_{x\in\ell_p} \frac{\|Tx\|_{B_p}}{\|x\|_{\ell_p}} = \sup_{x\in\ell_p} \frac{\|BTx\|_{\ell_p}}{\|x\|_{\ell_p}} = \|BT\|_{\ell_p}.$$

(ii) Since A_p and ℓ_p are isomorphic, hence

$$\|T\|_{A_p,\ell_p} = \sup_{x\in A_p} \frac{\|Tx\|_{\ell_p}}{\|x\|_{A_p}} = \sup_{x\in A_p} \frac{\|UAx\|_{\ell_p}}{\|Ax\|_{\ell_p}} = \sup_{y\in\ell_p} \frac{\|Uy\|_{\ell_p}}{\|y\|_{\ell_p}} = \|U\|_{\ell_p},$$

which gives the desired result. Part (iii) has a similar proof. □

3.1 Norm of Hilbert operator on Copson matrix domain

Recall the definition of the well-known Hilbert matrix $H = (h_{j,k})$, which was introduced (1894) by David Hilbert to study a question in approximation theory. For $j,k = 0,1,\dots$, the Hilbert matrix is defined by

$$h_{j,k} = \frac{1}{j+k+1}, \qquad H = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is a bounded operator on ℓ_p with ℓ_p-norm $\|H\|_{\ell_p} = \Gamma(1/p)\Gamma(1/p^*) = \pi \csc(\pi/p)$.

For a nonnegative integer n, we define the Hilbert matrix of order n, $H^n = (h^n_{j,k})$, by

$$h^n_{j,k} = \frac{1}{j+k+n+1} \quad (j,k = 0,1,\dots). \tag{3.1}$$

Note that for $n = 0$, $H^0 = H$ is the Hilbert matrix. For more examples:

$$H^1 = \begin{pmatrix} 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad H^2 = \begin{pmatrix} 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ 1/5 & 1/6 & 1/7 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For nonnegative integers n, j, and k, let us define the matrix $B^n = (b^n_{j,k})$ by

$$b^n_{j,k} = \frac{(j+1)\cdots(j+n)}{(j+k+1)\cdots(j+k+n+1)}.$$

Consider that, for $n = 0$, $B^0 = H$, where H is the Hilbert matrix.

Note that the matrix B^n has also the representation

$$b^n_{j,k} = \binom{n+j}{j}\beta(j+k+1,n+1) \quad (j,k=0,1,\ldots),$$

where the β function is

$$\beta(m,n) = \int_0^1 z^{m-1}(1-z)^{n-1}\,dz \quad (m,n=1,2,\ldots).$$

For computing the norm of Hilbert operator on the domain of Copson matrix, we need the following lemmas.

Lemma 3.2 *For $|z|<1$, we have*

$$(1-z)^{-n} = \sum_{j=0}^{\infty}\binom{n+j-1}{j}z^j.$$

Proof By differentiating $n-1$ times the identity $(1-z)^{-1} = \sum_{j=0}^{\infty} z^j$, we obtain the result. □

Lemma 3.3 ([22], Lemma 2.3) *The Hilbert matrix H and the Hilbert matrix of order n, H^n, have the following factorizations*:

(i) $H = C^n B^n$,

(ii) $H^n = B^n C^n$,

(iii) $HC^n = C^n H^n$,

(iv) B^n *is a bounded operator on* ℓ_p *and*

$$\|B^n\|_{\ell_p} = \frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)}.$$

Corollary 3.4 *Let H^n be the Hilbert operator of order n. Then*

(i) H^n *is a bounded operator from* C^n_p *into* ℓ_p *and*

$$\|H^n\|_{C^n_p,\ell_p} = \frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)}.$$

(ii) H^n *is a bounded operator from* C^n_p *into* C^n_p *and*

$$\|H^n\|_{C^n_p} = \pi\csc(\pi/p).$$

Proof According to Lemma 3.3, we have $H^n = B^n C^n$ and $C^n H^n = HC^n$. Now, by applying Lemma 3.1 parts (ii) and (iii), we gain the result. □

As an application of Lemma 3.3, we are ready to generalize the inequality

$$\|Hx\|_{\ell_p} \leq \pi\csc(\pi/p)\|x\|_{\ell_p},$$

also known as Hilbert's inequality.

