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Approximation on a class of Phillips operators generated by q-analogue

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Abstract

The main purpose of this article is to introduce a new generalization of q-Phillips operators generated by Dunkl exponential function. We establish some approximation results for these operators. We also determine the order of approximation, and the rate of convergence in terms of the modulus of continuity of order one and two. Moreover, we obtain some direct theorems.

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1 Introduction

Bernstein polynomials play a very important role in approximation process. For a positive integer $n \geqq 1$ and a function g defined on $[0,1]$, Bernstein defined the positive linear operators $B_n : C[0,1] \to C[0,1]$ by

$$B_n(g;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} g\left(\frac{k}{n}\right), \quad x \in [0,1]. \tag{1.1}$$

For some recent work on Bernstein operators, we refer to [21, 26, 27, 32, 46]. In 1950, Szász defined the operators [48]

$$S_m(g;x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} g\left(\frac{k}{m}\right) \quad (x \in [0,\infty), m \in \mathbb{N}) \tag{1.2}$$

for a continuous function g on $[0,\infty)$. The construction of Szász type operators is accomplished, by a newly parameter $\kappa \geqq 0$, and it is known as the Dunkl generalization. It was given by Sucu [47] with the help of [43]. The q-Hermite type polynomials were introduced by Cheikh et al. [13] by applying a new parameter $\kappa > -\frac{1}{2}$. The exponential functions and recursion formula on the Dunkl generalization are given by

$$e_{\kappa,q}(x) = \sum_{m=0}^{\infty} \frac{x^m}{\gamma_{\kappa,q}(m)}, \quad \text{and} \quad E_{\kappa,q}(x) = \sum_{m=0}^{\infty} \frac{q^{\frac{m(m-1)}{2}} x^m}{\gamma_{\kappa,q}(m)}, \tag{1.3}$$

$$\gamma_{\kappa,q}(m+1) = \left(\frac{1-q^{2\kappa\theta_{m+1}+m+1}}{1-q}\right)\gamma_{\kappa,q}(m), \quad m \in \mathbb{N}, \tag{1.4}$$

$$\theta_m = \begin{cases} 0 & \text{if } m = 2,4,6,\ldots, \\ 1 & \text{if } m = 1,3,5,\ldots. \end{cases} \tag{1.5}$$

We recall the basic information regarding the q-calculus:

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} & \text{for } q \neq 1, n \in \mathbb{N}, \\ 1 & \text{for } q = 1, \\ 0 & \text{for } n = 0, \end{cases} \qquad [n]_q! = \begin{cases} 1 & \text{for } n = 0, \\ \prod_{k=1}^{n}[k]_q & \text{for } n \in \mathbb{N}. \end{cases} \tag{1.6}$$

are the q-integer $[n]_q$ and q-factorial $[n]_q!$, respectively. İçöz and Çekim [17] wrote the Szász operators as follows:

$$D_{m,q}(g;x) = \frac{1}{e_{\kappa,q}([m]_q x)} \sum_{k=0}^{\infty} \frac{([m]_q x)^k}{\gamma_{\kappa,q}(k)} g\left(\frac{1-q^{2\kappa\theta_k+k}}{1-q^m}\right). \tag{1.7}$$

Recently, the Szász operators have many improvements and modifications in approximation process (see [1, 24, 25, 35, 42]). The q-analogue of some other interesting operators has been studied in [2, 37, 45, 49] and the references therein. An additional approach to improving the quantum calculus is post-quantum calculus via these types of generalizations; it was proposed in [3–5, 7, 8, 18, 23, 28, 33, 34, 36] (see also [39, 40]).

In this manuscript, we emphasize a new generalization of q-Phillips operators by introducing the new parameters and increasing and unbounded sequences of positive numbers. For more details of the approximation to classical Phillips operators via the Dunkl type version, see the recent article [38]. We study the convergence results in modulus of continuity of order one and two. Moreover, we investigate the rate of convergence for functions belonging to the Lipschitz class and also prove some direct theorems. For further information and the results used in this article we mention here some related articles (see [6, 19, 22, 44]).

2 Operators and their associated moments

Let $\{\alpha_{[m]_q}\}_{m\geq 1}$ and $\{\beta_{[m]_q}\}_{m\geq 1}$ be the increasing and unbounded sequences of positive numbers such that

$$\lim_{m\to\infty} \frac{1}{\beta_{[m]_q}} \to 0 \quad \text{and} \quad \frac{\alpha_{[m]_q}}{\beta_{[m]_q}} = 1 + O\left(\frac{1}{\beta_{[m]_q}}\right). \tag{2.1}$$

For $m = 1,2,\ldots$, we denote the nodes ∇_m by

$$\nabla_m = m + 2\kappa\theta_m, \quad \kappa \geq -\frac{1}{2}. \tag{2.2}$$

For all $x \in [0,\infty)$, $n \in \mathbb{N} \cup \{0\}$ and every $g \in C_\eta[0,\infty) = \{f \in C[0,\infty) : g(t) = O(t^\eta), t \to \infty\}$ with $\eta > n$, we define

$$\begin{aligned}&\mathcal{S}^*_{m,q}(g;\alpha_{[m]_q},\beta_{[m]_q};x)\\&\quad = \frac{\alpha_{[m]_q}}{e_{\kappa,q}(\alpha_{[m]_q}x)}\sum_{j=0}^{\infty}\mathcal{T}^\kappa_{m,q}(x)\int_0^{\infty/1-q}\frac{e_{\kappa,q}(-\alpha_{[m]_q}t)(\alpha_{[m]_q}t)^{\nabla_j}}{[\nabla_j]_q!}f\left(q^{\nabla_j}\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}t\right)\mathrm{d}_qt,\end{aligned}\tag{2.3}$$

where

$$\mathcal{T}^\kappa_{m,q}(x) = \frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)}q^{\frac{(\nabla_j)(\nabla_j+1)}{2}}.$$

Definition 2.1 For all $m > 0$ and $q \in (0,1)$, the generalized q-Gamma function is defined by

$$\varGamma_q(m) = \int_0^{1/1-q}x^{m-1}E_q(-qx)\,\mathrm{d}_qx, \quad m > 0, \tag{2.4}$$

$$\gamma^K_q(m) = \int_0^{\infty/K(1-q)}x^{m-1}e_q(-x)\,\mathrm{d}_qx, \quad m > 0, \tag{2.5}$$

where $\varGamma_q(m) = L(K;m)\gamma^K_q(m)$ and $L(K;m) = \frac{1}{1+K}K^m(1+\frac{1}{K})^m_q(1+K)^{m-1}_q$. Moreover, in particular for any positive integer m we have $L(K;m) = q^{\frac{m(m-1)}{2}}$ and $\varGamma_q(m) = q^{\frac{m(m-1)}{2}}\gamma^K_q(m)$, which also satisfies the following equation:

$$\varGamma_q(m+1) = \begin{cases}[m]_q\varGamma_q(m) & \text{for } m > 0,\\ 1 & \text{for } m = 0.\end{cases}\tag{2.6}$$

For more details, see [15].

