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On $\mathcal{I}_{\theta_2}$-convergence in fuzzy normed spaces

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Abstract

In this study, first, lacunary convergence of double sequences is introduced in fuzzy normed spaces, and basic definitions and theorems about lacunary convergence for double sequences are given in fuzzy normed spaces. Then, we introduce the concept of lacunary ideal convergence of double sequences in fuzzy normed spaces, and the relation between lacunary convergence and lacunary ideal convergence is investigated for double sequences in fuzzy normed spaces. Finally, in fuzzy normed spaces, we give the concept of limit point and cluster point for double sequences, and the theorems related to these concepts are given.

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1 Introduction and background

The statistical convergence was derived from the convergence of real sequences by Fast [1] and Schoenberg [2]. After the studies of Šalát [3], Fridy [4], and Connor [5] in this area, many studies have been conducted. Kostyrko et al. [6] introduced the concept of ideal convergence by expanding the concept of statistical convergence. After basic properties of $\mathcal{I}$-convergence were given by Kostyrko et al. [7], some studies [8–10] have been the basis of other studies.

Matloka [11] was the first scholar who introduced the notion of convergence of a sequence of fuzzy numbers, and he proved some basic theorems. In later years, Nanda [12] studied the sequences of fuzzy numbers again and Şençimen and Pehlivan [13] introduced the notions of a statistically convergent sequence and a statistically Cauchy sequence in a fuzzy normed linear space. The concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, and $\mathcal{I}$-Cauchy sequence were studied by Hazarika [14] in a fuzzy normed linear space. Especially, the studies of Savaş [15–17], Et et al. [18], Işık [19], Çınar [20], and Altınok et al. [21] on difference sequences and fuzzy numbers made important contributions to this field. Recently, Türkmen and Çınar [22] studied lacunary statistical convergence in fuzzy normed spaces. Türkmen and Dündar [23] scrutinized same concepts for double sequences, and Türkmen [24] reinterpreted these works in fuzzy n-normed spaces.

In this paper, we introduce and study the concepts of lacunary $\mathcal{I}_2$-convergence, lacunary convergence, $F\mathcal{I}_{\theta_2}$-limit point, and $F\mathcal{I}_{\theta_2}$-cluster point for double sequences in a fuzzy normed space. Also, we scrutinize their relations.

Now, we recall the concept of double sequence, statistical convergence, ideal convergence, double lacunary sequence, fuzzy norm, and some basic definitions (see [13, 18, 21–23, 25–49]).

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0,1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership.

According to Zadeh, a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \to [0,1]$. The function u itself is often used for the fuzzy set. The function u is called a fuzzy number under certain conditions. Also, all fuzzy numbers are denoted as $L(\mathbb{R})$ and the set of all nonnegative fuzzy numbers as $L^*(\mathbb{R})$.

For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1], \\ \sup u, & \text{if } \alpha = 0. \end{cases}$$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ is defined as

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\}.$$

A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to a fuzzy number x_0 if there exists a positive integer k_0 such that $D(x_k, x_0) < \varepsilon$, for every $\varepsilon > 0$ and $k > k_0$. A sequence $x = (x_k)$ of fuzzy numbers convergent to levelwise to x_0 if and only if $\lim_{k\to\infty}[x_k]_\alpha^- = [x_0]_\alpha^-$ and $\lim_{k\to\infty}[x_k]_\alpha^+ = [x_0]_\alpha^+$, where $[x_k]_\alpha = [(x_k)_\alpha^-, (x_k)_\alpha^+]$ and $[x_0]_\alpha = [(x_0)_\alpha^-, (x_0)_\alpha^+]$, for every $\alpha \in (0,1)$.

A sequence $x = (x_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number x_0 if every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \{k \leq n : \overline{d}(x_k, x_0) \geq \varepsilon\} \right| = 0.$$

Later, many mathematicians studied statistical convergence of fuzzy numbers and extended this notion to fuzzy normed spaces.

