Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Norm of Hilbert operator on sequence spaces

Hadi Roopaei[1*]

Dedicated to Prof. Maryam Mirzakhani who, in spite of a short lifetime, left a long standing impact on mathematics.

*Correspondence:
H.roopaei@gmail.com
[1]Young Researchers and Elite Club, Marvdasht Branch, Islamic Azad University, Marvdasht, Iran

Abstract
In this paper, we focus on the problem of finding the norm of Hilbert operator on some sequence spaces. Meanwhile, we obtain several interesting inequalities and inclusions.

MSC: 26D15; 40C05; 40G05; 47B37

Keywords: Hilbert matrix; Cesàro matrix; Copson matrix; Difference sequence space; Norm

1 Introduction

Let $p \geq 1$ and ω denote the set of all real-valued sequences. The space ℓ_p is the set of all real sequences $x = (x_k) \in \omega$ such that

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} < \infty.$$

Hilbert matrix The Hilbert matrix $H = (h_{j,k})$ was introduced by David Hilbert in 1894 to study a question in approximation theory. The finite and infinite Hilbert matrices are

$$H_{n\times n} = \begin{pmatrix} 1 & 1/2 & \cdots & 1/n \\ 1/2 & 1/3 & \cdots & 1/n+1 \\ \vdots & \vdots & \cdots & \vdots \\ 1/n & 1/n+1 & \cdots & 1/2n-1 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hilbert matrices are frequently used both in mathematics and computational sciences. In image processing, for example, Hilbert matrices are commonly used. Any two-dimensional array of natural numbers in the range $[0, n]$ for all $n \in \mathbb{N}$ can be viewed as a gray-scale digital image.

Cryptography is another example of applications of the Hilbert matrix. Cryptography is the science of using mathematics to encrypt and decrypt data. Cryptography enables you to store sensitive information or transmit it across insecure networks so that it cannot be read by anyone except the intended recipient. The objectives of the proposed work are to propose a new cryptographic method based on the special matrix called the Hilbert matrix for authentication and confidentiality and to propose a model for confidentiality and authentication using a combination of symmetric and public cryptosystems. In some studies related to cryptographic methods, the Hilbert matrix is used for authentication and confidentiality [16]. It is well known that the Hilbert matrix is very unstable [15] and so it can be used in security systems.

In this paper we only focus on the infinite version of Hilbert matrix $H = (h_{j,k})$, which is defined by

$$h_{j,k} = \frac{1}{j+k+1} = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad j,k = 0,1,\ldots,$$

and is a bounded operator on ℓ_p with ℓ_p-norm $\|H\|_{\ell_p} = \Gamma(1/p)\Gamma(1/p^*) = \pi \csc(\pi/p)$ by Theorem 323 in [8].

Matrix domain The matrix domain of an infinite matrix A in a sequence space X is defined as

$$A_p = \{x \in \omega : Ax \in X\},$$

which is also a sequence space. It is easy to see that, for an invertible matrix A, the matrix domain A_p is a normed space with $\|x\|_{A_p} := \|Ax\|_{\ell_p}$. By using matrix domains of special triangle matrices in classical spaces, we can define more general sequence spaces than the space ℓ_p.

More recently, the author and some other mathematicians have investigated the problem of finding the norm of operators on several matrix domains [5, 6, 10–14, 17, 20, 21, 23].

Throughout this research, we use the notations $\|\cdot\|_{A_p,\ell_p}$, $\|\cdot\|_{\ell_p,A_p}$ and $\|\cdot\|_{A_p,B_p}$ for the norm of operators from the matrix domain A_p into sequence space ℓ_p, for the norm of operators from ℓ_p into the matrix domain A_p and for the norm of operators from matrix domain A_p into the matrix domain B_p, respectively.

Motivation Although a variety of research has been done on the finite Hilbert operator, see [1, 4, 9, 22], and a lot of properties of this matrix have been discovered (determinant, inverse, …) there exists a few information about the infinite version of Hilbert matrix, specially in the area of finding the norm of this operator on sequence spaces. Recently the author [18, 19] has introduced some factorizations for the infinite Hilbert matrix based on the generalized Cesàro matrix and Cesàro and Gamma matrices of order n. Through this study the author has tried to compute the norm of Hilbert operator on several sequence spaces that have not been done before.