Lemma 2.2 *Let $\mathcal{S}^*_{m,q}(\cdot;\cdot)$ be the operators defined by* (2.3). *Then we have*:

(1) $\mathcal{S}^*_{m,q}(1;\alpha_{[m]_q},\beta_{[m]_q};x) = 1$,

(2) $\mathcal{S}^*_{m,q}(t;\alpha_{[m]_q},\beta_{[m]_q};x) = \left(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}\right)x + \frac{1}{q\beta_{[m]_q}}$,

(3) $$\begin{aligned}\mathcal{S}^*_{m,q}\left(t^2;\alpha_{[m]_q},\beta_{[m]_q};x\right) \leqq{}& \frac{(1+q)}{q^3(\beta_{[m]_q})^2} + \frac{\alpha_{[m]_q}}{(\beta_{[m]_q})^2}\left(\frac{1+2q}{q^2} + [1+2\kappa]_q\right)x\\&+\left(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}\right)^2x^2,\end{aligned}$$

(4) $$\begin{aligned}\mathcal{S}^*_{m,q}\left(t^3;\alpha_{[m]_q},\beta_{[m]_q};x\right) \leqq{}& \frac{(1+q)(1+q+q^2)}{q^6(\beta_{[m]_q})^3}\\&+\frac{\alpha_{[m]_q}}{q^5(\beta_{[m]_q})^3}\left\{\left(1+3q+4q^2+3q^3\right)\right.\\&\left.+q^2\left(1+2q+3q^2\right)[1+2\kappa]_q + q^5[1+2\kappa]^2_q\right\}x\end{aligned}$$

$$+\frac{\alpha_{[m]_q}}{q^4(\beta_{[m]_q})^3}\left\{q\left(1+2q+3q^2\right)+3q^4[1+2\kappa]_q\right\}x^2$$

$$+\left(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}\right)^3 x^3,$$

$$(5)\quad \mathcal{S}^*_{m,q}\left(t^4;\alpha_{[m]_q},\beta_{[m]_q};x\right)\leqq\frac{(1+q)(1+2q+3q^2+3q^3+2q^4+q^5)}{q^{10}(\beta_{[m]_q})^4}$$

$$+\frac{\alpha_{[m]_q}}{q^9(\beta_{[m]_q})^4}\left\{\left(1+4q+8q^2+12q^3+12q^4+9q^5+4q^6\right)\right.$$

$$+q^2\left(1+3q+7q^2+9q^3+9q^4+6q^5\right)[1+2\kappa]_q$$

$$\left.+q^5\left(1+2q+3q^2+4q^3\right)[1+2\kappa]_q^2+q^9[1+2\kappa]_q^3\right\}x$$

$$+\frac{(\alpha_{[m]_q})^2}{q^8(\beta_{[m]_q})^4}\left\{q\left(1+3q+7q^2+9q^3+9q^4+6q^5\right)\right.$$

$$\left.+q^4\left(1+2q+3q^2+4q^3\right)[1+2\kappa]_q+7q^8[1+2\kappa]_q^2\right\}x^2$$

$$+\frac{(\alpha_{[m]_q})^3}{q^7(\beta_{[m]_q})^4}\left\{q^3\left(1+2q+3q^2+4q^3\right)+6q^7[1+2\kappa]_q\right\}x^3$$

$$+\left(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}\right)^4 x^4.$$

Proof From the generalized q-Gamma function defined by Definition 2.1, we see that

$$\begin{aligned}
&\int_0^{\infty/1-q} q^{\frac{(\nabla_j)(\nabla_j+1)}{2}}\frac{e_{\kappa,q}(-\alpha_{[m]_q}t)(\alpha_{[m]_q}t)^{\nabla_j}}{[\nabla_j]_q!}\left(q^{\nabla_j}\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}t\right)^u \mathrm{d}_q t\\
&=\frac{1}{\alpha_{[m]_q}(\beta_{[m]_q})^u}\frac{1}{[\nabla_j]_q!}q^{\frac{(\nabla_j)(\nabla_j+1)}{2}+u(\nabla_j)}\int_0^{\infty/1-q}(\alpha_{[m]_q}t)^{\nabla_j+u}e_{\kappa,q}(-\alpha_{[m]_q}t)\alpha_{[m]_q}\,\mathrm{d}_q t\\
&=\frac{1}{\alpha_{[m]_q}(\beta_{[m]_q})^u}\frac{1}{[\nabla_j]_q!}q^{\frac{(\nabla_j)(\nabla_j+1)}{2}+u(\nabla_j)}\int_0^{\infty/1-q}t^{\nabla_j+u}e_{\kappa,q}(-t)\,\mathrm{d}_q t\\
&=\frac{1}{\alpha_{[m]_q}(\beta_{[m]_q})^u}\frac{1}{[\nabla_j]_q!}q^{\frac{(\nabla_j)(\nabla_j+1)}{2}+u(\nabla_j)}\gamma_q^1(\nabla_j+u+1)\\
&=\frac{1}{\alpha_{[m]_q}(\beta_{[m]_q})^u}\frac{1}{[\nabla_j]_q!}q^{\frac{(\nabla_j)(\nabla_j+1)}{2}+u(\nabla_j)}\frac{[\nabla_j+u]_q!}{q^{\frac{(\nabla_j+u)(\nabla_j+u+1)}{2}}}\\
&=\frac{1}{\alpha_{[m]_q}(\beta_{[m]_q})^u}\frac{[\nabla_j+u]_q!}{[\nabla_j]_q!}\frac{1}{q^{\frac{u(u+1)}{2}}}.
\end{aligned}$$

If $u=0$ then $g(t)=1$, and hence

$$\begin{aligned}
\mathcal{S}^*_{m,q}(1;\alpha_{[m]_q},\beta_{[m]_q};x)&=\frac{\alpha_{[m]_q}}{e_{\kappa,q}(\alpha_{[m]_q}x)}\sum_{j=0}^{\infty}\frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)}\frac{[\nabla_j]_q!}{\alpha_{[m]_q}[\nabla_j]_q!}\\
&=1.
\end{aligned}$$