Let X be a vector space over $\mathbb{R}$, $\|\cdot\| : X \to L^*(\mathbb{R})$, and let the mappings $L, R : [0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments, and satisfy $L(0,0) = 0$ and $R(1,1) = 1$. The quadruple $(X, \|\cdot\|, L, R)$ is called a fuzzy normed linear space (briefly, $(X, \|\cdot\|)$ *FNS*) and $\|\cdot\|$ is a fuzzy norm if the following axioms are satisfied:

(1) $\|x\| = \widetilde{0}$ if and only if $x = 0$,

(2) $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,

(3) for all $x, y \in X$,

  (a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$, whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s + t \leq \|x+y\|_1^-$,

  (b) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$, whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s + t \geq \|x+y\|_1^-$.

Let $(X, \|\cdot\|)$ be a fuzzy normed linear space. A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $L \in X$ with respect to the fuzzy norm on X and it is denoted by $x_n \overset{\text{FN}}{\to} L$, provided (D)- $\lim_{n\to\infty} \|x_n - L\| = \widetilde{0}$, i.e., for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$D(\|x_n - L\|, \widetilde{0}) < \varepsilon$, for all $n \geq N(\varepsilon)$. This means that for every $\varepsilon > 0$ there exists an integer $N(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{\alpha\in[0,1]} \|x_n - L\|_\alpha^+ = \|x_n - L\|_0^+ < \varepsilon$$

for all $n \geq N(\varepsilon)$.

Let $(X, \|\cdot\|)$ be an *FNS*. A sequence (x_n) in X is statistically convergent to $L \in X$ with respect to the fuzzy norm on X and it is denoted by $x_n \stackrel{\mathrm{FS}}{\to} L$, provided that for each $\varepsilon > 0$, we have $\delta(\{n \in \mathbb{N} : D(\|x_n - L\|, \tilde{0}) \geq \varepsilon\}) = 0$. This implies that for each $\varepsilon > 0$, the set $K(\varepsilon) = \{n \in \mathbb{N} : \|x_n - L\|_0^+ \geq \varepsilon\}$ has natural density zero, namely, for each $\varepsilon > 0$, $\|x_n - L\|_0^+ < \varepsilon$ for almost all n.

A double sequence $x = (x_{mn})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ is said to be Pringsheim's convergent (or P-convergent) if for given $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n \geq N(\varepsilon)$. It is denoted by $\lim_{m,n\to\infty} x_{mn} = L$, where m and n tend to infinity independent of each other.

Let $(X, \|\cdot\|)$ be an *FNS*. If for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that

$$D\bigl(\|x_{mn} - L\|, \widetilde{0}\bigr) < \varepsilon$$

for all $m, n \geq N$, then the double sequence $x = (x_{mn})$ is said to be convergent to $L \in X$ with respect to the fuzzy norm on X. In this case, it is denoted by $x_{mn} \stackrel{\mathrm{FN}}{\longrightarrow} L$. This means that for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that $\sup_{\alpha\in[0,1]} \|x_{mn} - L\|_\alpha^+ = \|x_{mn} - L\|_0^+ < \varepsilon$, for all $m, n \geq N$.

Let $K \subset \mathbb{N}\times\mathbb{N}$. Let K_{mn} be the number of $(j, k) \in K$ such that $j \leq m, k \leq n$. If the sequence $\{\frac{K_{mn}}{m.n}\}$ has a limit in Pringsheim's sense then we say that K has double natural density denoted by

$$\delta_2(K) = \lim_{m,n\to\infty} \frac{K_{mn}}{m.n}.$$

Let $(X, \|\cdot\|)$ be an *FNS*. If for every $\varepsilon > 0$,

$$\delta_2\bigl(\bigl\{(m, n) \in \mathbb{N}\times\mathbb{N} : D\bigl(\|x_{mn} - L\|, \widetilde{0}\bigr) \geq \varepsilon\bigr\}\bigr) = 0,$$

then the double sequence $x = (x_{mn})$ is said to be statistically convergent to $L \in X$ with respect to the fuzzy norm on X. This implies that $\|x_{mn} - L\|_0^+ < \varepsilon$ for almost all m, n and each $\varepsilon > 0$. In this case, it is denoted $\mathrm{FS}_2\text{-}\lim \|x_{mn} - L\| = \widetilde{0}$ or $x_{mn} \stackrel{\mathrm{FS}_2}{\longrightarrow} L$.