2 Norm of Hilbert operator on some sequence spaces

In this part of our study, we investigate the problem of finding the norm of well-known Hilbert operator on some sequence spaces. The following lemma plays a key role in finding the norm of operators between matrix domains.

Lemma 2.1 *Let U be a bounded operator on ℓ_p and A_p, B_p be two matrix domains such that $A_p \simeq \ell_p$.*

(i) *If T has a factorization of the form $T = UA$, then T is a bounded operator from the matrix domain A_p into ℓ_p and*

$$\|T\|_{A_p,\ell_p} = \|U\|_{\ell_p}.$$

(ii) *If $BT = UA$, then T is a bounded operator from the matrix domain A_p into B_p and*

$$\|T\|_{A_p,B_p} = \|U\|_{\ell_p}.$$

Proof (i) Since A_p and ℓ_p are isomorphic,

$$\|T\|_{A_p,\ell_p} = \sup_{x\in A_p} \frac{\|Tx\|_{\ell_p}}{\|x\|_{A_p}} = \sup_{x\in A_p} \frac{\|UAx\|_{\ell_p}}{\|Ax\|_{\ell_p}} = \sup_{y\in \ell_p} \frac{\|Uy\|_{\ell_p}}{\|y\|_{\ell_p}} = \|U\|_{\ell_p}.$$

(ii) Again by isomorphism between A_p and ℓ_p we have

$$\begin{aligned}\|T\|_{A_p,B_p} &= \sup_{x\in A_p} \frac{\|Tx\|_{B_p}}{\|x\|_{A_p}} = \sup_{x\in A_p} \frac{\|BTx\|_{\ell_p}}{\|Ax\|_{\ell_p}} \\ &= \sup_{x\in A_p} \frac{\|UAx\|_{\ell_p}}{\|Ax\|_{\ell_p}} = \sup_{y\in \ell_p} \frac{\|Uy\|_{\ell_p}}{\|y\|_{\ell_p}} = \|U\|_{\ell_p},\end{aligned}$$

which gives the desired result. □

2.1 Norm of Hilbert operator on Cesàro and Copson sequence spaces

Consider the Hausdorff matrix $H^\mu = (h_{j,k})_{j,k=0}^\infty$, with entries of the form

$$h_{j,k} = \begin{cases} \int_0^1 \binom{j}{k}\theta^k(1-\theta)^{j-k}\,d\mu(\theta) & j \geq k, \\ 0 & j < k, \end{cases}$$

where μ is a probability measure on $[0,1]$. The Hausdorff matrix contains several famous classes of matrices like Cesàro, Gamma, Hölder and Euler matrices. Hardy's formula ([7], Theorem 216) states that the Hausdorff matrix is a bounded operator on ℓ_p if and only if $\int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta) < \infty$ and

$$\|H^\mu\|_{\ell_p} = \int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta). \tag{2.1}$$

Cesàro matrix By letting $d\mu(\theta) = n(1-\theta)^{n-1}\,d\theta$ in the definition of the Hausdorff matrix, the Cesàro matrix of order n, $C^n = (c_{j,k}^n)$, is defined by

$$c_{j,k}^n = \begin{cases} \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}} & 0 \leq k \leq j, \\ 0 & \text{otherwise}, \end{cases} \tag{2.2}$$

which according to Eq. (2.1) has the ℓ_p-norm

$$\|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}, \tag{2.3}$$

where p^* is the conjugate of p i.e. $\frac{1}{p}+\frac{1}{p^*}=1$. Note that $C^1 = C$ is the well-known Cesàro matrix

$$c_{j,k} = \begin{cases} \frac{1}{j+1} & 0 \le k \le j, \\ 0 & \text{otherwise}, \end{cases} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which has the ℓ_p-norm $\|C\|_{\ell_p} = p^*$.

