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Fractional Ostrowski type inequalities for bounded functions

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Abstract

We first establish some results involving Riemann–Liouville fractional integrals for partially differentiable functions. Then we obtain some fractional Ostrowski type inequalities for functions in class of functions L_p , L_∞ and L_1 , respectively. We also give some midpoint type inequalities as special cases of our main results.

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1 Introduction

For well over a century, the study of various types of integral inequalities has been the focus of great attention by a number of mathematicians, interested both in pure and applied mathematics. One of the many fundamental mathematical discoveries of Ostrowski [1] is the following classical integral inequality associated with the differentiable mappings.

Theorem 1 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e. $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty,$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible.

The overall structure of the study takes the form of five sections including **Introduction**. The remainder of this work is organized as follows: we first give the definition of Riemann–Liouville fractional integrals and mention some work which focuses on Ostrowski inequality. In Sect. 2, we obtain some generalized identities for the twice partial differentiable functions. Using the equalities obtained in Sect. 2, we establish some Ostrowski type inequalities for the functions belong to L_∞ in Sect. 3, also we prove Ostrowski type inequalities for the mappings belong to L_p and L_1 in Sect. 4 and Sect. 5, respectively.

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Firstly, we give the definitions of Riemann–Liouville fractional integrals.

Definition 1 ([2]) *Let $f \in L_1[a, b]$. The Riemann–Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Now, we give the definitions Riemann–Liouville fractional integrals of two variable functions:

Definition 2 ([3]) *Let $f \in L_1([a, b] \times [c, d])$. The Riemann–Liouville fractional integrals $J_{a+,c+}^{\alpha,\beta}$, $J_{a+,d-}^{\alpha,\beta}$, $J_{b-,c+}^{\alpha,\beta}$ and $J_{b-,d-}^{\alpha,\beta}$ are defined by*

$$J_{a+,c+}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x - t)^{\alpha-1} (y - s)^{\beta-1} f(t, s) ds dt, \quad x > a, y > c,$$

$$J_{a+,d-}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x - t)^{\alpha-1} (s - y)^{\beta-1} f(t, s) ds dt, \quad x > a, y < d,$$

$$J_{b-,c+}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t - x)^{\alpha-1} (y - s)^{\beta-1} f(t, s) ds dt, \quad x < b, y > c,$$

and

$$J_{b-,d-}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t - x)^{\alpha-1} (s - y)^{\beta-1} f(t, s) ds dt, \quad x < b, y < d.$$

Ostrowski inequalities for fractional integrals of two variable functions are obtained in [4]. There are several papers on fractional Ostrowski type inequalities for one or two variable functions, you can find some of them in Refs. [5–22].

2 Some identities for double integrals

Some equalities including Riemann–Liouville fractional integrals of two variable functions are established in this section. These identities will be used to prove the inequalities developed throughout this study.

Firstly, we define the following functions which will be used frequently:

$$M_\alpha(a, b; x) := \frac{(x - a)^\alpha + (b - x)^\alpha}{\Gamma(\alpha + 1)},$$

$$N_\beta(c, d; y) := \frac{(y - c)^\beta + (d - y)^\beta}{\Gamma(\beta + 1)},$$

for $(x, y) \in \Lambda := [a, b] \times [c, d]$.

Now we prove the following equalities.

Lemma 1 *Let $f : \Lambda \rightarrow \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and is continuous on Λ in \mathbb{R}^2 . Then, for any $(x, y) \in \Lambda$, we have*

$$\begin{aligned}
 & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d \Omega(x, t, y, s) \left[\int_x^t \int_y^s \frac{\partial^2 f(\zeta, \tau)}{\partial \zeta \partial \tau} d\tau d\zeta \right] ds dt \\
 &= J_{a+,c+}^{\alpha,\beta} f(x, y) + J_{a+,d-}^{\alpha,\beta} f(x, y) + J_{b-,c+}^{\alpha,\beta} f(x, y) + J_{b-,d-}^{\alpha,\beta} f(x, y) \\
 &\quad - N_\beta(c, d; y) [J_{a+}^\alpha f(x, y) + J_{b-}^\alpha f(x, y)] \\
 &\quad - M_\alpha(a, b; x) [J_{c+}^\beta f(x, y) + J_{d-}^\beta f(x, y)] \\
 &\quad + M_\alpha(a, b; x) N_\beta(c, d; y) f(x, y) \\
 &=: \mathcal{G}_1(x, y; a, b, c, d),
 \end{aligned} \tag{2.1}$$