Let $X \neq \emptyset$ and (i) $\emptyset \in \mathcal{I}$, (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, and (iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

If class $\mathcal{I}$ satisfies (i), (ii), (iii), then $\mathcal{I}$ is an ideal in X. If $X \notin \mathcal{I}$, $\mathcal{I}$ is a nontrivial ideal. If also $\{x\} \in \mathcal{I}$ for each $x \in X$, the nontrivial ideal $\mathcal{I}$ in X is called admissible.

Let $X \neq \emptyset$ and $\mathcal{F}$ be a nonempty class subsets of X. If the following conditions are satisfied, $\mathcal{F}$ is a filter in X:

(i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, (iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Let $\mathcal{I}$ be a nontrivial ideal in $X \neq \emptyset$. The class $\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \setminus A)\}$ is a filter associated with $\mathcal{I}$ on X.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is also admissible. Throughout the paper, we consider $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Let $(X, \|\cdot\|)$ be a fuzzy normed space. A sequence $x = (x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $L \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - L\|_0^+ \geq \varepsilon\}$ belongs to $\mathcal{I}$. In this case, we write $x_n \xrightarrow{\mathrm{F}\mathcal{I}} L$. The element L is called the $\mathcal{I}$-limit of x in X.

A sequence (x_n) in X is said to be $\mathcal{I}^*$-convergent to L in X with respect to the fuzzy norm on X if there exists a set $M \in F(\mathcal{I})$, $M = \{t_1 < t_2 < \cdots\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} \|x_{t_k} - L\| = 0$.

Let $(X, \|\cdot\|)$ be a fuzzy normed space. A sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$ with respect to fuzzy norm on X, if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L\|_0^+ \geq \varepsilon\}$ belongs to $\mathcal{I}_2$. In this case, it is denoted $x_{mn} \xrightarrow{\mathrm{F}\mathcal{I}_2} L$. The element L is called the $\mathcal{I}_2$-limit of x in X.

The double sequence $\theta_2 = \{(r_k, t_l)\}$ is called a double lacunary sequence if there exist two increasing sequences of integers such that

$$r_0 = 0, \qquad h_k = r_k - r_{k-1} \to \infty \quad \text{and} \quad t_0 = 0, \qquad \overline{h}_l = t_l - t_{l-1} \to \infty, \quad \text{as } k, l \to \infty.$$

We use following notations in the sequel:

$$r_{kl} = r_k t_l, \qquad h_{kl} = h_k \overline{h}_l, \qquad J_{kl} = \big\{(r,t) : r_{k-1} < r \leq r_k \text{ and } t_{l-1} < t \leq t_l.\big\}$$

2 Lacunary $\mathcal{I}_2$-convergence

In this section, we introduce the concepts of θ_2-convergence and lacunary $\mathcal{I}_2$-convergence in fuzzy normed spaces. Also, we examine the relation between $\mathrm{F}\theta_2$-convergence and $\mathrm{F}\mathcal{I}_{\theta_2}$-convergence.

Throughout the paper, we let $(X, \|\cdot\|)$ be an FNS, $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $\theta_2 = \{(r_k, t_l)\}$ be a double lacunary sequence, and $x = (x_{rt})$ be a double sequence.

Definition 2.1 A sequence $x = (x_{rt})_{(r,t)\in\mathbb{N}\times\mathbb{N}}$ in X is said to be θ_2-convergent to $L_1 \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} D\big(\|x_{rt} - L_1\|, \tilde{0}\big) < \varepsilon$$

for all $k, l \geq n_0$. In this case, we write $x_{rt} \xrightarrow{\mathrm{F}\theta_2} L_1$ or $x_{rt} \to L_1(\mathrm{F}\theta_2)$, or $\mathrm{F}\theta_2\text{-}\lim_{r,t\to\infty} x_{rt} = L_1$. The element L_1 is called the $\mathrm{F}\theta_2$-limit of x in X.