Corollary 3.5 *Let $p > 1$ and $x \in \ell_p$. Then*

$$\|H^n x\|_{\ell_p} \leq \frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)} \|C^n x\|_{\ell_p}$$

or

$$\sum_{j=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{x_k}{j+k+n+1} \right|^p \leq \left(\frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)} \right)^p \sum_{j=0}^{\infty} \left| \sum_{k=j}^{\infty} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} x_k \right|^p.$$

In particular, for $n = 0$, Hilbert's inequality occurs, and for $n = 1$, we have the inequality

$$\|H^1 x\|_{\ell_p} \leq \pi/p \csc(\pi/p) \|Cx\|_{\ell_p}$$

or

$$\sum_{j=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{x_k}{j+k+2} \right|^p \leq \left(\pi/p \csc(\pi/p)\right)^p \sum_{j=0}^{\infty} \left| \sum_{k=j}^{\infty} \frac{x_k}{1+k} \right|^p.$$

Proof According to Lemma 3.3, $H^n = B^n C^n$, hence

$$\|H^n x\|_{\ell_p} = \|B^n C^n x\|_{\ell_p} \leq \frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)} \|C^n x\|_{\ell_p}.$$

Consider that, for $n = 0$, $C^0 = I$, and we have Hilbert's inequality. □

3.2 Norm of Copson operators on Copson matrix domains

Lemma 3.6 *Let α and n be two nonnegative integers that $\alpha \geq n \geq 0$. For $j = 1, 2, \ldots$, we have*

$$\sum_{k=0}^{j} (-1)^k \binom{n}{k} \binom{\alpha+j-k-1}{j-k} = \begin{cases} \binom{\alpha-n+j-1}{j}, & \alpha > n, \\ 1, & \alpha = n. \end{cases}$$

Proof Since $\alpha - n \geq 0$, hence Lemma 3.2 results in

$$(1-z)^{-(\alpha-n)} = \sum_{j=0}^{\infty} \binom{\alpha-n+j-1}{j} z^j.$$

On the other hand,

$$\begin{aligned}
(1-z)^{-(\alpha-n)} &= (1-z)^n (1-z)^{-\alpha} \\
&= \sum_{j=0}^{n} (-1)^j \binom{n}{j} z^j \sum_{j=0}^{\infty} \binom{\alpha+j-1}{j} z^j \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-1)^k \binom{n}{k} \binom{\alpha+j-k-1}{j-k} z^j.
\end{aligned}$$

Now, the result is obvious. □

Theorem 3.7 *Let α, n be two nonnegative integers that $\alpha \geq n \geq 0$. The Copson matrix of order α has a factorization of the form $C^\alpha = C^n S^{\alpha,n} = S^{\alpha,n} C^n$, where C^n is the Copson matrix of order n, $S^{\alpha,n} = (s^{\alpha,n}_{j,k})$ is a bounded operator on ℓ_p with the entries*

$$s^{\alpha,n}_{j,k} = \frac{\binom{n+j}{j}\binom{\alpha-n+k-j-1}{k-j}}{\binom{\alpha+k}{k}} \quad (j,k = 0,1,\ldots),$$

and ℓ_p-norm

$$\|S^{\alpha,n}\|_{\ell_p} = \frac{\Gamma(\alpha+1)\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(\alpha+1/p)}.$$

Proof For obtaining the matrix $S^{\alpha,n}$, it is sufficient to compute $C^{-n}C^\alpha$. By applying Lemma 3.6, we gain

$$\begin{aligned}
(C^{-n}C^\alpha)_{i,k} &= \frac{\binom{n+i}{i}}{\binom{\alpha+k}{k}} \sum_{j=i}^{k} (-1)^{j-i} \binom{n}{j-i} \binom{\alpha+k-j-1}{k-j} \\
&= \frac{\binom{n+i}{i}}{\binom{\alpha+k}{k}} \sum_{j=0}^{k-i} (-1)^{j} \binom{n}{j} \binom{\alpha+k-i-j-1}{k-i-j} \\
&= \frac{\binom{n+i}{i}\binom{\alpha-n+k-i-1}{k-i}}{\binom{\alpha+k}{k}} = s^{\alpha,n}_{i,k},
\end{aligned}$$

which proves the identity

$$C^\alpha = C^n S^{\alpha,n}. \tag{3.2}$$

Now, by letting $n = 0$ and $n = \alpha$ in relation (3.2), we gain $S^{\alpha,0} = C^\alpha$ and $S^{n,n} = I$, respectively.