If $u = 1$, then $g(t) = t$, hence,

$$\begin{aligned}
\mathcal{S}^*_{m,q}(t;\alpha_{[m]_q},\beta_{[m]_q};x) &= \frac{(\alpha_{[m]_q})^2}{\beta_{[m]_q} e_{\kappa,q}(\alpha_{[m]_q}x)} \sum_{j=0}^{\infty} \frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)} \frac{[\nabla_j+1]_q!}{q(\alpha_{[m]_q})^2[\nabla_j]_q!} \\
&= \frac{1}{q\beta_{[m]_q} e_{\kappa,q}(\alpha_{[m]_q}x)} \sum_{j=0}^{\infty} \frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)} [\nabla_j+1]_q \\
&= \frac{1}{q\beta_{[m]_q} e_{\kappa,q}(\alpha_{[m]_q}x)} \sum_{j=0}^{\infty} \frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)} \\
&\quad + \frac{1}{\beta_{[m]_q} e_{\kappa,q}(\alpha_{[m]_q}x)} \sum_{j=0}^{\infty} \frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)} [\nabla_j]_q \\
&= \left(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}\right)x + \frac{1}{q\beta_{[m]_q}}.
\end{aligned}$$

Take $u = 2$, then, for $g(t) = t^2$, we have

$$\begin{aligned}
\mathcal{S}^*_{m,q}\bigl(t^2;\alpha_{[m]_q},\beta_{[m]_q};x\bigr) &= \frac{(\alpha_{[m]_q})^3}{(\beta_{[m]_q})^2 e_{\kappa,q}(\alpha_{[m]_q}x)} \sum_{j=0}^{\infty} \frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)} \frac{[\nabla_j+2]_q!}{q^3(\alpha_{[m]_q})^3[\nabla_j]_q!} \\
&= \frac{1}{q^3(\beta_{[m]_q})^2 e_{\kappa,q}(\alpha_{[m]_q}x)} \sum_{j=0}^{\infty} \frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)} [\nabla_j+2]_q[\nabla_j+1]_q \\
&= \frac{1}{q^3(\beta_{[m]_q})^2 e_{\kappa,q}(\alpha_{[m]_q}x)} \sum_{j=0}^{\infty} \frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)} \\
&\quad \times \bigl\{(1+q) + q(1+2q)[\nabla_j]_q + q^3[\nabla_j]_q^2\bigr\} \\
&= \frac{(1+q)}{q^3(\beta_{[m]_q})^2} + \frac{\alpha_{[m]_q}(1+2q)}{q^2(\beta_{[m]_q})^2}x \\
&\quad + \frac{1}{(\beta_{[m]_q})^2 e_{\kappa,q}(\alpha_{[m]_q}x)} \sum_{j=0}^{\infty} \frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)} [\nabla_j]_q^2.
\end{aligned}$$

From [17] and by (1.7), we use

$$\begin{aligned}
&\frac{1}{e_{\kappa,q}(\alpha_{[m]_q}x)} \sum_{j=0}^{\infty} \frac{(\alpha_{[m]_q}x)^k}{\gamma_{\kappa,q}(j)} [\nabla_j]_q^2 \geq (\alpha_{[m]_q}x)^2 + q^{2\kappa}[1-2\kappa]_q \frac{e_{\kappa,q}(q\alpha_{[m]_q}x)}{e_{\kappa,q}(\alpha_{[m]_q}x)}\alpha_{[m]_q}x, \\
&\frac{1}{e_{\kappa,q}(\alpha_{[m]_q}x)} \sum_{j=0}^{\infty} \frac{(\alpha_{[m]_q}x)^k}{\gamma_{\kappa,q}(j)} [\nabla_j]_q^2 \leq (\alpha_{[m]_q}x)^2 + [1+2\kappa]_q\alpha_{[m]_q}x.
\end{aligned}$$

For $u = 3$, $g(t) = t^3$ and for $u = 4$, $g(t) = t^4$, we get

$$\mathcal{S}^*_{m,q}\bigl(t^3;\alpha_{[m]_q},\beta_{[m]_q};x\bigr) = \frac{1}{q^4(\beta_{[m]_q})^3 e_{\kappa,q}(\alpha_{[m]_q}x)} \sum_{j=0}^{\infty} \frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)} [\nabla_j+3]_q[\nabla_j+2]_q[\nabla_j+1]_q$$

and

$$\mathcal{S}^{*}_{m,q}\big(t^4;\alpha_{[m]_q},\beta_{[m]_q};x\big)=\frac{1}{q^{10}(\beta_{[m]_q})^4 e_{\kappa,q}(\alpha_{[m]_q}x)}\sum_{j=0}^{\infty}\frac{(\alpha_{[m]_q}x)^j}{\gamma_{\kappa,q}(j)}\\ \times[\nabla_j+4]_q[\nabla_j+3]_q[\nabla_j+2]_q[\nabla_j+1]_q.$$

From [37] we know that

$$\begin{aligned}&[\nabla_j+3]_q[\nabla_j+2]_q[\nabla_j+1]_q\\ &\quad=(1+q)\big(1+q+q^2\big)+\big\{q(1+2q)\big(1+q+q^2\big)+q^3(1+q)\big\}[\nabla_j]_q\\ &\qquad+\big\{q^3\big(1+q+q^2\big)+q^4(1+2q)\big\}[\nabla_j]_q^2+q^6[\nabla_j]_q^3,\\ &[\nabla_j+4]_q[\nabla_j+3]_q[k+2\kappa\theta_k+2]_q[\nabla_j+1]_q\\ &\quad=(1+q)\big(1+2q+3q^2+3q^3+2q^4+q^5\big)+\big\{q(1+2q)\big(1+2q+3q^2+3q^3+2q^4+q^5\big)\\ &\qquad+q^3(1+q)\big(1+2q+2q^2+2q^3\big)\big\}[\nabla_j]_q\\ &\qquad+\big\{q^3\big(1+2q+3q^2+3q^3+2q^4+q^5\big)+q^4(1+2q)\big(1+2q+2q^2+2q^3\big)\\ &\qquad+q^7(1+q)\big\}[\nabla_j]_q^2\\ &\qquad+\big\{q^6\big(1+2q+2q^2+2q^3\big)+q^8(1+2q)\big\}[\nabla_j]_q^3+q^{10}[\nabla_j]_q^4.\end{aligned}$$

Clearly by $D_{m,q}(f;x)$ in [17]) and from [37] for $g(t)=t^3$ and $g(t)=t^4$ we get the result. □