Theorem 2.1 *If $x = (x_{rt})$ in X is $\mathrm{F}\theta_2$-convergent, then $\mathrm{F}\theta_2\text{-}\lim x$ is unique.*

Proof Assume that $\mathrm{F}\theta_2\text{-}\lim x = L_1$ and $\mathrm{F}\theta_2\text{-}\lim x = L_2$. Then, for any $\varepsilon > 0$, there exists an $n_1 \in \mathbb{N}$ such that

$$\frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ < \frac{\varepsilon}{2}$$

for all $k, l \geq n_1$. Also, there exists an integer $n_2 \in \mathbb{N}$ such that

$$\frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_2\|_0^+ < \frac{\varepsilon}{2}$$

for all $k, l \geq n_2$.

Now, consider $n_0 = \max\{n_1, n_2\}$. For $k, l \geq n_0$, if we get a $(p,q) \in \mathbb{N} \times \mathbb{N}$, then we have

$$\|x_{pq} - L_1\|_0^+ < \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ < \frac{\varepsilon}{2}$$

and

$$\|x_{pq} - L_2\|_0^+ < \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_2\|_0^+ < \frac{\varepsilon}{2}.$$

Thus, we get

$$\|L_1 - L_2\|_0^+ \leq \|x_{pq} - L_1\|_0^+ + \|x_{pq} - L_2\|_0^+ < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\|L_1 - L_2\|_0^+ = 0$ which implies that $L_1 = L_2$. □

Definition 2.2 A sequence $x = (x_{rt})$ in X is said to be lacunary $\mathcal{I}_2$-convergent to $L_1 \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$, the set

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} D\bigl(\|x_{rt} - L_1\|, \tilde{0}\bigr) \geq \varepsilon\right\}$$

belongs to $\mathcal{I}_2$. In this case, we write $x_{rt} \overset{\mathrm{F}\mathcal{I}_{\theta_2}}{\longrightarrow} L_1$ or $x_{rt} \to L_1(\mathrm{F}\mathcal{I}_{\theta_2})$, or $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim_{r,t\to\infty} x_{rt} = L_1$. The element L_1 is called the $\mathrm{F}\mathcal{I}_{\theta_2}$-limit of (x_{rt}) in X.

Lemma 1 *For every $\varepsilon > 0$, the following statements are equivalent*:

(a) $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim_{r,t\to\infty} x_{rt} = L_1$,

(b) $\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2$,

(c) $\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ < \varepsilon\} \in \mathcal{F}(\mathcal{I}_2)$,

(d) $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim_{r,t\to\infty} \|x_{rt} - L_1\|_0^+ = 0$.

Theorem 2.2 *If $x = (x_{rt})$ in X is lacunary $\mathcal{I}_2$-convergent with respect to fuzzy norm on X, then $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim x$ is unique.*

Proof This theorem is an analogue of Theorem 2.1; the proof follows easily. □

Theorem 2.3 *Let $x = (x_{rt})$ and $y = (y_{rt})$ be two double sequences in X. Then,*

(i) *if $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim x_{rt} = L_1$ and $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim y_{rt} = L_2$, then $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim(x_{rt} + y_{rt}) = L_1 + L_2$;*

(ii) *if $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim x_{rt} = L_1$, then $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim cx_{rt} = cL_1$, for $c \in \mathbb{R} - \{0\}$.*

Proof (i) For any $\varepsilon > 0$, let we define the following sets:

$$B_1 = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ \geq \frac{\varepsilon}{2}\right\}$$

and

$$B_2 = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|y_{rt} - L_2\|_0^+ \geq \frac{\varepsilon}{2}\right\}.$$

Since $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim x_{rt} = L_1$ and $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim y_{rt} = L_2$, using Lemma 1, we have $B_1 \in \mathcal{I}_2$ and $B_2 \in \mathcal{I}_2$, for all $\varepsilon > 0$.

