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Well-posedness for 3D nematic liquid crystal flows with damping

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Abstract
The three-dimensional nematic liquid crystal flows with damping are considered in this paper. The existence and uniqueness of strong solutions for the 3D nematic liquid crystal flows with damping are proved for $\beta \geq 4$ with any $\alpha > 0$.

Keywords: Liquid crystal flows; Well-posedness; Strong solution

1 Introduction

In this paper, we consider the following three-dimensional nematic liquid crystal flows with damping:

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \alpha|u|^{\beta-1}u + \nabla p = -\lambda\nabla\cdot(\nabla d\odot\nabla d), \\ \partial_t d + (u\cdot\nabla)d = \gamma(\Delta d - f(d)), \\ \nabla\cdot u = 0, \\ u|_{\partial D} = d|_{\partial D} = 0, \\ u(x,0) = u_0(x), \qquad d(x,0) = d_0(x). \end{cases} \tag{1}$$

Here, $x \in D \subseteq \mathbb{R}^3$ is a bounded domain with the boundary ∂D and $t > 0$. $u = u(x,t)$ is the velocity field of the flow, $d = d(x,t)$ represents the (averaged) macroscopic/continuum molecule orientation and p is the pressure. ν, λ, γ, α are positive constants, $\beta \geq 1$ and $f(d) = \frac{1}{\eta^2}(|d|^2 - 1)d$ $(\eta > 0)$. The 3×3 matrix is given by $(\nabla d\odot\nabla d)_{ij} = \partial_i d\cdot\partial_j d$ for $(1 \leq i,j \leq 3)$. For simplicity, we set $\nu = \gamma = \lambda = \eta = 1$.

Recently, the 3D nematic liquid crystal flows were proposed by Lin ([1, 2]) and have been extensively investigated. The damping term describes many physical situations such as drag or friction effects, porous media flow, some dissipative mechanisms. When $d = 0$, the problem (1) reduces to the three-dimensional Navier–Stokes equations with damping. In [3–6], the well-posedness of the three-dimensional Navier–Stokes equations with damping is proved for $\beta > 3$ with any $\alpha > 0$ and $\alpha \geq \frac{1}{4}$ as $\beta = 3$. The global existence of weak solutions of the 3D nematic liquid crystal flow was proved in [7]. In [8], the existence and uniqueness of strong solutions for the 3D magneto-micropolar equations were proved for $\beta \geq 4$ with any $\alpha > 0$.

This paper is organized as follows. In Sect. 2, we will prove the existence and uniqueness of strong solutions for the 3D nematic liquid crystal flows with damping for $\beta \geq 4$ with any $\alpha > 0$. Moreover, we get the following main result.

Theorem 1.1 *Let $(u_0, d_0) \in H^1(D) \times H^2(D)$ such that $\nabla \cdot u_0 = 0$, for $\beta \geq 4$ with any $\alpha > 0$. Then the problem (1) has a unique strong solution (u, d) satisfying for any given $T > 0$*

$$\begin{aligned} &u \in L^\infty\big(0,T;H^1\big) \cap L^2\big(0,T;H^2\big) \cap L^{\beta+1}\big(0,T;L^{\beta+1}(D)\big), \\ &d \in L^\infty\big(0,T;H^1\big) \cap L^\infty\big(0,T;H^2\big) \cap L^2\big(0,T;H^3\big), \\ &|u|^{\frac{\beta-1}{2}} \nabla u \in L^2\big(0,T;L^2(D)\big), \qquad \nabla |u|^{\frac{\beta+1}{2}} \in L^2\big(0,T;L^2(D)\big). \end{aligned}$$

2 Proof of Theorem 1.1

In this section, C represents a nonnegative constant whose value may be different from line to line. Multiplying the first equation of Eq. (1) by u and the second equation of (1) by $-\Delta d + f(d)$, integrating the result over D, and summing their results, then we have

$$\frac{1}{2}\frac{d}{dt}\int_D \big(|u|^2 + |\nabla d|^2 + 2F(d)\big)\,dx + \int_D \big(|\nabla u|^2 + \alpha |u|^{\beta+1} + \big|\Delta d - f(d)\big|^2\big)\,dx = 0, \tag{2}$$

here $f(d) = \nabla F(d)$, $((u \cdot \nabla)u, u) = (u, \nabla p) = ((u \cdot \nabla)d, f(d)) = (u, \nabla \frac{|\nabla d|^2}{2}) = 0$ and $\nabla \cdot (\nabla d \odot \nabla d) = \nabla(\frac{|\nabla d|^2}{2}) + \Delta d \cdot \nabla d$, i.e., $F(d) = \frac{|d|^4}{4} - \frac{|d|^2}{2}$. Then it is easy to get