Lemma 2.3 *Take* $\mho_j=(t-x)^j$ *for* $j=1,2,3,4$ *and* $\delta_{m,q}=\sqrt{\mathcal{S}^{*}_{m,q}(\mho_j;\alpha_{[m]_q},\beta_{[m]_q};x)}$. *Let* $\mathcal{S}^{*}_{m,q}(\cdot;\cdot)$ *be the operators defined by* (2.3). *Then, for all* $x\in[0,\infty)$ *and* $0<q<1$, *we have* $\mathcal{S}^{*}_{m,q}(\mho_1;\alpha_{[m]_q},\beta_{[m]_q};x)=(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}-1)x+\frac{1}{q\beta_{[m]_q}}$ *and*

$$(\delta_{m,q})^2=\begin{cases}\frac{(1+q)}{q^3(\beta_{[m]_q})^2}+\frac{\alpha_{[m]_q}}{(q\beta_{[m]_q})^2}(1+2q+q^2[1+2\kappa]_q-2q\beta_{[m]_q})x+(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}-1)^2x^2\\ \quad\textit{for } j=2,\\ \frac{(1+q)^2}{q^6(\beta_{[m]_q})^4}+\frac{2(1+q)\alpha_{[m]_q}}{(q^5\beta_{[m]_q})^4}(1+2q+q^2[1+2\kappa]_q-2q\beta_{[m]_q})x\\ \quad+[\frac{(\alpha_{[m]_q})^2}{(q\beta_{[m]_q})^4}(1+2q+q^2[1+2\kappa]_q-2q\beta_{[m]_q})^2+\frac{2(1+q)}{q^3(\beta_{[m]_q})^2}(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}-1)^2]x^2\\ \quad+\frac{2\alpha_{[m]_q}}{(q\beta_{[m]_q})^2}(1+2q+q^2[1+2\kappa]_q-2q\beta_{[m]_q})(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}-1)^2x^3+(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}-1)^4x^4\\ \quad\textit{for } j=4.\end{cases}$$

3 Approximation by Korovkin's theorem

Korovkin's theorem has many applications and useful connections between the branches of mathematics and classical approximation theory (see [9]). In a very general context it is possible to define the Korovkin theorem presented in [9], so that it can be used in applications for the best approximation. Now we approximate the operators $\mathcal{S}^{*}_{m,q_m}(\cdot;\cdot)$ by using Korovkin's theorem. Let $q=q_m$ with $q_m\in(0,1)$ and let c be a fixed positive constant such that

$$\lim_{n\to\infty}q_m=1\quad\text{and}\quad\lim_{n\to\infty}q_m^n=c. \tag{3.1}$$

Theorem 3.1 *Let* $\{\alpha_{m_{q_m}}\}_{m\geqq 1}$ *and* $\{\beta_{m_{q_m}}\}_{m\geqq 1}$ *be the sequences satisfying* (3.1). *Then, for every function g such that* $\{g : g \in C[0,\infty) \cap x \in [0,\infty),$ *and* $\frac{g(x)}{1+x^2}$ *is finite when* $x \to \infty\}$,

$$\lim_{m\to\infty} \mathcal{S}^*_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x) = g(x)$$

uniformly on each compact subset of $[0,\infty)$.

Proof The well-known Korovkin theorem implies that

$$\lim_{m\to\infty} \mathcal{S}^*_{m,q_m}\bigl(t^i;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\bigr) = x^i, \quad i = 0,1,2.$$

Clearly, from (2.1) and (3.1), we see that

$$\lim_{m\to\infty} \mathcal{S}^*_{m,q_m}(t;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x) = x, \qquad \lim_{m\to\infty} \mathcal{S}^*_{m,q_m}\bigl(t^2;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\bigr) = x^2.$$

This proves the theorem. □

We recall that

$$\begin{aligned}
&\mathcal{X}_{(1+x^2)}[0,\infty) = \bigl\{g : |g(x)| \leqq \mathcal{C}_g\bigl(1+x^2\bigr)\bigr\},\\
&\mathcal{Y}_{(1+x^2)}[0,\infty) = \bigl\{g : g \in C[0,\infty) \cap \mathcal{X}_{(1+x^2)}[0,\infty)\bigr\},\\
&\mathcal{Y}^{\sigma}_{(1+x^2)}[0,\infty) = \left\{g : g \in \mathcal{Y}_{(1+x^2)}[0,\infty) \text{ such that } \lim_{x\to\infty}\frac{g(x)}{1+x^2} = \sigma\right\},
\end{aligned}$$

where σ is positive constant and $\mathcal{C}_g$ is a constant depends upon g.

Theorem 3.2 *For all* $g \in \mathcal{Y}^{\sigma}_{(1+x^2)}[0,\infty)$, *we have*

$$\lim_{m\to\infty} \bigl\|\mathcal{S}^*_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}}) - g\bigr\|_{(1+x^2)} = 0.$$

Proof Take the test function $g(t) = t^p$ for $p = 0,1,2$ and use Lemma 2.2. From the Korovkin theorem we know, for every $g(t) \in \mathcal{Y}^{\sigma}_{(1+x^2)}[0,\infty)$, $\mathcal{S}^*_{m,q_m}(t^p;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x) \to x^p$ uniformly on $[0,\infty)$, as $m \to \infty$. When $\mathcal{S}^*_{m,q_m}(1;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x) = 1$, then clearly

$$\lim_{m\to\infty} \bigl\|\mathcal{S}^*_{m,q_m}(1;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}}) - 1\bigr\|_{(1+x^2)} = 0. \tag{3.2}$$

In the case of $g(t) = t^2$

$$\begin{aligned}
\bigl\|\mathcal{S}^*_{m,q_m}(t;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}}) - x\bigr\|_{(1+x^2)} &= \sup_{x\geqq 0}\frac{|\mathcal{S}^*_{m,q_m}(t;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x) - x|}{1+x^2}\\
&= \left(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}} - 1\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2} + \frac{1}{q_m\beta_{q_m}}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}.
\end{aligned}$$

Clearly, in the view of the results by (2.1), we see that

$$\lim_{m\to\infty} \bigl\|\mathcal{S}^*_{m,q_m}(t;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}}) - x\bigr\|_{(1+x^2)} = 0. \tag{3.3}$$