Now, let $B_3 = B_1 \cup B_2$. Then, $B_3 \in \mathcal{I}_2$. This implies that its complement $(B_3)^c$ is a nonempty set in $\mathcal{F}(\mathcal{I}_2)$. We claim that

$$(B_3)^c \subset \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1 + y_{rt} - L_2\|_0^+ < \varepsilon\right\}.$$

Let $(k,l) \in (B_3)^c$, then we have

$$\frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|y_{rt} - L_2\|_0^+ < \frac{\varepsilon}{2}.$$

Now, If we get a $(p,q) \in \mathbb{N} \times \mathbb{N}$, then we have

$$\|x_{pq} - L_1\|_0^+ < \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ < \frac{\varepsilon}{2}$$

and

$$\|y_{pq} - L_2\|_0^+ < \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|y_{rt} - L_2\|_0^+ < \frac{\varepsilon}{2}.$$

Then, we have

$$\|x_{rt} - L_1 + y_{rt} - L_2\|_0^+ \leq \|x_{rt} - L_1\|_0^+ + \|y_{rt} - L_2\|_0^+ < \varepsilon.$$

Hence,

$$(B_3)^c \subset \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1 + y_{rt} - L_2\|_0^+ < \varepsilon\right\}.$$

Since $(B_3)^c \in \mathcal{F}(\mathcal{I}_2)$, we have

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \left\|(x_{rt} + y_{rt}) - (L_1 + L_2)\right\|_0^+ \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Therefore, $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim(x_{rt} + y_{rt}) = L_1 + L_2$.

(ii) Let $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim x_{rt} = L_1$. Then, for each $\varepsilon > 0$ and $c \in \mathbb{R} \setminus \{0\}$, we define the following set:

$$C = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ < \frac{\varepsilon}{|c|}\right\}.$$

So, $C \in \mathcal{F}(\mathcal{I}_2)$. Let $(k,l) \in C$, then we have

$$\begin{aligned}\frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ < \frac{\varepsilon}{|c|} &\Longrightarrow \frac{|c|}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ < |c|\frac{\varepsilon}{|c|} \\ &\Longrightarrow \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} |c|\|x_{rt} - L_1\|_0^+ < \varepsilon \\ &\Longrightarrow \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|cx_{rt} - cL_1\|_0^+ < \varepsilon.\end{aligned}$$

Therefore,

$$C \subset \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|cx_{rt} - cL_1\|_0^+ < \varepsilon\right\}$$

and

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|cx_{rt} - cL_1\|_0^+ < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Hence $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim cx_{rt} = cL_1$. □

Theorem 2.4 *Let $x = (x_{rt})$ be a double sequence in X. If $\mathrm{F}\theta_2\text{-}\lim x = L_1$, then $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim x = L_1$.*

Proof Let $\mathrm{F}\theta_2\text{-}\lim x = L_1$. Then, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ < \varepsilon$$

for all $k, l \geq n_0$. Therefore, the set

$$\begin{aligned}K &= \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ \geq \varepsilon\right\} \\ &\subset \big(\mathbb{N}\times\{1,2,\ldots,(n_0-1)\}\big) \cup \big(\{1,2,\ldots,(n_0-1)\}\times\mathbb{N}\big).\end{aligned}$$

So, we have $K \in \mathcal{I}_2$. Hence, $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim x = L_1$. □

Theorem 2.5 *Let $x = (x_{rt})$ be a double sequence in X. If $\mathrm{F}\theta_2\text{-}\lim x = L_1$, then there exists a subsequence $(x_{r_i t_j})$ such that $x_{r_i t_j} \xrightarrow{\mathrm{FN}} L_1$.*

Proof Let $\mathrm{F}\theta_2\text{-}\lim x = L_1$. Then, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ < \varepsilon$$

for all $k, l \geq n_0$. Clearly, for each $k, l \geq n_0$, we can select a $(r_i, t_j) \in J_{kl}$ such that

$$\|x_{r_i t_j} - L_1\|_0^+ < \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L_1\|_0^+ < \varepsilon.$$

It follows that $x_{r_i t_j} \xrightarrow{\text{FN}} L_1$. □

3 Limit point and cluster point

In this section, we introduce the notions of F$\mathcal{I}_{\theta_2}$-limit point and F$\mathcal{I}_{\theta_2}$-cluster point for double sequences in a fuzzy normed space. Also, we examine the relations between F$\mathcal{I}_{\theta_2}$-limit point and F$\mathcal{I}_{\theta_2}$-cluster point of double sequences in a fuzzy normed space.