where $\Omega(x, t, y, s)$ is defined by

$$\begin{aligned}
 & \Omega(x, t, y, s) \\
 &:= \begin{cases} (x-t)^{\alpha-1}(y-s)^{\beta-1}, & a \leq t < x \text{ and } c \leq s < y, \\ (x-t)^{\alpha-1}(s-y)^{\beta-1}, & a \leq t < x \text{ and } y \leq s \leq d, \\ (t-x)^{\alpha-1}(y-s)^{\beta-1}, & x \leq t \leq b \text{ and } c \leq s < y, \\ (t-x)^{\alpha-1}(s-y)^{\beta-1}, & x \leq t \leq b \text{ and } y \leq s \leq d. \end{cases}
 \end{aligned} \tag{2.2}$$

Proof It is easy to see that

$$\begin{aligned}
 & \int_x^t \int_y^s \frac{\partial^2 f(\zeta, \tau)}{\partial \zeta \partial \tau} d\tau d\zeta \\
 &= f(t, s) - f(t, y) - f(x, s) + f(x, y) \\
 &=: F(x, t, y, s).
 \end{aligned} \tag{2.3}$$

By the above equality and the definition of $\Omega(x, t, y, s)$, we get

$$\begin{aligned}
 & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d Q(x, t, y, s) \left[\int_x^t \int_y^s \frac{\partial^2 f(\zeta, \tau)}{\partial \zeta \partial \tau} d\tau d\zeta \right] ds dt \\
 &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1} F(x, t, y, s) ds dt \\
 &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1} F(x, t, y, s) ds dt \\
 &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1} F(x, t, y, s) ds dt \\
 &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1} F(x, t, y, s) ds dt.
 \end{aligned}$$

Applying the fundamental integral rules for the first integral in the right side of the above identity, we find that

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1} F(x,t,y,s) \, ds \, dt \\ &= J_{a+,c+}^{\alpha,\beta} f(x,y) - \frac{(y-c)^\beta}{\Gamma(\beta+1)} J_{a+}^\alpha f(x,y) \\ & \quad - \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} J_{c+}^\beta f(x,y) + \frac{(x-a)^\alpha(y-c)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} f(x,y). \end{aligned}$$

Adding the resulting identities after calculating the other integrals, then the desired equality (2.1) can be attained. \square

Lemma 2 *Suppose that all the assumptions of the Lemma 1 hold. Then, for any $(x,y) \in \Lambda$, we have*

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d \Omega(t,s) \left[\int_x^t \int_y^s \frac{\partial^2 f(\zeta,\tau)}{\partial \zeta \partial \tau} \, d\tau \, d\zeta \right] \, ds \, dt \\ &= J_{x+,y+}^{\alpha,\beta} f(b,d) + J_{x+,y-}^{\alpha,\beta} f(b,c) + J_{x-,y+}^{\alpha,\beta} f(a,d) + J_{x-,y-}^{\alpha,\beta} f(a,c) \\ & \quad - N_\beta(c,d;y) [J_{x+}^\alpha f(b,y) + J_{x-}^\alpha f(a,y)] \\ & \quad - M_\alpha(a,b;x) [J_{y+}^\beta f(x,d) + J_{y-}^\beta f(x,c)] \\ & \quad + M_\alpha(a,b;x) N_\beta(c,d;y) f(x,y) \\ &=: \mathcal{G}_2(x,y;a,b,c,d), \end{aligned} \tag{2.4}$$

where $\Omega(t,s)$ is defined by

$$\begin{aligned} & \Omega(t,s) \\ &:= \begin{cases} (t-a)^{\alpha-1}(s-c)^{\beta-1}, & a \leq t < x \text{ and } c \leq s < y, \\ (t-a)^{\alpha-1}(d-s)^{\beta-1}, & a \leq t < x \text{ and } y \leq s \leq d, \\ (b-t)^{\alpha-1}(s-c)^{\beta-1}, & x \leq t \leq b \text{ and } c \leq s < y, \\ (b-t)^{\alpha-1}(d-s)^{\beta-1}, & x \leq t \leq b \text{ and } y \leq s \leq d. \end{cases} \end{aligned} \tag{2.5}$$