Similarly,

$$
\begin{aligned}
\left\|\mathcal{S}^*_{m,q_m}\left(t^2;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}}\right)-x^2\right\|_{(1+x^2)} &= \sup_{x\geqq 0}\frac{|\mathcal{S}^*_{m,q_m}(t^2;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-x^2|}{1+x^2}\\
&= \left(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1\right)^2\sup_{x\geqq 0}\frac{x^2}{1+x^2}\\
&\quad+\frac{\alpha_{[m]_{q_m}}}{(\beta_{[m]_{q_m}})^2}\left(\frac{1+2q_m}{q_m^2}+[1+2\kappa]_{q_m}\right)\sup_{x\geqq 0}\frac{x}{1+x^2}\\
&\quad+\frac{(1+q_m)}{q_m^3(\beta_{[m]_{q_m}})^2}\sup_{x\geqq 0}\frac{1}{1+x^2}.
\end{aligned}
$$

Thus,

$$
\lim_{m\to\infty}\left\|\mathcal{S}^*_{m,q_m}\left(t^2;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}}\right)-x^2\right\|_{\sigma}=0. \tag{3.4}
$$

This completes the proof of Theorem 3.2. □

4 Order of approximation

Let $g\in C_H[0,\infty)$ denote the set of all continuous functions on $[0,\infty)$ satisfying $|g(x)|\leqq ae^{bx}$ for all $x\in[0,\infty)$ and where a, b are positive constants. For a given $\delta^*>0$, the modulus of continuity of the function g is defined as

$$
\omega^*\left(g;\delta^*\right)=\sup_{|x_1-x_2|\leqq\delta^*,x_1,x_2\in[0,\infty)}\left|g(x_1)-g(x_2)\right|. \tag{4.1}
$$

Note that

$$
\left|g(x_1)-g(x_2)\right|\leqq\left(\frac{|x_1-x_2|}{\delta^*}+1\right)\omega^*\left(g;\delta^*\right). \tag{4.2}
$$

For all $g\in C_B[0,\infty)$, the modulus of continuity of order two is defined by

$$
\omega_2^*\left(g;\delta^*\right)=\sup_{0<t\leqq\delta^*,y\in[0,\infty)}\left\|g(y+2t)-2g(y+t)+g(y)\right\|_{C_B[0,\infty)}, \tag{4.3}
$$

where $C_B[0,\infty)$ is defined as a class of all real valued functions on $[0,\infty)$ which are bounded and uniformly continuous with the sup norm defined as $\|g\|_{C_B[0,\infty)}=\sup_{x\geqq 0}|g(x)|$.

Theorem 4.1 *For all $g\in C_B[0,\infty)$ and $q=q_m$ with the property $q_m\in(0,1)$, we have*

$$
\begin{aligned}
&\left|\mathcal{S}^*_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-g(x)\right|\\
&\quad\leqq\left(1+\sqrt{\Theta_{m,q_m}(x)}\right)\omega^*\left(g;\frac{1}{\sqrt{\beta_{[m]_{q_m}}}}\right),
\end{aligned}
$$

where $\Theta_{m,q_m}(x)=\frac{(1+q)}{q^3}+\frac{\alpha_{[m]_q}}{q^2}(1+2q+q^2[1+2\kappa]_q-2q\beta_{[m]_q})x+(\alpha_{[m]_q}-\beta_{[m]_q})^2x^2$.

Proof In the light of (4.1), (4.2) and the Cauchy–Schwarz inequality, we see that

$$\begin{aligned}
&\left|\mathcal{S}^*_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-g(x)\right|\\
&\leqq \frac{\alpha_{[m]_{q_m}}}{e_{\kappa,q_m}(\alpha_{[m]_{q_m}}x)}\sum_{j=0}^{\infty}\mathcal{T}^{\kappa}_{m,q_m}(x)\int_0^{\infty/1-q_m}\frac{e_{\kappa,q_m}(-[m]_{q_m}t)([m]_{q_m}t)^{\nabla_j}}{[\nabla_j]_q!}\left|g(t)-g(x)\right|\mathrm{d}_{q_m}t\\
&\leqq \frac{\alpha_{[m]_{q_m}}}{e_{\kappa,q_m}(\alpha_{[m]_{q_m}}x)}\sum_{j=0}^{\infty}\mathcal{T}^{\kappa}_{m,q_m}(x)\\
&\quad\times\int_0^{\infty/1-q_m}\frac{e_{\kappa,q_m}(-\alpha_{[m]_{q_m}}t)(\alpha_{[m]_{q_m}}t)^{\nabla_j}}{[\nabla_j]_q!}\left(1+\frac{1}{\delta^*}|t-x|\right)\mathrm{d}_{q_m}t\,\omega^*(g;\delta^*)\\
&=\left\{\frac{1}{\delta^*}\left(\frac{\alpha_{[m]_{q_m}}}{e_{\kappa,q_m}(\alpha_{[m]_{q_m}}x)}\sum_{j=0}^{\infty}\mathcal{T}^{\kappa}_{m,q_m}(x)\right.\right.\\
&\quad\left.\left.\times\int_0^{\infty/1-q_m}\frac{e_{\kappa,q_m}(\alpha_{[m]_{q_m}}t)(\alpha_{[m]_{q_m}}t)^{\nabla_j}}{[\nabla_j]_q!}\left(|t-x|\right)\mathrm{d}_{q_m}t\right)+1\right\}\omega^*(g;\delta^*)\\
&\leq\left\{1+\frac{1}{\delta^*}\left[\frac{\alpha_{[m]_{q_m}}}{e_{\kappa,q_m}(\alpha_{[m]_{q_m}}x)}\sum_{j=0}^{\infty}\mathcal{T}^{\kappa}_{m,q_m}(x)\right.\right.\\
&\quad\left.\left.\times\int_0^{\infty/1-q_m}\frac{e_{\kappa,q_m}(-\alpha_{[m]_{q_m}}t)(\alpha_{[m]_{q_m}}t)^{\nabla_j}}{[\nabla_j]_q!}\left(1+\frac{1}{\delta^*}(t-x)^2\right)\mathrm{d}_{q_m}t\right]^{\frac{1}{2}}\right\}\omega^*(g;\delta^*)\\
&=\left\{1+\frac{1}{\delta^*}\left(\mathcal{S}^*_{m,q_m}\left((t-x)^2;\alpha_{[m]_q},\beta_{[m]_q};x\right)\right)^{\frac{1}{2}}\right\}\omega^*(g;\delta^*).
\end{aligned}$$

If we take $\delta^*=\delta^*_{m,q_m}=\frac{1}{\sqrt{\beta_{[m]_{q_m}}}}$, then we easily get the results. □

Corollary 4.2 *If we choose* $\delta^*_{m,q_m}=\mathcal{S}^*_{m,q_m}((t-x)^2;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)$, *then*

$$\left|\mathcal{S}^*_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-g(x)\right|\leqq 2\omega^*(g;\delta^*_{m,q_m}).$$