Definition 3.1 An element $L \in X$ is said to be an F$\mathcal{I}_{\theta_2}$-limit point of $x = (x_{rt})$ if there is a set $M_1 = \{r_1 < r_2 < \cdots < r_i < \cdots\} \subset \mathbb{N}$ and $M_2 = \{t_1 < t_2 < \cdots < t_j < \cdots\} \subset \mathbb{N}$ such that the set $M' = \{(k,l) \in \mathbb{N} \times \mathbb{N} : (r_i, t_j) \in J_{kl}\} \notin \mathcal{I}_2$ and Fθ_2-$\lim x_{r_i t_j} = L$. We denote the set of all F$\mathcal{I}_{\theta_2}$-limit points of x as $\Lambda_F^{\mathcal{I}_{\theta_2}}(x)$.

Definition 3.2 An element $L \in X$ is said to be an F$\mathcal{I}_{\theta_2}$-cluster point of $x = (x_{rt})$ if for every $\varepsilon > 0$, we have

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L\|_0^+ < \varepsilon\right\} \notin \mathcal{I}_2.$$

We denote the set of all F$\mathcal{I}_{\theta_2}$-cluster points of x as $\Gamma_F^{\mathcal{I}_{\theta_2}}(x)$.

Theorem 3.1 *For each $x = (x_{rt})$ in X, we have $\Lambda_F^{\mathcal{I}_{\theta_2}}(x) \subset \Gamma_F^{\mathcal{I}_{\theta_2}}(x)$.*

Proof Let $L \in \Lambda_F^{\mathcal{I}_{\theta_2}}(x)$, then there exist two sets $M_1, M_2 \subset \mathbb{N}$ such that $M' \notin \mathcal{I}_2$, where M_1, M_2, and M' are as in Definition 3.1, and also Fθ_2-$\lim x_{r_i t_j} = L$. Thus, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{kl}} \sum_{(r_i,t_j)\in J_{kl}} \|x_{r_i t_j} - L\|_0^+ < \varepsilon$$

for all $k, l \geq n_0$. Then we get

$$D = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L\|_0^+ < \varepsilon\right\}$$
$$\supseteq M' \setminus \left\{\{r_1, r_2, \ldots, r_{n_0}\} \times \{t_1, t_2, \ldots, t_{n_0}\}\right\}.$$

Therefore, we have $M' \setminus \{\{r_1, r_2, \ldots, r_{n_0}\} \times \{t_1, t_2, \ldots, t_{n_0}\}\} \notin \mathcal{I}_2$ and as such $D \notin \mathcal{I}_2$. Consequently, $L \in \Gamma_F^{\mathcal{I}_{\theta_2}}(x)$. □

Theorem 3.2 *For every double sequence $x = (x_{rt})$, the following statements are equivalent:*

(i) *L is an F$\mathcal{I}_{\theta_2}$-limit points of x,*

(ii) *there exist two double sequences $y = (y_{rt})$ and $z = (z_{rt})$ in X such that $x = y + z$, Fθ_2-$\lim y = L$ and $\{(k,l) \in \mathbb{N} \times \mathbb{N} : (r,t) \in J_{kl}, z_{rt} \neq 0\} \in \mathcal{I}_2$.*

Proof Suppose that (i) holds. Then, there exist sets M_1, M_2, and M' as in Definition 3.1 such that $M' \notin \mathcal{I}_2$ and $\mathrm{F}\theta_2\text{-}\lim x_{r_i t_j} = L$. Define the sequences y and z as follows:

$$y_{rt} = \begin{cases} x_{rt}, & \text{if } (r,t) \in J_{kl}, (k,l) \in M', \\ L, & \text{otherwise} \end{cases}$$

and

$$z_{rt} = \begin{cases} 0, & \text{if } (r,t) \in J_{kl}, (k,l) \in M', \\ x_{rt} - L, & \text{otherwise.} \end{cases}$$