Some more examples are

$$C^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 2/3 & 1/3 & 0 & \cdots \\ 3/6 & 2/6 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 3/4 & 1/4 & 0 & \cdots \\ 6/10 & 3/10 & 1/10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Cesàro matrix domain C_p^n is the set of all sequences whose C^n-transforms are in the space ℓ_p; that is,

$$C_p^n = \left\{ x = (x_j) \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \right|^p < \infty \right\},$$

which is a Banach space with the norm

$$\|x\|_{C_p^n} = \left(\sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \right|^p \right)^{1/p}.$$

Transposing the Cesàro matrix of order n results the Copson matrix of order n, $C^{nt} = (c_{j,k}^{nt})$, which has the entries

$$c_{j,k}^{nt} = \begin{cases} \frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}} & j \le k, \\ 0 & \text{otherwise}, \end{cases}$$

and the ℓ_p-norm

$$\|C^{nt}\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p)}{\Gamma(n+1/p)}, \tag{2.4}$$

by the Hellinger–Toeplitz theorem, which is the following.

Theorem 2.2 ([2], Proposition 7.2) *Suppose that* $1 < p, q < \infty$. *A matrix* A *maps* ℓ_p *into* ℓ_q *if and only if the transposed matrix,* A^t, *maps* ℓ_{q^*} *into* ℓ_{p^*}. *We then have* $\|A\|_{\ell_p,\ell_q} = \|A^t\|_{\ell_{q^*},\ell_{p^*}}$.

For a positive integer n, we define the Hilbert matrix of order n, $H^n = (h^n_{j,k})$, by

$$h^n_{j,k} = \frac{1}{j+k+n+1} \quad (j,k = 0,1,\ldots). \tag{2.5}$$

Note that, for $n = 0$, $H^0 = H$ is the well-known Hilbert matrix. For more examples:

$$H^1 = \begin{pmatrix} 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad H^2 = \begin{pmatrix} 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ 1/5 & 1/6 & 1/7 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For non-negative integers n, j and k, let us define the matrix $B^n = (b^n_{j,k})$ by

$$b^n_{j,k} = \frac{(k+1)\cdots(k+n)}{(j+k+1)\cdots(j+k+n+1)}.$$

Consider that, for $n = 0$, $B^0 = H$, where H is the Hilbert matrix.

Note that the matrix B^n has also the representation

$$b^n_{j,k} = \binom{n+k}{k}\beta(j+k+1, n+1) \quad (j,k = 0,1,\ldots),$$

where the β function is

$$\beta(m,n) = \int_0^1 z^{m-1}(1-z)^{n-1}\,dz \quad (m,n = 1,2,\ldots).$$

For computing the norm of Hilbert operator on Cesàro and Copson matrix domains we need the following lemma.

Lemma 2.3 *The Hilbert matrix H and the Hilbert matrix of order n, H^n, have the following factorizations based on the Cesàro matrix of order n:*

(i) $H = B^n C^n$,

(ii) $H^n = C^n B^n$,

(iii) $C^n H = H^n C^n$,

(iv) B^n *is a bounded operator on* ℓ_p *and*

$$\|B^n\|_{\ell_p} = \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}.$$

Proof (i) By applying the identity $\sum_{j=0}^{\infty}\binom{n+j-1}{j}z^j = (1-z)^{-n}$ for $|z| < 1$, we deduce that

$$\begin{aligned}(B^n C^n)_{j,k} &= \sum_{i=k}^{\infty}\binom{n+i}{i}\beta(j+i+1, n+1)\frac{\binom{n+i-k-1}{i-k}}{\binom{n+i}{i}} \\ &= \sum_{i=0}^{\infty}\binom{n+i-1}{i}\beta(j+i+k+1, n+1)\end{aligned}$$

$$= \int_0^1 \sum_{i=0}^{\infty} \binom{n+i-1}{i} z^i z^{j+k} (1-z)^n \, dz$$
$$= \int_0^1 z^{j+k} \, dz = \frac{1}{j+k+1} = h_{j,k}.$$