5 Rate of convergence

For all $g\in C[0,\infty)$ and $\lambda_1,\lambda_2\in[0,\infty)$, the set $\mathrm{Lip}_{\mathcal{C}}(\xi)$ is defined as

$$\mathrm{Lip}_{\mathcal{C}}(\xi)=\left\{g:\left|g(\lambda_1)-g(\lambda_2)\right|\leqq\mathcal{C}|\lambda_1-\lambda_2|^{\xi}\right\}. \tag{5.1}$$

Moreover, for any $\chi\in C_B[0,\infty)$ one has the supremum norm

$$\|\chi\|_{C_B[0,\infty)}=\sup_{x\geqq 0}\left|\chi(x)\right|. \tag{5.2}$$

Let

$$C_B^2[0,\infty)=\left\{\chi:\chi,\chi',\chi''\in C_B[0,\infty)\right\}, \tag{5.3}$$

$$\|\chi\|_{C_B^2[0,\infty)}=\|\chi\|_{C_B[0,\infty)}+\left\|\chi'\right\|_{C_B[0,\infty)}+\left\|\chi''\right\|_{C_B[0,\infty)}. \tag{5.4}$$

Theorem 5.1 *Let the sequences of positive numbers* $\{\alpha_{[m]_{q_m}}\}_{n\geqq 1}$ *and* $\{\beta_{[m]_{q_m}}\}_{m\geqq 1}$ *be defined by* (2.1). *Then, for each* $g \in \operatorname{Lip}_{\mathcal{C}}(\xi)$ *the operators* $\mathcal{S}^{*}_{m,q_m}(\cdot\,;\cdot)$ *satisfy*

$$\begin{aligned}&\bigl|\mathcal{S}^{*}_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-g(x)\bigr|\\&\quad\leqq \mathcal{C}\left\{\frac{(1+q_m)}{q_m^3(\beta_{[m]_{q_m}})^2}+\frac{\alpha_{[m]_{q_m}}}{(q\beta_{[m]_{q_m}})^2}\bigl(1+2q_m+q_m^2[1+2\kappa]_{q_m}-2q_m\beta_{[m]_{q_m}}\bigr)x\right.\\&\qquad\left.+\left(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1\right)^2x^2\right\}^{\frac{\xi}{2}}.\end{aligned}$$

Proof We use the result by (5.1):

$$\begin{aligned}\bigl|\mathcal{S}^{*}_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-g(x)\bigr| &\leq \bigl|\mathcal{S}^{*}_{m,q_m}\bigl(g(t)-g(x);\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\bigr)\bigr|\\&\leq \mathcal{S}^{*}_{m,q_m}\bigl(\bigl|g(t)-g(x)\bigr|;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\bigr)\\&\leq \mathcal{C}\mathcal{S}^{*}_{m,q_m}\bigl(|t-x|^{\xi};x\bigr).\end{aligned}$$

Therefore,

$$\begin{aligned}&\bigl|\mathcal{S}^{*}_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-g(x)\bigr|\\&\quad\leqq \mathcal{C}\frac{\alpha_{[m]_{q_m}}}{e_{\kappa,q_m}(\alpha_{[m]_{q_m}})}\sum_{j=0}^{\infty}\mathcal{T}^{\kappa}_{m,q_m}(x)\int_0^{\infty/1-q_m}\frac{e_{\kappa,q_m}(-\alpha_{[m]_{q_m}}t)(\alpha_{[m]_{q_m}}t)^{\nabla_j}}{[\nabla_j]_q!}|t-x|\,\mathrm{d}_{q_m}t\\&\quad\leqq \mathcal{C}\frac{\alpha_{[m]_{q_m}}}{e_{\kappa,q_m}(\alpha_{[m]_{q_m}})}\sum_{j=0}^{\infty}\bigl(\mathcal{T}^{\kappa}_{m,q_m}(x)\bigr)^{\frac{2-\xi}{2}}\bigl(\mathcal{T}^{\kappa}_{m,q_m}(x)\bigr)^{\frac{\xi}{2}}\\&\qquad\times\int_0^{\infty/1-q_m}\frac{e_{q_m}(-\alpha_{[m]_{q_m}}t)(\alpha_{[m]_{q_m}}t)^{\nabla_j}}{[\nabla_j]_q!}|t-x|\,\mathrm{d}_{q_m}t\\&\quad\leqq \mathcal{C}\left(\frac{\alpha_{[m]_{q_m}}}{e_{\kappa,q_m}(\alpha_{[m]_{q_m}}x)}\sum_{j=0}^{\infty}\mathcal{T}^{\kappa}_{m,q_m}(x)\int_0^{\infty/1-q_m}\frac{e_{q_m}(-\alpha_{[m]_{q_m}}t)(\alpha_{[m]_{q_m}}t)^{\nabla_j}}{[\nabla_j]_q!}\,\mathrm{d}_{q_m}t\right)^{\frac{2-\xi}{2}}\\&\qquad\times\left(\frac{\alpha_{[m]_{q_m}}}{e_{\kappa,q_m}(\alpha_{[m]_{q_m}}x)}\sum_{j=0}^{\infty}\mathcal{T}^{\kappa}_{m,q_m}(x)\int_0^{\infty/1-q_m}\frac{e_{q_m}(-\alpha_{[m]_{q_m}}t)(\alpha_{[m]_{q_m}}t)^{\nabla_j}}{[\nabla_j]_q!}|t-x|^2\,\mathrm{d}_{q_m}t\right)^{\frac{\xi}{2}}\\&\quad=\mathcal{C}\bigl(\mathcal{S}^{*}_{m,q_m}(t-x)^2;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\bigr)^{\frac{\xi}{2}}.\end{aligned}$$

This completes the proof. □

Theorem 5.2 *For all* $\chi\in C_B^2[0,\infty)$ *defined by* (5.3), *we have*

$$\bigl|\mathcal{S}^{*}_{m,q_m}(\chi;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-\chi(x)\bigr|\leqq\left(\frac{\varLambda_{m,q_m}(x)}{2}+\varTheta_{m,q_m}(x)\right)\|\chi\|_{C_B^2[0,\infty)},$$

where $\varLambda_{m,q_m}(x)=\frac{(1+q_m)}{q_m^3(\beta_{[m]_{q_m}})^2}+\frac{\alpha_{[m]_{q_m}}}{(q_m\beta_{[m]_{q_m}})^2}(1+2q_m+q_m^2[1+2\kappa]_{q_m}-2q_m\beta_{[m]_{q_m}})x+(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1)^2x^2$ *and* $\varTheta_{m,q_m}(x)=(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1)x+\frac{1}{q\beta_{[m]_{q_m}}}$.