For computing the ℓ_p-norm of $S^{\alpha,n}$, by inserting $n = 0$ in identity (3.2), we gain

$$\|S^{\alpha,n}\|_{\ell_p} = \frac{\Gamma(\alpha+1)\Gamma(1/p)}{\Gamma(\alpha+1/p)} g(n), \quad g(0) \neq 0. \tag{3.3}$$

Now, inserting $n = \alpha$ in (3.3) results in

$$\|S^{\alpha,n}\|_{\ell_p} = \frac{\Gamma(\alpha+1)\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(\alpha+1/p)},$$

which completes the proof. □

Corollary 3.8 *Let $p > 1$, $\alpha \geq n \geq 0$, and $x \in \ell_p$. Then*

(i) $\|C^\alpha x\|_{\ell_p} \leq \frac{\Gamma(\alpha+1)\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(\alpha+1/p)} \|C^n x\|_{\ell_p}$,

(ii) $\|Cx\|_{\ell_p} \leq p\|x\|_{\ell_p}$,

(iii) $C^n_p \subset C^\alpha_p$.

Proof (i) Since $C^\alpha = S^{\alpha,\beta}C^n$, according to Theorem 3.7, we have

$$\|C^\alpha x\|_{\ell_p} = \|S^{\alpha,n}C^n x\|_{\ell_p} \le \frac{\Gamma(\alpha+1)\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(\alpha+1/p)}\|C^n x\|_{\ell_p}.$$

(ii) Consider that, for $\alpha = 1$ and $n = 0$, $C^1 = C$ and $C^0 = I$ in part (i), hence we have the inequality.

(iii) is a straightforward result of part (i). □

Theorem 3.9 *Let $\alpha \ge n \ge 0$. The Copson operator of order α, C^α, is a bounded operator from C_p^n into ℓ_p and*

$$\|C^\alpha\|_{C_p^n,\ell_p} = \frac{\Gamma(\alpha+1)\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(\alpha+1/p)}.$$

Proof According to Lemma 3.1 and Theorem 3.7, we have

$$\|C^\alpha\|_{C_p^n,\ell_p} = \|S^{\alpha,n}C^n\|_{C_p^n,\ell_p} = \|S^{\alpha,n}\|_{\ell_p} = \frac{\Gamma(\alpha+1)\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(\alpha+1/p)}.$$ □

4 Norm of Copson operators on some sequence spaces

In this section, we investigate the problem of finding the norm of Copson operators on several sequence spaces.

4.1 Norm of Copson operators on difference sequence spaces

In this part of study, we investigate the norm of both the generalized Copson matrix and the Copson matrix of order n on the difference sequence spaces. In so doing we need the following preliminaries.

Let $n \in \mathbb{N}$ and $\Delta^{n_F} = (\delta_{j,k}^{n_F})$ be the forward difference operator of order n with entries

$$\delta_{j,k}^{n_F} = \begin{cases} (-1)^{k-j}\binom{n}{k-j}, & j \le k \le n+j, \\ 0 & \text{otherwise}. \end{cases}$$

We define the sequence space $\ell_p(\Delta^{n_F})$ as the set $\{x = (x_k) : \Delta^{n_F}x \in \ell_p\}$ or

$$\ell_p(\Delta^{n_F}) = \left\{ x = (x_k) : \sum_{j=0}^{\infty}\left|\sum_{k=0}^{n}(-1)^k\binom{n}{k}x_{k+j}\right|^p < \infty \right\},$$

with semi-norm $\|\cdot\|_{\ell_p(\Delta^{n_F})}$, which is defined by

$$\|x\|_{\ell_p(\Delta^{n_F})} = \left(\sum_{j=0}^{\infty}\left|\sum_{k=0}^{n}(-1)^k\binom{n}{k}x_{k+j}\right|^p\right)^{\frac{1}{p}}.$$

Note that this function will be not a norm since if $x = (1,1,1,\dots)$, then $\|x\|_{\ell_p(\Delta^{n_F})} = 0$ while $x \ne 0$. The definition of backward difference sequence space $\ell_p(\Delta^{n_B})$ is similar to that of $\ell_p(\Delta^{n_F})$, except $\|\cdot\|_{\ell_p(\Delta^{n_B})}$ is a norm.