(ii) For convenience, let $\lambda = \frac{\binom{n+k}{k}}{\binom{n+i}{i}}$. The factorization will be obtained by the following calculations:

$$\begin{aligned}
(C^n B^n)_{i,k} &= \sum_{j=0}^{i} \frac{\binom{n+i-j-1}{i-j}}{\binom{n+i}{i}} \binom{n+k}{k} \beta(j+k+1, n+1) \\
&= \lambda \left\{ \binom{n+i-1}{i} \beta(k+1, n+1) + \cdots \right. \\
&\quad \left. + \binom{n}{1} \beta(i+k, n+1) + \binom{n-1}{0} \beta(i+k+1, n+1) \right\} \\
&= \lambda \left\{ \binom{n+i-1}{i} \beta(k+1, n+1) + \cdots \right. \\
&\quad \left. + \binom{n+1}{2} \beta(i+k-1, n+1) + \frac{(n+1)!(i+k-1)!}{(i+k+n-1)!(i+k+n+1)} \right\} \\
&= \lambda \left\{ \binom{n+i-1}{i} \beta(k+1, n+1) + \cdots \right. \\
&\quad \left. + \binom{n+2}{3} \beta(i+k-2, n+1) + \frac{(n+2)!(i+k-2)!}{2!(i+k+n-2)!(i+k+n+1)} \right\} \\
&\vdots \\
&= \lambda \left\{ \binom{n+i-1}{i} \beta(k+1, n+1) + \frac{(n+i-1)!(k+1)!}{(i-1)!(k+n+1)!(i+k+n+1)} \right\} \\
&= \lambda \left\{ \frac{(n+i-1)!k!}{i!(n+k+1)!} \frac{(n+k+1)(n+i)}{i+k+n+1} \right\} = \frac{1}{i+k+n+1} = h^n_{i,k}.
\end{aligned}$$

(iii) This is obvious by parts (i) and (ii). (iv) For computing the ℓ_p-norm of B^n, we introduce a family of matrices, $B(w)$, $0 < w \leq 1$, given by

$$b(w)_{j,k} = \binom{j+k}{k} w^j (1-w)^{n+k}.$$

Since

$$\begin{aligned}
\sum_{k=0}^{\infty} b(w)_{j,k} &= w^j (1-w)^n \sum_{k=0}^{\infty} \binom{j+1+k-1}{k} (1-w)^k \\
&= w^j (1-w)^n \bigl(1-(1-w)\bigr)^{-(j+1)} = \frac{(1-w)^n}{w}
\end{aligned}$$

and

$$\sum_{j=0}^{\infty} b(w)_{j,k} = (1-w)^{n+k} \sum_{j=0}^{\infty} \binom{k+1+j-1}{j} w^j = (1-w)^{n-1},$$

the row sums are all $\frac{(1-w)^n}{w}$ and the column sums are all $(1-w)^{n-1}$. Thus Schur's theorem results in

$$\|B(w)\|_{\ell_p} \le (1-w)^{n-\frac{1}{p}} w^{\frac{-1}{p^*}}.$$

On the other hand,

$$\begin{aligned}\int_0^1 b(w)_{j,k}\,dw &= \binom{j+k}{k}\int_0^1 w^j(1-w)^{n+k}\,dw\\ &= \binom{j+k}{k}\beta(j+1,n+k+1)\\ &= \binom{n+k}{k}\beta(j+k+1,n+1) = b^n_{j,k}.\end{aligned}$$

Now

$$\begin{aligned}\|B^n\|_{\ell_p} &= \left\|\int_0^1 B(w)\,dw\right\| \le \int_0^1 \|B(w)\|\,dw\\ &\le \int_0^1 (1-w)^{n-1/p}w^{-1/p^*}\,dw = \beta\big(n-1/p+1, 1-1/p^*\big)\\ &= \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}.\end{aligned}$$

Also the factorization $H = B^nC^n$ results $\|H\|_{\ell_p} \le \|B^n\|_{\ell_p}\|C^n\|_{\ell_p}$. Therefore

$$\|B^n\|_{\ell_p} \ge \frac{\|H\|_{\ell_p}}{\|C^n\|_{\ell_p}} = \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)},$$

which completes the proof. □

Remark 2.4 In parallel, the two Hilbert and Hilbert matrices of order n, H and H^n, have the following factorizations based on the Copson matrix:

(i) $H = C^{nt}B^{nt}$,
(ii) $H^n = B^{nt}C^{nt}$,
(iii) $HC^{nt} = C^{nt}H^n$,
(iv) B^{nt} is a bounded operator on ℓ_p and

$$\|B^{nt}\|_{\ell_p} = \frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)}.$$

Theorem 2.5 *Let H^n be the Hilbert matrix of order n. Then*

(i) *H is a bounded operator from C^n_p into ℓ_p and*

$$\|H\|_{C^n_p,\ell_p} = \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}.$$

(ii) *H is a bounded operator from C^n_p into C^n_p and*

$$\|H\|_{C^n_p} = \pi\csc(\pi/p).$$

(iii) *H^n is a bounded operator from C_p^{nt} into ℓ_p and*

$$\|H^n\|_{C_p^{nt},\ell_p} = \frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)}.$$

(iv) *H^n is a bounded operator from C_p^{nt} into C_p^{nt} and*

$$\|H^n\|_{C_p^{nt}} = \pi\csc(\pi/p).$$

Proof (i) According to Lemma 2.3 we have $H = B^nC^n$. Now, by applying Lemma 2.1 part (ii) we gain the result. (ii) From Lemma 2.3 we have $C^nH^n = HC^n$ that by Lemma 2.1 part (iii) the proof is obvious. □

As an application of Lemma 2.3, we are ready to generalize the inequality

$$\|Hx\|_{\ell_p} \le \pi\csc(\pi/p)\|x\|_{\ell_p},$$

also known as Hilbert's inequality.

Corollary 2.6 *Let $p > 1$ and $x \in \ell_p$. Then*

(i)

$$\|Hx\|_{\ell_p} \le \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}\|C^nx\|_{\ell_p},$$

or

$$\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right|^p \le \left(\frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}\right)^p\sum_{j=0}^{\infty}\left|\sum_{k=0}^{j}\frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}}x_k\right|^p.$$

In particular, for $n = 0$, the Hilbert inequality occurs.

(ii)

$$\|H^nx\|_{\ell_p} \le \frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)}\|C^{nt}x\|_{\ell_p},$$

or

$$\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+n+1}\right|^p \le \left(\frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)}\right)^p\sum_{j=0}^{\infty}\left|\sum_{k=j}^{\infty}\frac{\binom{n+k-j-1}{k-j}}{\binom{n+k}{k}}x_k\right|^p.$$

In particular, for $n = 0$, the Hilbert inequality occurs and for $n = 1$ we have the inequality

$$\|H^1x\|_{\ell_p} \le \pi/p\csc(\pi/p)\|C^tx\|_{\ell_p},$$

or

$$\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+2}\right|^p \le \left(\pi/p\csc(\pi/p)\right)^p\sum_{j=0}^{\infty}\left|\sum_{k=j}^{\infty}\frac{x_k}{1+k}\right|^p.$$

Proof According to Lemma 2.3, $H = B^n C^n$, hence

$$\|Hx\|_{\ell_p} = \|B^n C^n x\|_{\ell_p} \leq \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)} \|C^n x\|_{\ell_p}.$$

Consider that, for $n = 0$, $C^0 = I$ and we have the Hilbert inequality. Also by applying Lemma 2.3, $H^n = B^n C^n$, hence we have

$$\|H^n x\|_{\ell_p} = \|B^{nt} C^{nt} x\|_{\ell_p} \leq \frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)} \|C^{nt} x\|_{\ell_p}.$$

Consider that, for $n = 0$, $C^0 = I$ and we have the Hilbert inequality. □

Let H_p^n be the sequence space associated with the Hilbert matrix H^n, which is

$$H_p^n = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{x_k}{j+k+n+1} \right|^p < \infty \right\}$$

and has the norm

$$\|x\|_{H_p^n} = \left(\sum_{j=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{x_k}{j+k+n+1} \right|^p \right)^{\frac{1}{p}}.$$

As another application of Lemma 2.3, we have the following inclusions.