Proof Take $\chi \in C_B^2[0,\infty)$ and $\phi \in (x,t)$. On applying the linearity on generalized mean value theorem of Taylor series, we conclude that

$$\begin{aligned}\mathcal{S}^*_{m,q_m}(\chi;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-\chi(x) &= \chi'(x)\mathcal{S}^*_{m,q_m}\big((t-x);\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\big)\\ &\quad+\frac{\chi''(\phi)}{2}\mathcal{S}^*_{m,q_m}\big((t-x)^2;;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\big|\mathcal{S}^*_{m,q_m}(\chi;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-\chi(x)\big|\\ &\quad\leqq\bigg\{\bigg(\frac{\alpha_{[m]_q}}{\beta_{[m]_q}}-1\bigg)x+\frac{1}{q\beta_{[m]_q}}\bigg\}\|\chi'\|_{C_B[0,\infty)}\\ &\qquad+\bigg\{\frac{(1+q_m)}{q_m^3(\beta_{[m]_{q_m}})^2}+\frac{\alpha_{[m]_{q_m}}}{(q_m\beta_{[m]_{q_m}})^2}\big(1+2q_m+q_m^2[1+2\kappa]_{q_m}-2q_m\beta_{[m]_{q_m}}\big)x\\ &\qquad+\bigg(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1\bigg)^2x^2\bigg\}\frac{\|\chi''\|_{C_B[0,\infty)}}{2}.\end{aligned}$$

From (5.4) we easily see that

$$\begin{aligned}&\big|\mathcal{S}^*_{m,q_m}(\chi;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-\chi(x)\big|\\ &\quad\leqq\bigg\{\bigg(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1\bigg)x+\frac{1}{q_m\beta_{[m]_{q_m}}}\bigg\}\|\chi\|_{C_B^2[0,\infty)}\\ &\qquad+\bigg\{\frac{(1+q_m)}{q_m^3(\beta_{[m]_{q_m}})^2}+\frac{\alpha_{[m]_{q_m}}}{(q_m\beta_{[m]_{q_m}})^2}\big(1+2q_m+q_m^2[1+2\kappa]_{q_m}-2q_m\beta_{[m]_{q_m}}\big)x\\ &\qquad+\bigg(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1\bigg)^2x^2\bigg\}\frac{\|\chi\|_{C_B^2[0,\infty)}}{2}.\end{aligned}$$

This completes the proof. □

6 Some direct theorem

Let g and $\Phi \in C_B^2[0,\infty)$. For a given $\delta^* > 0$, the Peetre's K-functional [41] is defined as

$$K(g;\delta^*)=\inf\big\{\|g-\Phi\|_{C_B[0,\infty)}+\delta^*\|\Phi\|_{C_B^2[0,\infty)}\big\}, \tag{6.1}$$

$$K(g;\delta^*)\leqq\mathcal{W}\omega_2^*\big(g;(\delta^*)^{\frac{1}{2}}\big), \tag{6.2}$$

where ω_2^* is the modulus of continuity of order two defined in (4.3). By [14], there exists an absolute constant $\mathcal{C} > 0$ such that

$$K(g;\delta^*)\leqq\mathcal{C}\big\{\omega_2^*\big(g;\sqrt{\delta^*}\big)+\min(1,\delta^*)\|g\|\big\}. \tag{6.3}$$

Moreover, in the spaces of weighted modulus of continuity for each arbitrary $g \in \mathcal{Y}^{\sigma}_{(1+x^2)}[0,\infty)$ we have [10]

$$\Omega^*(g;\delta^*)=\sup_{x\geqq0,|u|\leqq\delta^*}\frac{|g(x+u)-g(x)|}{(1+u^2)(1+x^2)}, \tag{6.4}$$

$$\lim_{\delta\to 0}\Omega^*(g;\delta^*) = 0, \tag{6.5}$$

$$\left|g(t)-g(x)\right| \leqq 2\left(1+\frac{|t-x|}{\delta^*}\right)\left(1+(\delta^*)^2\right)\left(1+x^2\right)\left(1+(t-x)^2\right)\Omega^*(g;\delta^*), \tag{6.6}$$

where $t,x\in[0,\infty)$.

Theorem 6.1 *For all $g\in C_B[0,\infty)$ and $q=q_m$ with the number $q_m\in(0,1)$, we have*

$$\begin{aligned}&\left|\mathcal{S}^*_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-g(x)\right| \\ &\quad\leqq 2\mathcal{C}\left\{\omega_2^*\left(g;\sqrt{\frac{\Lambda_{m,q_m}(x)}{4}+\frac{\Theta_{m,q_m}(x)}{2}}\right)+\min\left(1,\frac{\Lambda_{m,q_m}(x)}{4}+\frac{\Theta_{m,q_m}(x)}{2}\right)\|g\|_{C_B[0,\infty)}\right\},\end{aligned}$$

where $\mathcal{C}$ is an absolute positive constant.

Proof Let $\Phi\in C_B[0,\infty)$. Applying Theorem 5.2, we get the result asserted by Theorem 6.1. Therefore,

$$\begin{aligned}&\left|\mathcal{S}^*_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-g(x)\right| \\ &\quad\leqq \left|\mathcal{S}^*_{m,q_m}\left((g-\Phi);\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\right)\right| \\ &\qquad+\left|\mathcal{S}^*_{m,q_m}(\Phi;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-\Phi(x)\right|+\left|g(x)-\Phi(x)\right| \\ &\quad\leqq 2\|g-\Phi\|_{C_B[0,\infty)}+\left(\frac{\Lambda_{m,q_m}(x)}{2}+\Theta_{m,q_m}(x)\right)\|\Phi\|_{C_B^2[0,\infty)} \\ &\quad= 2\left(\|g-\Phi\|_{C_B[0,\infty)}+\frac{\Lambda_{m,q_m}(x)}{4}+\frac{\Theta_{m,q_m}(x)}{2}\|\Phi\|_{C_B^2[0,\infty)}\right).\end{aligned}$$

On taking the infimum over all $\Phi\in C_B^2[0,\infty)$ and applying the result (6.2), we get

$$\left|\mathcal{S}^*_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-g(x)\right|\leqq 2K\left(g;\frac{\Lambda_{m,q_m}(x)}{4}+\frac{\Theta_{m,q_m}(x)}{2}\right).$$