Proof The proof of the equality (2.4) follows the same lines as the proof of Lemma 1. \square

Lemma 3 *Suppose that all the assumptions of the Lemma 1 hold. Then, for any $(x,y) \in \Lambda$, we have*

$$\begin{aligned} & \frac{1}{4\Gamma(\alpha)\Gamma(\beta)} \left\{ \int_a^b \int_c^d [(t-a)^{\alpha-1} + (b-t)^{\alpha-1}] \right. \\ & \quad \times [(s-c)^{\beta-1} + (d-s)^{\beta-1}] \left[\int_x^t \int_y^s \frac{\partial^2 f(\zeta,\tau)}{\partial \zeta \partial \tau} \, d\tau \, d\zeta \right] \, ds \, dt \Big\} \\ &= \frac{J_{b-,d-}^{\alpha,\beta} f(a,c) + J_{b-,c+}^{\alpha,\beta} f(a,d) + J_{a+,d-}^{\alpha,\beta} f(b,c) + J_{a+,c+}^{\alpha,\beta} f(b,d)}{4} \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{2} \frac{(d-c)^\beta}{\Gamma(\beta+1)} [J_{b-}^\alpha f(a,y) + J_{a+}^\alpha f(b,y)] \\
 & - \frac{1}{2} \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} [J_{d-}^\beta f(x,c) + J_{c+}^\beta f(x,d)] + \frac{(b-a)^\alpha (d-c)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} f(x,y) \\
 & = \mathcal{G}_3(x,y;a,b,c,d). \tag{2.6}
 \end{aligned}$$

Proof If we handle integral of the first expression in left side of (2.6), from the equality (2.3), then we have

$$\begin{aligned}
 & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (t-a)^{\alpha-1} (y-c)^{\beta-1} F(x,t,y,s) \, ds \, dt \\
 & = J_{b-d}^{\alpha,\beta} f(a,c) - \frac{(d-c)^\beta}{\Gamma(\beta+1)} J_{b-}^\alpha f(a,y) \\
 & \quad - \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} J_{d-}^\beta f(x,c) + \frac{(b-a)^\alpha (d-c)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} f(x,y).
 \end{aligned}$$

Adding the resulting identities side by side after the other expressions have been similarly examined, the required equality can be easily derived. □

3 The case when $f_{\zeta\tau} \in L_\infty(\Lambda)$

In this section, we observe some double integral inequalities involving Riemann–Liouville fractional expressions by considering identities given in the previous section and the functions that are element of L_∞ .

Theorem 2 *Let $f : \Lambda \rightarrow \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and is bounded, i.e.,*

$$\|f_{\zeta\tau}\|_\infty = \sup_{(\zeta,\tau) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f(\zeta,\tau)}{\partial \zeta \partial \tau} \right| < \infty$$

for all $(\zeta, \tau) \in \Lambda$. Then one has

$$|\mathcal{G}_1(x,y;a,b,c,d)| \leq M_{\alpha+1}(a,b;x) N_{\beta+1}(c,d;y) \|f_{\zeta\tau}\|_\infty \tag{3.1}$$

for all $(x,y) \in \Lambda$.