Corollary 2.7 *Let $p > 1$. Then*

(i) $C_p^n \subset H_p$,
(ii) $C_p^{nt} \subset H_p^n$,
(iii) $H_p^n \subset H_p^{n-1} \subset \cdots \subset H_p$.

2.2 Norm of Hilbert operator on the generalized Cesàro matrix domain

Suppose that $N \geq 1$ is a real number. The generalized Cesàro matrix, $C^N = (c_{j,k}^N)$,

$$c_{j,k}^N = \begin{cases} \frac{1}{j+N} & 0 \leq k \leq j, \\ 0 & \text{otherwise}, \end{cases}$$

has the ℓ_p-norm $\|C^N\|_{\ell_p} = p^*$ ([3], Lemma 2.3), and the entries

$$c_{j,k}^N = \begin{pmatrix} \frac{1}{N} & 0 & 0 & \cdots \\ \frac{1}{1+N} & \frac{1}{1+N} & 0 & \cdots \\ \frac{1}{2+N} & \frac{1}{2+N} & \frac{1}{2+N} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that C^1 is the well-known Cesàro matrix C. Some more examples are

$$C^2 = \begin{pmatrix} 1/2 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 0 & \cdots \\ 1/4 & 1/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} 1/3 & 0 & 0 & \cdots \\ 1/4 & 1/4 & 0 & \cdots \\ 1/5 & 1/5 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix domain associated with this matrix is the set

$$C_p^N = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=0}^{j} \frac{x_k}{j+N} \right|^p < \infty \right\},$$

which has the following norm:

$$\|x\|_{C_p^N} = \left(\sum_{j=0}^{\infty} \left| \sum_{k=0}^{j} \frac{x_k}{j+N} \right|^p \right)^{\frac{1}{p}}.$$

Theorem 2.8 *The Hilbert operator H is a bounded operator from C_p^N into ℓ_p and*

$$\|H\|_{C_p^N, \ell_p} \le \frac{\pi N}{p^*} \csc(\pi/p).$$

In particular, the Hilbert operator H is a bounded operator from C_p into ℓ_p and $\|H\|_{C_p,\ell_p} = \frac{\pi}{p^*} \csc(\pi/p)$.

Proof The author in [18] has proved that the Hilbert matrix admits a factorization of the form $H = B^N C^N$, where B^N is a bounded operator on ℓ_p and

$$\frac{\pi}{p^*} \csc(\pi/p) \le \left\| B^N \right\|_{\ell_p} \le \frac{\pi N}{p^*} \csc(\pi/p).$$

Now, according to Lemma 2.1 we have the result. □

2.3 Norm of Hilbert operator on Gamma sequence spaces

By letting $d\mu(\theta) = n\theta^{n-1}\,d\theta$ in the definition of the Hausdorff matrix, the Gamma matrix of order n, $\Gamma^n = (\gamma_{j,k}^n)$, is

$$\gamma_{j,k}^n = \begin{cases} \frac{\binom{n+k-1}{k}}{\binom{n+j}{j}} & 0 \le k \le j, \\ 0 & \text{otherwise,} \end{cases}$$

which according to the Hardy formula has the norm

$$\left\| \Gamma^n \right\|_{\ell_p} = \frac{np}{np-1}. \tag{2.6}$$

Note that Γ^1 is the well-known Cesàro matrix. Some more examples are

$$\Gamma^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/3 & 2/3 & 0 & \cdots \\ 1/6 & 2/6 & 3/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \Gamma^3 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/4 & 3/4 & 0 & \cdots \\ 1/10 & 3/10 & 6/10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The sequence space associated with Γ^n, is the set

$$\Gamma_p^n = \{x = (x_k) \in \omega : \Gamma^n x \in \ell_p\} = \left\{x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right|^p < \infty \right\},$$

which is called the Gamma space of order n.