Therefore, from (6.3), we get an absolute constant $\mathcal{C}>0$ such that the result holds. □

Theorem 6.2 *For each $g\in\mathcal{Y}^{\sigma}_{(1+x^2)}[0,\infty)$,*

$$\sup_{x\in[0,\Psi_{\kappa,q_m}(m))}\frac{|\mathcal{S}^*_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-g(x)|}{1+x^2}\leqq\mathcal{N}\left(1+\Psi_{\kappa,q_m}(m)\right)\Omega^*(g;\sqrt{\Psi_{\kappa,q_m}}),$$

where $\mathcal{N}=2(1+\mathcal{N}_1+2\sqrt{2}\mathcal{N}_2)$ and $\Psi_{\kappa,q_m}(m)=\max\{\gamma_{1,n_{q_m}},\gamma_{2,n_{q_m}},\gamma_{3,n_{q_m}}\}$ with $\gamma_{1,n_{q_m}}=\frac{(1+q_m)}{q_m^3(\beta_{[m]_{q_m}})^2}$, $\gamma_{2,n_{q_m}}=\frac{\alpha_{[m]_{q_m}}}{(q_m\beta_{[m]_{q_m}})^2}(1+2q_m+q_m^2[1+2\kappa]_{q_m}-2q_m\beta_{[m]_{q_m}})$ and $\gamma_{3,n_{q_m}}=(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1)^2$.

Proof From (6.6), we have

$$\begin{aligned}&\left|\mathcal{S}^*_{m,q_m}(g;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x)-g(x)\right| \\ &\quad\leqq 2\left(1+(\delta^*)^2\right)\left(1+x^2\right)\Omega^*(g;\delta^*)\left(1+\mathcal{S}^*_{m,q_m}\left((t-x)^2;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\right)\right.\end{aligned}$$

$$+\mathcal{S}^{*}_{m,q_m}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta^{*}};\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\right)\bigg). \tag{6.7}$$

From the Cauchy–Schwartz inequality obviously

$$\begin{aligned}
&\mathcal{S}^{*}_{m,q_m}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta^{*}};\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\right)\\
&\quad\leqq 2\left(\mathcal{S}^{*}_{m,q_m}\left(1+(t-x)^4;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\right)\right)^{\frac{1}{2}}\\
&\quad\times\left(\mathcal{S}^{*}_{m,q_m}\left(\frac{(t-x)^2}{(\delta^{*})^2};\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\right)\right)^{\frac{1}{2}}.
\end{aligned} \tag{6.8}$$

From Lemma 2.3, we easily conclude that

$$\mathcal{S}^{*}_{m,q_m}\left((t-x)^2;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\right)\leqq \Psi_{\kappa,q_m}(m)\left(1+x+x^2\right),$$

where

$$\begin{aligned}
\Psi_{\kappa,q_m}(m)=\max\bigg\{&\frac{(1+q_m)}{q_m^3(\beta_{[m]_{q_m}})^2},\frac{\alpha_{[m]_{q_m}}}{(q_m\beta_{[m]_{q_m}})^2}\left(1+2q_m+q_m^2[1+2\kappa]_{q_m}-2q_m\beta_{[m]_{q_m}}\right),\\
&\left(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1\right)^2\bigg\}.
\end{aligned}$$

There exists a positive constant $\mathcal{C}_1$ satisfying

$$\mathcal{S}^{*}_{m,q_m}\left((t-x)^2;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\right)\leqq \mathcal{N}_1\left(1+x+x^2\right). \tag{6.9}$$

Similarly,

$$\mathcal{S}^{*}_{m,q_m}\left((t-x)^4;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\right)\leqq \varphi_{\kappa,q_m}(m)\left(1+x+x^2+x^3+x^4\right), \tag{6.10}$$

where

$$\begin{aligned}
&\xi_{\kappa,q_m}(m)\\
&\quad=\max\bigg\{\frac{(1+q_m)^2}{q_m^6(\beta_{[m]_q})^4},\left[\frac{2(1+q_m)\alpha_{[m]_{q_m}}}{(q_m^5\beta_{[m]_{q_m}})^4}\left(1+2q_m+q_m^2[1+2\kappa]_{q_m}-2q_m\beta_{[m]_{q_m}}\right)\right],\\
&\quad\left[\frac{(\alpha_{[m]_{q_m}})^2}{(q_m\beta_{[m]_{q_m}})^4}\left(1+2q_m+q_m^2[1+2\kappa]_{q_m}-2q_m\beta_{[m]_{q_m}}\right)^2\right.\\
&\quad\left.+\frac{2(1+q_m)}{q_m^3(\beta_{[m]_{q_m}})^2}\left(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1\right)^2\right],\\
&\quad\frac{2\alpha_{[m]_{q_m}}}{(q_m\beta_{[m]_{q_m}})^2}\left(1+2q_m+q_m^2[1+2\kappa]_{q_m}-2q_m\beta_{[m]_{q_m}}\right)\left(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1\right)^2,\\
&\quad\left(\frac{\alpha_{[m]_{q_m}}}{\beta_{[m]_{q_m}}}-1\right)^4\bigg\}.
\end{aligned}$$

Since $\{\alpha_{[m]_q}\}$ and $\{\beta_{[m]_q}\}$ are the sequences satisfying (2.1) and $\lim_{n\to\infty} q_m = 1$, there exists a constant $\mathcal{N}_2 > 0$, such that

$$\left(\mathcal{S}^*_{m,q_m}\left(1+(t-x)^4;\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\right)\right)^{\frac{1}{2}} \leqq \mathcal{N}_2\left(2+x+x^2+x^3+x^4\right)^{\frac{1}{2}}. \tag{6.11}$$

Clearly in the light of (6.7)–(6.11), we conclude that

$$\left(\mathcal{S}^*_{m,q_m}\left(\frac{(t-x)^2}{(\delta^*)^2};\alpha_{[m]_{q_m}},\beta_{[m]_{q_m}};x\right)\right)^{\frac{1}{2}} \leqq \frac{1}{\delta^*}\left(\Psi_{\kappa,q_m}(m)\right)^{\frac{1}{2}}\left(1+x+x^2\right)^{\frac{1}{2}}. \tag{6.12}$$

Thus by combining (6.8)–(6.12) in (6.7) and if we put $\delta^* = \sqrt{\Psi_{\kappa,q_m}(m)}$ and taking the supremum over all $x \in [0, \Psi_{\kappa,q_m}(m))$, we get the result. □

Remark 6.3 For future work, some convergence properties of operators through summability techniques (see [11, 12, 16, 20, 29–31]) can be examined.

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