Proof Taking the absolute value of both sides of the equality (2.1), because $f_{\zeta\tau}$ is a bounded function on Λ , it follows that

$$\begin{aligned}
 & |\mathcal{G}_1(x,y;a,b,c,d)| \\
 & \leq \|f_{\zeta\tau}\|_\infty \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d |\Omega(x,t,y,s)| |t-x| |s-y| \, ds \, dt.
 \end{aligned}$$

Using the definition of $\Omega(x,t,y,s)$ and elementary analysis operations, the desired result can be easily obtained. □

Corollary 1 *If we choose $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (3.1), then we have the midpoint inequality*

$$\begin{aligned} & \left| J_{a+,c+}^{\alpha,\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + J_{a+,d-}^{\alpha,\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\ & \quad + J_{b-,c+}^{\alpha,\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + J_{b-,d-}^{\alpha,\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \quad - \frac{(d-c)^\beta}{2^{\beta-1}\Gamma(\beta+1)} \left[J_{a+}^\alpha f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + J_{b-}^\alpha f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] \\ & \quad - \frac{(b-a)^\alpha}{2^{\alpha-1}\Gamma(\alpha+1)} \left[J_{c+}^\beta f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + J_{d-}^\beta f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] \\ & \quad \left. + \frac{(b-a)^\alpha(d-c)^\beta}{2^{\alpha+\beta-2}\Gamma(\alpha+1)\Gamma(\beta+1)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \\ & \leq \frac{(b-a)^{\alpha+1}(d-c)^{\beta+1}}{2^{\alpha+\beta}\Gamma(\alpha+2)\Gamma(\beta+2)} \|f_{\zeta\tau}\|_\infty. \end{aligned}$$

Theorem 3 *Suppose that all the assumptions of Theorem 2 hold. Then we have*

$$|\mathcal{G}_2(x, y; a, b, c, d)| \leq M_{\alpha+1}(a, b; x) N_{\beta+1}(c, d; y) \|f_{\zeta\tau}\|_\infty \tag{3.2}$$

for any $(x, y) \in \Lambda$.

Proof This proof follows the same strategy which was used in the proof of Theorem 2 by considering the equality (2.4). □

Corollary 2 *With the assumption of Theorem 3, one has the midpoint type inequality*

$$\begin{aligned} & \left| J_{\frac{a+b}{2}+, \frac{c+d}{2}+}^{\alpha,\beta} f(b, d) + J_{\frac{a+b}{2}+, \frac{c+d}{2}-}^{\alpha,\beta} f(b, c) + J_{\frac{a+b}{2}-, \frac{c+d}{2}+}^{\alpha,\beta} f(a, d) + J_{\frac{a+b}{2}-, \frac{c+d}{2}-}^{\alpha,\beta} f(a, c) \right. \\ & \quad - \frac{(d-c)^\beta}{2^{\beta-1}\Gamma(\beta+1)} \left[J_{\frac{a+b}{2}+}^\alpha f\left(b, \frac{c+d}{2}\right) + J_{\frac{a+b}{2}-}^\alpha f\left(a, \frac{c+d}{2}\right) \right] \\ & \quad - \frac{(b-a)^\alpha}{2^{\alpha-1}\Gamma(\alpha+1)} \left[J_{\frac{c+d}{2}+}^\beta f\left(\frac{a+b}{2}, d\right) + J_{\frac{c+d}{2}-}^\beta f\left(\frac{a+b}{2}, c\right) \right] \\ & \quad \left. + \frac{(b-a)^\alpha(d-c)^\beta}{2^{\alpha+\beta-2}\Gamma(\alpha+1)\Gamma(\beta+1)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \\ & \leq \frac{(b-a)^{\alpha+1}(d-c)^{\beta+1}}{2^{\alpha+\beta}\Gamma(\alpha+2)\Gamma(\beta+2)} \|f_{\zeta\tau}\|_\infty. \end{aligned}$$

Theorem 4 *Suppose that all the assumptions of Theorem 2 hold. Then, for all $(x, y) \in \Lambda$, one has*

$$\begin{aligned} & |\mathcal{G}_3(x, y; a, b, c, d)| \\ & \leq \frac{1}{4\Gamma(\alpha)\Gamma(\beta)} [R_\alpha(x; a, b) + S_\alpha(x; a, b)] \\ & \quad \times [R_\beta(y; c, d) + S_\beta(y; c, d)] \|f_{\zeta\tau}\|_\infty, \end{aligned} \tag{3.3}$$

where

$$R_\lambda(\xi; u, v) = 2 \frac{(\xi - u)^{\lambda+1}}{\lambda(\lambda + 1)} + (v - u)^\lambda \left(\frac{v - u}{\lambda + 1} - \frac{\xi - u}{\lambda} \right)$$

and

$$S_\lambda(\xi; u, v) = 2 \frac{(v - \xi)^{\lambda+1}}{\lambda(\lambda + 1)} + (v - u)^\lambda \left(\frac{v - u}{\lambda + 1} - \frac{v - \xi}{\lambda} \right).$$