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^{\beta+1}(0,T;L^{\beta+1})} + \|u\|_{L^2(0,T;H^1)} \leq C. \tag{3}$$

Multiplying the second equation of (1) by $|d|^2 d$, it is easy to get

$$\frac{1}{4}\frac{d}{dt}\int_D |d|^4\,dx + \frac{1}{2}\big\|\nabla |d|^2\big\|_{L^2}^2 + \int_D \big(|d|^2|\nabla d|^2 + |d|^6\big)\,dx = \int_D |d|^4\,dx. \tag{4}$$

Applying the Gronwall inequality, then we have

$$\big\|d(t)\big\|_{L^4}^4 + \int_0^t \big\|\nabla |d|^2\big\|_{L^2}^2\,ds + \int_0^t \int_D \big(|d|^2|\nabla d|^2 + |d|^6\big)\,dx\,ds \leq C(t, d_0). \tag{5}$$

Multiplying the second equation of (1) by $f(d)$, we deduce

$$\frac{d}{dt}\int_D F(d)\,dx = \int_D \big(\Delta d f(d) - \big|f(d)\big|^2\big)\,dx. \tag{6}$$

Adding (2)–(6) and using the Gronwall inequality and $f(d) = (|d|^2 - 1)d$, we have

$$\|d\|_{L^\infty(0,T;H^1)} + \|d\|_{L^2(0,T;H^2)} \leq C. \tag{7}$$

Multiplying the first equation of (1) by $-\Delta u$, it is easy to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \alpha \big\||u|^{\frac{\beta-1}{2}}\nabla u\big\|_{L^2}^2 + \frac{4\alpha(\beta-1)}{(\beta+1)^2}\big\|\nabla |u|^{\frac{\beta+1}{2}}\big\|_{L^2}^2 \\ &\quad = \int_D (u \cdot \nabla)u \cdot \Delta u\,dx + \int_D \nabla d \Delta d \Delta u\,dx. \end{aligned} \tag{8}$$

Taking Δ on the second equation of (1) and dotting with Δd, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta d\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2={}&-\int_D\Delta f(d)\Delta d\,dx-\int_D\Delta u\nabla d\Delta d\,dx\\&-2\sum_{i,k=1}^{3}\int_D\nabla u_i\partial_i\nabla d_k\Delta d_k\,dx.\end{aligned}\tag{9}$$

Adding (8) and (9), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)+\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2+\alpha\big\||u|^{\frac{\beta-1}{2}}\nabla u\big\|_{L^2}^2\\&\quad+\frac{4\alpha(\beta-1)}{(\beta+1)^2}\big\|\nabla|u|^{\frac{\beta+1}{2}}\big\|_{L^2}^2\\&=\int_D(u\cdot\nabla)u\cdot\Delta u\,dx-\int_D\Delta f(d)\Delta d\,dx-2\sum_{i,k=1}^{3}\int_D\nabla u_i\partial_i\nabla d_k\Delta d_k\,dx\\&=\sum_{i=1}^{3}I_i(t).\end{aligned}\tag{10}$$

For $I_1(t)$, using the Young inequality and the Hölder inequality, it is easy to get, for any $\beta>3$,

$$\begin{aligned}\big|I_1(t)\big|&\le\frac{1}{4}\|\Delta u\|_{L^2}^2+C\int_D|u|^2|\nabla u|^{\frac{4}{\beta-1}}|\nabla u|^{2-\frac{4}{\beta-1}}\,dx\\&\le\frac{1}{4}\|\Delta u\|_{L^2}^2+\frac{\alpha}{2}\int_D|u|^{\beta-1}|\nabla u|^2\,dx+C\|\nabla u\|_{L^2}^2.\end{aligned}\tag{11}$$

Inspired by [3–5] and exists $\theta>0$, we get $1-\frac{1}{2\theta}\ge 0$ and $\alpha-\frac{\theta}{2}\ge 0$. Then we get the above estimate easily for $\alpha\ge\frac{1}{4}$ as $\beta=3$.