For special case $n = 1$, we use the notations Δ^B and Δ^F to indicate the backward and forward difference matrices of order 1, respectively. These matrices are defined by

$$\delta^B_{j,k} = \begin{cases} 1, & k = j, \\ -1, & k = j-1, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \delta^F_{j,k} = \begin{cases} 1, & k = j, \\ -1, & k = j+1, \\ 0 & \text{otherwise}, \end{cases}$$

and their associated sequence spaces $\ell_p(\Delta^B)$ and $\ell_p(\Delta^F)$ are

$$\ell_p(\Delta^B) = \left\{ x = (x_n) : \sum_{n=1}^{\infty} |x_n - x_{n-1}|^p < \infty \right\}$$

and

$$\ell_p(\Delta^F) = \left\{ x = (x_n) : \sum_{n=1}^{\infty} |x_n - x_{n+1}|^p < \infty \right\},$$

respectively.

The idea of difference sequence spaces was introduced by Kizmaz [18] in 1981. Although topological properties and inclusion relations of these spaces have been studied till now, the problem of finding the norm of operators on difference sequence spaces has not been studied extensively. More recently, Roopaei and Foroutannia investigated this problem for the difference sequence spaces $\ell_p(\Delta^F)$, $\ell_p(\Delta^B)$, and $\ell_p(\Delta^{n_F})$ in [9, 21, 24].

Theorem 4.1 *The Copson matrix of order n, C^n, is a bounded operator from ℓ_p into $\ell_p(\Delta^{n_F})$ and*

$$\|C^n\|_{\ell_p,\ell_p(\Delta^{n_F})} = 1.$$

In particular, the Copson matrix is a bounded operator from ℓ_p into $\ell_p(\Delta^F)$ and $\|C\|_{\ell_p,\ell_p(\Delta^F)} = 1$.

Proof Let $\Delta^{n_F} C^n = D^n$. By the definition of Copson matrix, the matrix $D^n = (d^n_{i,j})$ has the entries

$$\begin{aligned} d^n_{i,j} &= \frac{1}{\binom{n+j}{j}} \sum_{k=i}^{j} (-1)^{k-i} \binom{n}{k-i} \binom{n+j-k-1}{j-k} \\ &= \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j-i} (-1)^{k} \binom{n}{k} \binom{n+j-i-k-1}{j-i-k}. \end{aligned}$$

If $i = j$, then $j - i = 0$, which results in $k = 0$, hence $d^n_{j,j} = 1/\binom{n+j}{j}$. If $j > i$, then Lemma 3.6 will result in $d^n_{i,j} = 0$. Therefore $d^n_{i,j} = I_{i,j}/\binom{n+j}{j}$, where I is the identity matrix. Now, since D^n is diagonal, Lemma 3.1 results in

$$\|C^n\|_{\ell_p,\ell_p(\Delta^{n_F})} = \|D^n\|_{\ell_p} = \sup_j d^n_{j,j} = 1.$$

□

Theorem 4.2 *The generalized Copson matrix C^N is a bounded operator from ℓ_p into matrix domain $\ell_p(\Delta^F)$ and*

$$\|C^N\|_{\ell_p,\ell_p(\Delta^F)} = \frac{1}{N}.$$

In particular, the Copson matrix is a bounded operator from ℓ_p into $\ell_p(\Delta^F)$ and $\|C\|_{\ell_p,\ell_p(\Delta^F)} = 1$.