Theorem 2.9 *The Hilbert operator H is a bounded operator from Γ_p^n into ℓ_p and*

$$\|H\|_{\Gamma_p^n,\ell_p} = \pi\left(1 - \frac{1}{np}\right)\csc(\pi/p).$$

In particular, the Hilbert operator H is a bounded operator from C_p into ℓ_p and $\|H\|_{C_p,\ell_p} = \frac{\pi}{p^*}\csc(\pi/p)$.

Proof The author in [19] has proved that the Hilbert matrix has a factorization of the form $H = S^n\Gamma^n$, where the matrix S^n has the ℓ_p-norm

$$\|S^n\|_{\ell_p} = \pi\left(1 - \frac{1}{np}\right)\csc(\pi/p).$$

Now, by applying part (ii) of Lemma 2.1 the proof is obvious. □

2.4 Norm of Hilbert operator on difference sequence spaces

Let $n \in \mathbb{N}$ and $\Delta^{n_F} = (\delta_{j,k}^{n_F})$ be the forward difference operator of order n with entries

$$\delta_{j,k}^{n_F} = \begin{cases} (-1)^{k-j}\binom{n}{k-j} & j \le k \le n+j, \\ 0 & \text{otherwise.} \end{cases}$$

We define the sequence space $\ell_p(\Delta^{n_F})$ as the set $\{x = (x_k) : \Delta^{n_F}x \in \ell_p\}$ or

$$\ell_p(\Delta^{n_F}) = \left\{x = (x_k) : \sum_{j=0}^{\infty} \left| \sum_{k=0}^{n} (-1)^k \binom{n}{k} x_{k+j} \right|^p < \infty \right\},$$

with semi-norm, $\|\cdot\|_{\ell_p(\Delta^{n_F})}$, which is defined by

$$\|x\|_{\ell_p(\Delta^{n_F})} = \left(\sum_{j=0}^{\infty} \left| \sum_{k=0}^{n} (-1)^k \binom{n}{k} x_{k+j} \right|^p \right)^{\frac{1}{p}}.$$

Note that this function will not be a norm, since if $x = (1,1,1,\ldots)$ then $\|x\|_{\ell_p(\Delta^{n_F})} = 0$, while $x \neq 0$. The definition of backward difference sequence space $\ell_p(\Delta^{n_B})$ is similar to $\ell_p(\Delta^{n_F})$, except $\|\cdot\|_{\ell_p(\Delta^{n_B})}$ is a norm.

Theorem 2.10 *The Hilbert matrix H, is a bounded operator from $\ell_p(\Delta^{n_B})$ into $\ell_p(\Delta^{n_F})$ and*

$$\|H\|_{\ell_p(\Delta^{n_B}),\ell_p(\Delta^{n_F})} = \frac{\pi}{\sin(\pi/p)}.$$

Proof Let $A = \Delta^{n_F} H$. By using the identity $\sum_{j=0}^{n}(-1)^j\binom{n}{j}z^j = (1-z)^n$, we have

$$\begin{aligned}
a_{i,k} &= \sum_{j=i}^{n+i} \delta_{i,j}^{n_F} h_{j,k} = \sum_{j=i}^{n+i}(-1)^{j-i}\binom{n}{j-i}\frac{1}{j+k+1} \\
&= \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{j+i+k+1} = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\int_0^1 z^{j+i+k}\,dz \\
&= \int_0^1 \sum_{j=0}^{n}(-1)^j\binom{n}{j} z^j z^{i+k}\,dz = \int_0^1 z^{i+k}(1-z)^n\,dz \\
&= \beta(i+k+1, n+1).
\end{aligned}$$

Obviously, A is a symmetric matrix which implies that $\Delta^{n_F} H = H\Delta^{n_B}$. Now by using Lemma 2.1 part (ii), $\|H\|_{\ell_p(\Delta^{n_B}),\ell_p(\Delta^{n_F})} = \|H\|_{\ell_p} = \pi\csc(\pi/p)$. □

Acknowledgements
The author would like to express his gratitude to the referee for his/her corrections and improvements to the results.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The author declares that they have no competing interests.

Authors' contributions
This manuscript has been written by only one author. The author read and approved the current manuscript.