For $I_2(t)$, integrating by parts, applying the Hölder inequality and the Young inequality, we get

$$\begin{aligned}I_2(t)&=\sum_{i=1}^{3}\int_D\partial_i\big(|d|^2d\big)\partial_i\Delta d\,dx-\sum_{i=1}^{3}\int_D\partial_i d\partial_i\Delta d\,dx\\&\le C\big(\|\nabla d\|_{L^6}\|\nabla\Delta d\|_{L^2}\|d\|_{L^6}^2+\|\nabla d\|_{L^2}\|\nabla\Delta d\|_{L^2}\big)\\&\le\frac{1}{4}\|\nabla\Delta d\|_{L^2}^2+C\big(\|\nabla d\|_{L^2}^2+\|\nabla d\|_{L^2}^4\|\Delta d\|_{L^2}^2\big)\\&\le\frac{1}{4}\|\nabla\Delta d\|_{L^2}^2+C\|\Delta d\|_{L^2}^2+C.\end{aligned}\tag{12}$$

For $I_3(t)$, since $\frac{2}{\beta-2}\le 1$ for $\beta\ge 4$, by using the Hölder, Gagliardo-Nirenberg and Young inequalities, we get

$$\begin{aligned}I_3(t)&=2\sum_{i=1}^{3}\bigg(\int_D u_i\partial_i\nabla d\nabla\Delta d\,dx+\int_D u_i\partial_i\nabla^2 d\Delta d\,dx\bigg)\\&\le C\|u\|_{L^{\beta+1}}\|\Delta d\|_{L^{\frac{2(\beta+1)}{\beta-1}}}\|\nabla\Delta d\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&\le C\|u\|_{L^{\beta+1}}\|\Delta d\|_{L^2}^{\frac{\beta-2}{\beta+1}}\|\nabla\Delta d\|_{L^2}^{\frac{\beta+4}{\beta+1}}\\
&\le \frac{1}{4}\|\nabla\Delta d\|_{L^2}^2 + C\|u\|_{L^{\beta+1}}^{\frac{2(\beta+1)}{\beta-2}}\|\Delta d\|_{L^2}^2\\
&\le \frac{1}{4}\|\nabla\Delta d\|_{L^2}^2 + C\bigl(1+\|u\|_{L^{\beta+1}}^{\beta+1}\bigr)\|\Delta d\|_{L^2}^2. \qquad (13)
\end{aligned}
$$

Adding (10)–(13), it is easy to get

$$
\begin{aligned}
&\frac{d}{dt}\bigl(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\bigr)+\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2+\bigl\||u|^{\frac{\beta-1}{2}}\nabla u\bigr\|_{L^2}^2+\bigl\|\nabla|u|^{\frac{\beta+1}{2}}\bigr\|_{L^2}^2\\
&\quad\le C\bigl(1+\|u\|_{L^{\beta+1}}^{\beta+1}\bigr)\bigl(\|\Delta d\|_{L^2}^2+\|\nabla u\|_{L^2}^2\bigr)+C. \qquad (14)
\end{aligned}
$$

Applying the Gronwall inequality and (3), we have

$$
\begin{aligned}
&\bigl\|\nabla u(t)\bigr\|_{L^2}^2+\bigl\|\Delta d(t)\bigr\|_{L^2}^2\\
&\qquad+\int_0^t\bigl(\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2+\bigl\||u|^{\frac{\beta-1}{2}}\nabla u\bigr\|_{L^2}^2+\bigl\|\nabla|u|^{\frac{\beta+1}{2}}\bigr\|_{L^2}^2\bigr)\,ds\\
&\quad\le C(t,u_0,d_0). \qquad (15)
\end{aligned}
$$

Next, we will prove the uniqueness of the strong solutions of the problem (1). Let (u,d) and $(\bar{u},\bar{d})$ be the two solutions for the problem (1) with the same u_0, d_0. Assume that $(\hat{u},\hat{d})=(\bar{u}-u,\bar{d}-d)$. Then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigl(\|\hat{u}\|_{L^2}^2+\|\nabla\hat{d}\|_{L^2}^2\bigr)+\|\nabla\hat{u}\|_{L^2}^2+\|\Delta\hat{d}\|_{L^2}^2+\alpha\int_D\bigl(|\bar{u}|^{\beta-1}\bar{u}-|u|^{\beta-1}u\bigr)(\bar{u}-u)\,dx\\
&\quad\le\int_D|\hat{u}|^2|\nabla\bar{u}|\,dx+\int_D|\hat{u}||\nabla\hat{d}||\Delta\bar{d}|\,dx\\
&\qquad+\int_D|\bar{u}||\nabla\hat{d}||\Delta\hat{d}|\,dx+\int_D\bigl|f(\bar{d})-f(d)\bigr||\Delta\hat{d}|\,dx\\
&\quad=\sum_{i=1}^4 J_i(t). \qquad (16)
\end{aligned}
$$