Proof It is not difficult to verify the identity $\Delta^F C^N = D^N$, where the diagonal matrix $D^N = (d^N_{j,k})$ has the entries

$$D^N = \begin{pmatrix} \frac{1}{N} & 0 & 0 & \cdots \\ 0 & \frac{1}{N+1} & 0 & \cdots \\ 0 & 0 & \frac{1}{N+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4.1}$$

and the ℓ_p-norm $1/N$. Now, by applying Lemma 3.1, we have

$$\|C^N\|_{\ell_p,\ell_p(\Delta^F)} = \|\Delta^F C^N\|_{\ell_p} = \|D^N\|_{\ell_p} = \frac{1}{N}. \qquad \square$$

Theorem 4.3 *The Copson matrix of order n, C^n, is a bounded operator from $\ell_p(\Delta^{n_B})$ into $\ell_p(\Delta^{n_F})$ and*

$$\|C^n\|_{\ell_p(\Delta^{n_B}),\ell_p(\Delta^{n_F})} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}.$$

In particular, the Copson matrix is a bounded operator from $\ell_p(\Delta^B)$ into $\ell_p(\Delta^F)$ and

$$\|C\|_{\ell_p(\Delta^B),\ell_p(\Delta^F)} = p^*.$$

Proof The fact that Δ^{n_B} is the transpose of Δ^{n_F} and $\Delta^{n_F} C^n$ is a diagonal matrix, according to Theorem 4.1, results in the identity $\Delta^{n_F} C^n = C^{nt} \Delta^{n_B}$. Now, Lemma 3.1 and relation (1.2) complete the proof. $\square$

Theorem 4.4 *The generalized Copson matrix C^N is a bounded operator from $\ell_p(\Delta^B)$ into matrix domain $\ell_p(\Delta^F)$ and*

$$\|C^N\|_{\ell_p(\Delta^B),\ell_p(\Delta^F)} = p^*.$$

In particular, the Copson matrix is a bounded operator from $\ell_p(\Delta^B)$ into $\ell_p(\Delta^F)$ and

$$\|C\|_{\ell_p(\Delta^B),\ell_p(\Delta^F)} = p^*.$$

Proof Through the proof of Theorem 4.2 we showed that $\Delta^F C^N = D^N$, where D^N is a diagonal matrix as defined by relation (4.1). Similar to the proof of Theorem 4.3, $\Delta^F C^N =$

$C^{Nt}\Delta^B$, where C^{Nt} is the transpose of the generalized Copson matrix. Now, according to Lemma 3.1, we have

$$\|C^N\|_{\ell_p(\Delta^B),\ell_p(\Delta^F)} = \|C^{Nt}\|_{\ell_p} = p^*.$$

This completes the proof. □

4.2 Norm of Copson operator on the Hilbert matrix domain

Let H_p^n be the sequence space associated with Hilbert matrix H^n, which is

$$H_p^n = \left\{x = (x_k) \in \omega : \sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+n+1}\right|^p < \infty\right\},$$

and has the norm

$$\|x\|_{H_p^n} = \left(\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+n+1}\right|^p\right)^{\frac{1}{p}}.$$

Corollary 4.5 *The Copson operator of order n, C^n, is a bounded operator from H_p^n into H_p and*

$$\|C^n\|_{H_p^n,H_p} = \frac{\Gamma(n+1/p)\Gamma(1/p)}{\Gamma(n+1)}.$$

Proof According to Lemma 3.3, we have $HC^n = C^nH^n$. Now, Lemma 3.1 completes the proof. □

As another application of Lemma 3.3, we have the following inclusions.

Corollary 4.6 *Let $p > 1$. Then*

(i) $C_p^n \subset H_p^n$,
(ii) $H_p^n \subset H_p^{n-1} \subset \cdots \subset H_p$.

Acknowledgements
There are no competing interests, there is only one author and no funding on this manuscript.

Funding
There is no funding on this research.

Availability of data and materials
No data have been used in this study.

Competing interests
The author declares that there are no competing interests.

Authors' contributions
This manuscript has only one author and nobody has collaborated in writing that. All authors read and approved the final manuscript.