Proof Should we take the absolute value of (2.6), from the modulus property of the integral, then we have the inequality

$$\begin{aligned} & |\mathcal{G}_3(x, y; a, b, c, d)| \tag{3.4} \\ & \leq \|f_{\zeta\tau}\|_\infty \frac{1}{4} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left\{ \int_a^b \int_c^d [(t - a)^{\alpha-1} + (b - t)^{\alpha-1}] \right. \\ & \quad \left. \times [(s - c)^{\beta-1} + (d - s)^{\beta-1}] |t - x| |s - y| \, ds \, dt \right\}. \end{aligned}$$

Calculating the double integral given in the right side of (3.4), the desired inequality (3.3) can be easily obtained. \square

Corollary 3 *Suppose that all the assumptions of Theorem 4 hold. If we choose $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, then we have*

$$\begin{aligned} & \left| \frac{J_{b-,d}^{\alpha,\beta} f(a, c) + J_{b-,c+}^{\alpha,\beta} f(a, d) + J_{a+,d-}^{\alpha,\beta} f(b, c) + J_{a+,c+}^{\alpha,\beta} f(b, d)}{4} \right. \\ & \quad - \frac{1}{2} \frac{(d - c)^\beta}{\Gamma(\beta + 1)} \left[J_{b-}^\alpha f\left(a, \frac{c + d}{2}\right) + J_{a+}^\alpha f\left(b, \frac{c + d}{2}\right) \right] \\ & \quad - \frac{1}{2} \frac{(b - a)^\alpha}{\Gamma(\alpha + 1)} \left[J_{d-}^\beta f\left(\frac{a + b}{2}, c\right) + J_{c+}^\beta f\left(\frac{a + b}{2}, d\right) \right] \\ & \quad \left. + \frac{(b - a)^\alpha (d - c)^\beta}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) \right| \\ & \leq \frac{(b - a)^{\alpha+1} (d - c)^{\beta+1}}{\Gamma(\alpha + 2)\Gamma(\beta + 2)} \left[\frac{1}{2^\alpha} + \frac{\alpha - 1}{2} \right] \left[\frac{1}{2^\beta} + \frac{\beta - 1}{2} \right] \|f_{\zeta\tau}\|_\infty. \end{aligned}$$

4 The case when $f_{\zeta\tau} \in L_p(\Lambda)$

In this section, we examine how to obtain inequalities for mappings which are elements of the space L_p .

Theorem 5 *Let $f : \Lambda \rightarrow \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists on $(a, b) \times (c, d)$. If $\frac{\partial^2 f(\zeta, \tau)}{\partial \zeta \partial \tau} \in L_p(\Lambda)$ for $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, i.e.,*

$$\|f_{\zeta\tau}\|_p = \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(\zeta, \tau)}{\partial \zeta \partial \tau} \right|^p \, d\tau \, d\zeta \right)^{\frac{1}{p}} < \infty$$

for all $(\varsigma, \tau) \in \Lambda$, then we have the inequality

$$\begin{aligned}
 & |\mathcal{G}_1(x, y; a, b, c, d)| \\
 & \leq \frac{[(x-a)^{\alpha+\frac{1}{q}} + (b-x)^{\alpha+\frac{1}{q}}][(y-c)^{\beta+\frac{1}{q}} + (d-y)^{\beta+\frac{1}{q}}]}{\Gamma(\alpha)\Gamma(\beta)(\alpha+1/q)(\beta+1/q)} \|f_{\varsigma\tau}\|_p
 \end{aligned} \tag{4.1}$$

for all $(x, y) \in \Lambda$.