Since $g(u)=\alpha|u|^{\beta-1}u$ is a monotonic function in D, it is easy to get

$$
\alpha\int_D\bigl(|\bar{u}|^{\beta-1}\bar{u}-|u|^{\beta-1}u\bigr)(\bar{u}-u)\,dx\ge 0. \qquad (17)
$$

For $J_1(t)$, using the Gagliardo–Nirenberg and Young inequalities, we have

$$
\begin{aligned}
J_1(t)&\le C\|\hat{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\hat{u}\|_{L^2}^{\frac{3}{2}}\|\nabla\bar{u}\|_{L^2}\\
&\le\frac{1}{4}\|\nabla\hat{u}\|_{L^2}^2+C\|\nabla\bar{u}\|_{L^2}^4\|\hat{u}\|_{L^2}^2. \qquad (18)
\end{aligned}
$$

For $J_2(t)$ and $J_3(t)$, similarly, we also get

$$\begin{aligned} J_2(t) &\leq C\|\hat{u}\|_{L^4}\|\nabla\hat{d}\|_{L^4}\|\Delta\bar{d}\|_{L^2} \\ &\leq C\|\hat{u}\|_{L^2}^{\frac{1}{4}}\|\nabla\hat{u}\|_{L^2}^{\frac{3}{4}}\|\nabla\hat{d}\|_{L^2}^{\frac{1}{4}}\|\Delta\hat{d}\|_{L^2}^{\frac{3}{4}}\|\Delta\bar{d}\|_{L^2} \\ &\leq \frac{1}{8}\|\nabla\hat{u}\|_{L^2}^2 + \frac{1}{8}\|\Delta\hat{d}\|_{L^2}^2 + C\|\Delta\bar{d}\|_{L^2}^4\big(\|\hat{u}\|_{L^2}^2 + \|\nabla\hat{d}\|_{L^2}^2\big) \end{aligned} \tag{19}$$

and

$$\begin{aligned} J_3(t) &\leq C\|\bar{u}\|_{L^6}\|\nabla\hat{d}\|_{L^2}^{\frac{1}{2}}\|\Delta\hat{d}\|_{L^2}^{\frac{3}{2}} \\ &\leq \frac{1}{8}\|\Delta\hat{d}\|_{L^2}^2 + C\|\nabla\bar{u}\|_{L^2}^4\|\nabla\hat{d}\|_{L^2}^2. \end{aligned} \tag{20}$$

For $J_4(t)$, applying the Hölder inequality, we get

$$\begin{aligned} J_4(t) &\leq \big\|f(\bar{d}) - f(d)\big\|_{L^2}\|\Delta\hat{d}\|_{L^2} \\ &\leq \frac{1}{4}\|\Delta\hat{d}\|_{L^2}^2 + C\big(1 + \|\nabla\bar{d}\|_{L^2}^4 + \|\nabla d\|_{L^2}^4\big)\|\nabla\hat{d}\|_{L^2}^2. \end{aligned} \tag{21}$$

Summing (16)–(21), we have

$$\begin{aligned} &\frac{d}{dt}\big(\|\hat{u}\|_{L^2}^2 + \|\nabla\hat{d}\|_{L^2}^2\big) + \|\nabla\hat{u}\|_{L^2}^2 + \|\Delta\hat{d}\|_{L^2}^2 \\ &\quad\leq C\big(1 + \|\nabla\bar{u}\|_{L^2}^4 + \|\nabla\bar{d}\|_{L^2}^4 + \|\nabla d\|_{L^2}^4 + \|\Delta\bar{d}\|_{L^2}^4\big)\big(\|\hat{u}\|_{L^2}^2 + \|\nabla\hat{d}\|_{L^2}^2\big). \end{aligned} \tag{22}$$

Applying the Gronwall inequality and (7) and (15), then we have

$$\big\|\hat{u}(t)\big\|_{L^2}^2 + \big\|\nabla\hat{d}(t)\big\|_{L^2}^2 \leq \big(\big\|\hat{u}(0)\big\|_{L^2}^2 + \big\|\nabla\hat{d}(0)\big\|_{L^2}^2\big)e^{C\int_0^t H(s)\,ds}, \tag{23}$$

where, $H(s) = 1 + \|\nabla\bar{u}(s)\|_{L^2}^4 + \|\nabla\bar{d}(s)\|_{L^2}^4 + \|\nabla d(s)\|_{L^2}^4 + \|\Delta\bar{d}(s)\|_{L^2}^4$. The uniqueness of the strong solutions of the problem (1) is proved. This completes the proof of Theorem 1.1.

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Authors' contributions
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