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A note on degenerate Genocchi and poly-Genocchi numbers and polynomials

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Abstract

Recently, Dolgy–Jang introduced the poly-Genocchi polynomials and numbers arising from the modified polyexponential function. In this paper, we study the degenerate poly-Genocchi polynomials and numbers constructed from the modified degenerate polyexponential function. We derive explicit expressions for those polynomials and numbers. Also, we obtain identities involving those polynomials and numbers and some other special numbers and polynomials. In addition, we investigate the higher-order degenerate Genocchi polynomials and find identities involving those polynomials and the higher-order Changhee polynomials.

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1 Introduction

In [2], Carlitz initiated a study of degenerate versions of some special polynomials and numbers, namely the degenerate Bernoulli and Euler polynomials and numbers. We have witnessed in recent years that much research has been done for various degenerate versions of many special polynomials and numbers. These include the degenerate Stirling numbers of the first and second kinds, degenerate Bernoulli numbers of the second kind, degenerate Bell numbers and polynomials, degenerate complete Bell polynomials and numbers, degenerate central factorial numbers of the second kind, degenerate Bernstein polynomials, degenerate random variables, and so on (see [6, 7, 11–15, 17–19, 21, 22, 25, 26] and the references therein).

The aim of this paper is to introduce a degenerate version of the poly-Genocchi polynomials and numbers, so-called degenerate poly-Genocchi polynomials and numbers, constructing from the modified degenerate polyexponential function. We derive some explicit expressions and identities for those numbers and polynomials.

This paper is organized as follows. In Sect. 1, we recall some necessary stuffs that are needed throughout this paper. These include the degenerate exponential functions, the degenerate Genocchi polynomials, the degenerate Euler polynomials, and the degenerate Stirling numbers of the first and second kinds. In Sect. 2, we introduce the degenerate

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poly-Genocchi polynomials by making use of the modified degenerate polyexponential function. We express those polynomials in terms of the degenerate Genocchi polynomials and the degenerate Stirling numbers of the first kind and also of the degenerate Euler polynomials and the Stirling numbers of the first kind. We represent the generating function of the degenerate poly-Genocchi numbers by iterated integrals from which we obtain an expression of those numbers in terms of the degenerate Bernoulli numbers of the second kind. We show that a zeta function connected with the degenerate poly-Genocchi numbers and originally defined for $\mathrm{Re}(s) > 0$ can be continued to an entire function. Also, we determine its special values at nonpositive integers. We obtain identities relating the degenerate Genocchi numbers and Changhee numbers. In addition, we get identities involving the higher-order degenerate Genocchi polynomials and the higher-order Changhee polynomials. Finally, we conclude this paper in Sect. 3.

In 1905, Hardy considered the polyexponential function given by

$$e(x,a|s) = \sum_{n=0}^{\infty} \frac{x^n}{(n+a)^s n!}, \quad \bigl(\mathrm{Re}(a) > 0\bigr) \text{ (see [14])}. \tag{1}$$

Recently, Kim–Kim introduced the modified polyexponential function as

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k} \quad (k \in \mathbb{Z}) \text{ (see [9])}. \tag{2}$$

As was noted in [14], it is immediate to see from (1) and (2) that we have

$$xe(x,1|k) = Ei_k(x).$$

The Euler polynomials are defined by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad \text{(see [1–33])}. \tag{3}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

As is well known, the Genocchi polynomials of order α are defined by

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \widehat{G}_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \text{(see [1, 4, 5, 18, 33])}. \tag{4}$$

When $\alpha = 1$, $G_n = \widehat{G}_n^{(1)}(0)$ are called the Genocchi numbers.

The reader may refer to [31] as an important work focusing on the Genocchi polynomials and their relations with different areas.