Publisher's Note

Received: 24 November 2019 Accepted: 16 April 2020 Published online: 29 April 2020

References

1. Beckermann, B.: The condition number of real Vandermonde, Krylov and positive definite Hankel matrices. Numer. Math. **85**, 553–577 (2000)
2. Bennett, G.: Factorizing the Classical Inequalities. Mem. Amer. Math. Soc., vol. 576 (1996)
3. Chen, C.P., Luor, D.C., Ou, Z.y.: Extensions of Hardy inequality. J. Math. Anal. Appl. **273**, 160–171 (2000)
4. Choi, M.D.: Tricks or treats with the Hilbert matrix. Am. Math. Mon. **90**(5), 301–312 (1983)
5. Foroutannia, D., Roopaei, H.: The norms and the lower bounds for matrix operators on weighted difference sequence spaces. UPB Sci. Bull., Ser. A **79**(2), 151–160 (2017)
6. Foroutannia, D., Roopaei, H.: Bounds for the norm of lower triangular matrices on the Cesàro weighted sequence space. J. Inequal. Appl. **2017**(67), 1 (2017)
7. Hardy, G.H.: Divergent Series. Oxford University Press, Cambridge (1973)
8. Hardy, G.H., Polya, G., Littlewood, J.E.: Inequalities, 2nd edn. Cambridge University Press, Cambridge (2001)

9. Hilbert, D.: Ein Beitrag zur Theorie des Legendre'schen Polynoms. Acta Math. **18**, 155–159 (1894)
10. İlkhan, M.: Norms and lower bounds of some matrix operators on Fibonacci weighted difference sequence space. Math. Methods Appl. Sci. **42**(16), 5143–5153 (2019)
11. İlkhan, M., Kara, E.E.: A new Banach space defined by Euler totient matrix operator. Oper. Matrices **13**(2), 527–544 (2019)
12. İlkhan, M., Kara, E.E., Demiriz, S.: A new paranormed sequence space defined by Euler totient matrix. Karaelmas Fen Mühendis. Derg. **9**(2), 277–282 (2019)
13. Kara, E.E., Başarir, M.: On compact operators and some Euler $B^{(m)}$ difference sequence spaces. J. Math. Anal. Appl. **379**(2), 499–511 (2011)
14. Kara, E.E., İlkhan, M.: Some properties of generalized Fibonacci sequence spaces. Linear Multilinear Algebra **64**(11), 2208–2223 (2016)
15. Matsuo, Y.: Matrix theory, Hilbert scheme and integrable system. Mod. Phys. Lett. A **13**, 2731–2742 (1998). https://doi.org/10.1142/S0217732398002904
16. Raja, P.V.K., Chakravarthy, A.S.N., Avadhani, P.S.: A cryptosystem based on Hilbert matrix using cipher block chaining mode. Int. J. Math. Trends Technol. **2**(1), 17–22 (2011)
17. Roopaei, H.: Norms of summability and Hausdorff mean matrices on difference sequence spaces. Math. Inequal. Appl. **22**(3), 983–987 (2019)
18. Roopaei, H.: Factorization of Cesàro and Hilbert matrices based on generalized Cesàro matrix. Linear Multilinear Algebra **68**(1), 193–204 (2020)
19. Roopaei, H.: Factorization of the Hilbert matrix based on Cesàro and Gamma matrices. Results Math. **75**, 3 (2020). https://doi.org/10.1007/s00025-019-1129-1
20. Roopaei, H., Foroutannia, D.: The norm of matrix operators on Cesàro weighted sequence space. Linear Multilinear Algebra **67**(1), 175–185 (2019)
21. Roopaei, H., Foroutannia, D.: The norms of certain matrix operators from ℓ_p spaces into $\ell_p(\Delta^n)$ spaces. Linear Multilinear Algebra **67**(4), 767–776 (2019)
22. Todd, J.: The condition number of the finite segment of the Hilbert matrix. Natl. Bur. Stand., Appl. Math. Ser. **39**, 109–116 (1954)
23. Zengin Alp, P., İlkhan, M.: On the difference sequence space $\ell_p(\widehat{T})^q$. Math. Sci. Appl. E-Notes **7**(2), 161–173 (2019)