Proof If we take the absolute value of both sides of the equality (2.1) and later apply the Hölder inequality, from the assumption of the function f , we have

$$\begin{aligned}
 & \Gamma(\alpha)\Gamma(\beta)|\mathcal{G}_1(x, y; a, b, c, d)| \\
 & \leq \int_a^b \int_c^d |\Omega(x, t, y, s)| \left| \int_x^t \int_y^s \frac{\partial^2 f(\varsigma, \tau)}{\partial \varsigma \partial \tau} d\tau d\varsigma \right| ds dt \\
 & \leq \int_a^b \int_c^d |\Omega(x, t, y, s)| |t-x|^{\frac{1}{q}} |s-y|^{\frac{1}{q}} \left| \int_x^t \int_y^s \left| \frac{\partial^2 f(\varsigma, \tau)}{\partial \varsigma \partial \tau} \right|^p d\tau d\varsigma \right|^{\frac{1}{p}} ds dt \\
 & = \int_a^b \int_c^d |\Omega(x, t, y, s)| |t-x|^{\frac{1}{q}} |s-y|^{\frac{1}{q}} \|f_{\varsigma\tau}\|_{[t,x] \times [s,y], p} ds dt \\
 & \leq \|f_{\varsigma\tau}\|_p \int_a^b \int_c^d |\Omega(x, t, y, s)| |t-x|^{\frac{1}{q}} |s-y|^{\frac{1}{q}} ds dt.
 \end{aligned}$$

If the last integral of above expression is observed by considering the definition of $\Omega(x, t, y, s)$, then one attains the required inequality (4.1) which completes the proof. \square

Theorem 6 Assume that all the assumptions of Theorem 2 hold. If $\frac{\partial^2 f(\varsigma, \tau)}{\partial \varsigma \partial \tau} \in L_p(\Lambda)$ for $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then one has

$$\begin{aligned}
 & |\mathcal{G}_2(x, y; a, b, c, d)| \\
 & \leq \frac{[\Gamma(1+1/q)]^2}{\Gamma(\alpha+1+1/q)\Gamma(\beta+1+1/q)} \|f_{\varsigma\tau}\|_p \\
 & \quad \times [(x-a)^{\alpha+\frac{1}{q}} + (b-x)^{\alpha+\frac{1}{q}}][(y-c)^{\beta+\frac{1}{q}} + (d-y)^{\beta+\frac{1}{q}}]
 \end{aligned}$$

for any $(x, y) \in \Lambda$.

Proof If similar methods to the proof of Theorem 5 are followed by taking into account the equality (2.4), because of the definition of $\Omega(t, s)$, then one has

$$\begin{aligned}
 & \Gamma(\alpha)\Gamma(\beta)|\mathcal{G}_2(x, y; a, b, c, d)| \\
 & \leq \|f_{\varsigma\tau}\|_p \int_a^b \int_c^d |\Omega(t, s)| |t-x|^{\frac{1}{q}} |s-y|^{\frac{1}{q}} ds dt \\
 & = \|f_{\varsigma\tau}\|_p \int_a^x \int_c^y (t-a)^{\alpha-1} (s-c)^{\beta-1} |t-x|^{\frac{1}{q}} |s-y|^{\frac{1}{q}} ds dt \\
 & \quad + \|f_{\varsigma\tau}\|_p \int_a^x \int_y^d (t-a)^{\alpha-1} (d-s)^{\beta-1} |t-x|^{\frac{1}{q}} |s-y|^{\frac{1}{q}} ds dt
 \end{aligned}$$

$$\begin{aligned}
 &+ \|f_{\zeta\tau}\|_p \int_x^b \int_c^y (b-t)^{\alpha-1} (s-c)^{\beta-1} |t-x|^{\frac{1}{q}} |s-y|^{\frac{1}{q}} \, ds \, dt \\
 &+ \|f_{\zeta\tau}\|_p \int_x^b \int_y^d (b-t)^{\alpha-1} (d-s)^{\beta-1} |t-x|^{\frac{1}{q}} |s-y|^{\frac{1}{q}} \, ds \, dt.
 \end{aligned}$$