It suffices to consider the case $(r,t) \in J_{kl}$ such that $(k,l) \in \mathbb{N} \times \mathbb{N} \setminus M'$. For each $\varepsilon > 0$, we have $\|y_{rt} - L\|_0^+ = 0 < \varepsilon$. Thus,

$$\frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|y_{rt} - L\|_0^+ = 0 < \varepsilon.$$

Therefore, $\mathrm{F}\theta_2\text{-}\lim y = L$. Now

$$\bigl\{(k,l) \in \mathbb{N} \times \mathbb{N} : (r,t) \in J_{kl}, z_{rt} \neq 0\bigr\} \subset \mathbb{N} \times \mathbb{N} \setminus M'.$$

But, $\mathbb{N} \times \mathbb{N} \setminus M' \in \mathcal{I}_2$, and so

$$\bigl\{(k,l) \in \mathbb{N} \times \mathbb{N} : (r,t) \in J_{kl}, z_{rt} \neq 0\bigr\} \in \mathcal{I}_2.$$

Now, suppose that (ii) holds. Let $M' = \{(k,l) \in \mathbb{N} \times \mathbb{N} : (r,t) \in J_{kl}, z_{rt} = 0\}$. Then, clearly $M' \in F(\mathcal{I}_2)$ and so it is an infinite set. Construct the sets $M_1 = \{r_1 < r_2 < \cdots < r_i < \cdots\} \subset \mathbb{N}$ and $M_2 = \{t_1 < t_2 < \cdots < t_j < \cdots\} \subset \mathbb{N}$ such that $(r_i, t_j) \in J_{kl}$ and $z_{r_i t_j} = 0$. Since $x_{r_i t_j} = y_{r_i t_j}$ and $\mathrm{F}\theta_2\text{-}\lim y = L$, we obtain $\mathrm{F}\theta_2\text{-}\lim x_{r_i t_j} = L$. This completes the proof. □

Theorem 3.3 *If there is an* $\mathrm{F}\mathcal{I}_{\theta_2}$*-convergent sequence* $y = (y_{rt})$ *in* X *such that* $\{(r,t) \in \mathbb{N} \times \mathbb{N} : y_{rt} \neq x_{rt}\} \in \mathcal{I}_2$, *then* $x = (x_{rt})$ *is also* $\mathrm{F}\mathcal{I}_{\theta_2}$*-convergent.*

Proof Suppose that $\{(r,t) \in \mathbb{N} \times \mathbb{N} : y_{rt} \neq x_{rt}\} \in \mathcal{I}_2$ and $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim y = L$. Then, for every $\varepsilon > 0$, the set

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|y_{rt} - L\|_0^+ \geq \varepsilon\right\} \in \mathcal{I}_2.$$

For every $\varepsilon > 0$, we get

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|x_{rt} - L\|_0^+ \geq \varepsilon\right\}$$
$$\subseteq \bigl\{(r,t) \in \mathbb{N} \times \mathbb{N} : y_{rt} \neq x_{rt}\bigr\} \cup \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{kl}} \sum_{(r,t)\in J_{kl}} \|y_{rt} - L\|_0^+ \geq \varepsilon\right\}.$$

Therefore, we have

$$\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{kl}}\sum_{(r,t)\in J_{kl}}\|x_{rt}-L\|_0^+\geq\varepsilon\right\}\in\mathcal{I}_2.$$

This completes the proof of the theorem. □

4 Conclusions

In this paper, we introduced the concepts of $\mathrm{F}\theta_2$-convergence and $\mathrm{F}\mathcal{I}_{\theta_2}$-convergence. So, we saw that these limits are unique and if $\mathrm{F}\theta_2\text{-}\lim x = L$, then $\mathrm{F}\mathcal{I}_{\theta_2}\text{-}\lim x = L$. Later, we gave the definitions of $\mathrm{F}\mathcal{I}_{\theta_2}$-limit point and $\mathrm{F}\mathcal{I}_{\theta_2}$-cluster point, and we proved that every limit point was also a cluster point. In further studies, the lacunary ideal Cauchy sequence and lacunary Cauchy sequence of double sequences can be defined and examined in fuzzy normed spaces.

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