Publisher's Note

Received: 20 February 2020 Accepted: 17 April 2020 Published online: 29 April 2020

References

1. Alexiewicz, A.: On Cauchy's condensation theorem. Stud. Math. **16**, 80–85 (1957)
2. Başar, F.: Summability Theory and Its Applications. Bentham Science Publishers, İstanbul (2012)
3. Başar, F.: Domain of the composition of some triangles in the space of p-summable sequences. AIP Conf. Proc. **1611**, 348–356 (2014)
4. Başar, F., Altay, B.: On the space of sequences of p-bounded variation and related matrix mappings. Ukr. Math. J. **55**(1), 136–147 (2003)
5. Başar, F., Kirişçi, M.: Almost convergence and generalized difference matrix. Comput. Math. Appl. **61**(3), 602–611 (2011)
6. Bennett, G.: Factorizing the Classical Inequalities. Mem. Amer. Math. Soc., vol. 576 (1996)
7. Braha, N.L., Başar, F.: On the domain of the triangle $A(\lambda)$ on the spaces of null, convergent and bounded sequences. Abstr. Appl. Anal. **2013**, Article ID 476363 (2013). https://doi.org/10.1155/2013/476363
8. Chen, C.-P., Luor, D.-C., Ou, Z.: Extensions of Hardy inequality. J. Math. Anal. Appl. **273**, 160–171 (2002)
9. Foroutannia, D., Roopaei, H.: The norms and the lower bounds for matrix operators on weighted difference sequence spaces. UPB Sci. Bull., Ser. A **79**(2), 151–160 (2017)
10. Foroutannia, D., Roopaei, H.: Bounds for the norm of lower triangular matrices on the Cesàro weighted sequence space. J. Inequal. Appl. **2017**, Article ID 67 (2017)
11. Hardy, G.H.: Divergent Series. Oxford University Press, Oxford (1973)
12. İlkhan, M.: Norms and lower bounds of some matrix operators on Fibonacci weighted difference sequence space. Math. Methods Appl. Sci. **42**(16), 5143–5153 (2019)
13. İlkhan, M., Kara, E.E.: A new Banach space defined by Euler totient matrix operator. Oper. Matrices **13**(2), 527–544 (2019)
14. Jagers, A.A.: A note on Cesàro sequence spaces. Nieuw Arch. Wiskd. **22**, 113–124 (1974)
15. Jarrah, A.M., Malkowsky, E.: BK spaces, bases and linear operators. Rend. Circ. Mat. Palermo **52**, 177–191 (1990)
16. Kara, E.E., Başarır, M.: On compact operators and some Euler $B^{(m)}$ difference sequence spaces. J. Math. Anal. Appl. **379**(2), 499–511 (2011)
17. Kirişçi, M., Başar, F.: Some new sequence spaces derived by the domain of generalized difference matrix. Comput. Math. Appl. **60**(5), 1299–1309 (2010)
18. Kizmaz, H.: On certain sequence spaces I. Can. Math. Bull. **25**(2), 169–176 (1981)
19. Luxemburg, W.A.J., Zaanen, A.C.: Some examples of normed Köthe spaces. Math. Ann. **162**, 337–350 (1966)
20. Mursaleen, M.: Applied Summability Methods. Springer, Berlin (2014)
21. Roopaei, H.: Norms of summability and Hausdorff mean matrices on difference sequence spaces. Math. Inequal. Appl. **22**(3), 983–987 (2019)
22. Roopaei, H.: Norm of Hilbert operator on sequence spaces. J. Inequal. Appl. (2020). https://doi.org/10.1186/s13660-020-02380-2
23. Roopaei, H., Foroutannia, D.: The norm of matrix operators on Cesàro weighted sequence space. Linear Multilinear Algebra **67**(1), 175–185 (2019)
24. Roopaei, H., Foroutannia, D.: The norms of certain matrix operators from ℓ_p spaces into $\ell_p(\Delta^n)$ spaces. Linear Multilinear Algebra **67**(4), 767–776 (2019)
25. Sönmez, A., Başar, F.: Generalized difference spaces of non-absolute type of convergent and null sequences. Abstr. Appl. Anal. **2012**, Article ID 435076 (2012). https://doi.org/10.1155/2012/435076
26. Stieglitz, M., Tietz, H.: Matrix transformationen von folgenraumen eineergebnisübersicht. Math. Z. **154**, 1–16 (1977)