The above integrals can be readily calculated by utilizing the fact that

$$\begin{aligned}
 \int_{\lambda}^{\mu} (\xi - \lambda)^{\rho-1} (\mu - \xi)^{\sigma-1} \, d\xi &= (\mu - \lambda)^{\rho+\sigma-1} \int_0^1 u^{\rho-1} (1-u)^{\sigma-1} \, du \\
 &= (\mu - \lambda)^{\rho+\sigma-1} B(\rho, \sigma),
 \end{aligned}$$

which is obtained by using the change of variable $\xi = (1-u)\lambda + u\mu$, and where $B(\cdot, \cdot)$ is Beta function. Hence, the proof is finished. \square

We also note that if we choose $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in two inequalities presented in this section, then we reach new midpoint type results different from inequalities given in corollaries of the previous section.

5 The case when $f_{\zeta\tau} \in L_1(\Lambda)$

Now, we investigate how to results in the case when f element of L_1 .

Theorem 7 *Let $f : \Lambda \rightarrow \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists on $(a, b) \times (c, d)$. If $\frac{\partial^2 f(\zeta, \tau)}{\partial \zeta \partial \tau} \in L_1(\Lambda)$, i.e.,*

$$\|f_{\zeta\tau}\|_1 = \int_a^b \int_c^d \left| \frac{\partial^2 f(\zeta, \tau)}{\partial \zeta \partial \tau} \right| \, d\tau \, d\zeta < \infty$$

for all $(\zeta, \tau) \in \Lambda$, then we have the inequalities

$$|\mathcal{G}_1(x, y; a, b, c, d)| \leq M_{\alpha}(a, b; x) N_{\beta}(c, d; y) \|f_{\zeta\tau}\|_1 \tag{5.1}$$

and

$$|\mathcal{G}_2(x, y; a, b, c, d)| \leq M_{\alpha}(a, b; x) N_{\beta}(c, d; y) \|f_{\zeta\tau}\|_1 \tag{5.2}$$

for all $(x, y) \in \Lambda$.

Proof If we take the absolute value of both sides of the equality (2.1), due to the assumption of the function f , we have

$$\begin{aligned}
 &|\mathcal{G}_1(x, y; a, b, c, d)| \\
 &\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d |\Omega(x, t, y, s)| \left| \int_x^t \int_y^s \frac{\partial^2 f(\zeta, \tau)}{\partial \zeta \partial \tau} \, d\tau \, d\zeta \right| \, ds \, dt \\
 &\leq \|f_{\zeta\tau}\|_1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d |\Omega(x, t, y, s)| \, ds \, dt.
 \end{aligned}$$

Later, utilizing the definition of $\Omega(x, t, y, s)$, the desired inequality (5.1) can be readily attained. If we follow the same line as the proof of (5.1) by taking into account the equality (2.4), then we can also obtain the inequality (5.2). The proof is thus completed. \square

Theorem 8 *Suppose that all the assumptions of Theorem 7 hold. If $\frac{\partial^2 f(\varsigma, \tau)}{\partial \varsigma \partial \tau} \in L_1(\Lambda)$, then one has*

$$|\mathcal{G}_3(x, y; a, b, c, d)| \leq \frac{(b-a)^\alpha (d-c)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} \|f_{\varsigma\tau}\|_1 \tag{5.3}$$

for all $(x, y) \in \Lambda$.

Proof Taking the absolute value of (2.6), from the modulus property of the integral, we find that

$$\begin{aligned} & |\mathcal{G}_3(x, y; a, b, c, d)| \\ & \leq \frac{1}{4\Gamma(\alpha)\Gamma(\beta)} \|f_{\varsigma\tau}\|_1 \\ & \quad \times \int_a^b \int_c^d [(t-a)^{\alpha-1} + (b-t)^{\alpha-1}][(s-c)^{\beta-1} + (d-s)^{\beta-1}] ds dt. \end{aligned}$$

Calculating the above double integral, the required inequality (5.3) can be easily obtained. \square

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Authors’ contributions

The main idea of this paper was proposed by SE, HB, and PK. MZS and SI prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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