As was observed in [7], it is easy to see from (3) and (4) that we have

$$\frac{G_{n+1}}{n+1} = E_n \quad (n \ge 0), \ G_0 = 0. \tag{5}$$

In [3], Dolgy–Jang introduced poly-Genocchi polynomials arising from polyexponential function as

$$\frac{2\mathrm{Ei}_k(\log(1+t))}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}. \tag{6}$$

When $x = 0$, $G_n^{(k)} = G_n^{(k)}(0)$ are called the poly-Genocchi numbers. Note that $G_n(x) = G_n^{(1)}(x) = \widehat{G}_n^{(1)}(x)$, $(n \geq 0)$, are the Genocchi polynomials.

In [2], Carlitz considered the degenerate Bernoulli polynomials given by

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{7}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

Note that $\lim_{\lambda\to 0}\beta_{n,\lambda}(x) = B_n(x)$ $(n \geq 0)$, where $B_n(x)$ are the ordinary Bernoulli polynomials given by

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{(see [1–33])}. \tag{8}$$

The degenerate exponential functions are defined as

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \qquad e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}} \quad \text{(see [11, 15, 16])}. \tag{9}$$

Here we note that

$$e_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} \quad \text{(see [16])}, \tag{10}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$ $(n \geq 1)$.

Let $\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1)$ be the compositional inverse function of $e_\lambda(t)$ such that

$$e_\lambda\big(\log_\lambda(t)\big) = \log_\lambda\big(e_\lambda(t)\big) = t.$$

In [26], Lim considered the degenerate Genocchi polynomials of order $r(r \in \mathbb{N})$, which are given by

$$\left(\frac{2t}{e_\lambda(t)+1}\right)^r e_\lambda^x(t) = \sum_{n=0}^{\infty}\widehat{G}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{11}$$

Note here that $\widehat{G}_{0,\lambda}^{(r)}(x) = \widehat{G}_{1,\lambda}^{(r)}(x) = \cdots = \widehat{G}_{r-1,\lambda}^{(r)}(x) = 0$. When $x = 0$, $\widehat{G}_{n,\lambda}^{(r)} = \widehat{G}_{n,\lambda}^{(r)}(0)$ are called the degenerate Genocchi numbers of order r.

In particular, for $r = 1$, the degenerate Genocchi polynomials are given by

$$\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty}\widehat{G}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{12}$$

When $x = 0$, $\widehat{G}_{n,\lambda} = \widehat{G}_{n,\lambda}(0)$ are called the degenerate Genocchi numbers.

In [2], Carlitz introduced the degenerate Euler polynomials given by

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{13}$$

When $x=0$, $\mathcal{E}_{n,\lambda}=\mathcal{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers.

As was noted in [7], it is immediate to see from (12) and (13) that we have

$$\widehat{G}_{0,\lambda}(x)=0,\mathcal{E}_{n,\lambda}(x)=\frac{\widehat{G}_{n+1,\lambda}(x)}{n+1}\quad (n\geq 0). \tag{14}$$

In [14], the degenerate polyexponential functions are defined by

$$e_\lambda(x,\delta|k)=\sum_{n=0}^{\infty}\frac{(1)_{n,\lambda}}{n!(n+\delta)^k}x^n, \tag{15}$$

where $k\in\mathbb{N}\cup\{0\}$, and $\delta\in\mathbb{C}$, with $\operatorname{Re}(\delta)>0$.

The degenerate Stirling numbers of the first kind are defined by (see [17])

$$\frac{1}{k!}\bigl(\log_\lambda(1+t)\bigr)^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}\quad (k\geq 0). \tag{16}$$

Note here that $\lim_{\lambda\to 0}S_{1,\lambda}(n,l)=S_1(n,l)$, where $S_1(n,l)$ are the Stirling numbers of the first kind given by

$$\frac{1}{k!}\bigl(\log(1+t)\bigr)^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!}\quad (k\geq 0)\ (\text{see [1–33]}).$$

The degenerate Stirling numbers of the second kind are given by (see [11])

$$\frac{1}{k!}\bigl(e_\lambda(t)-1\bigr)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}\quad (k\geq 0). \tag{17}$$

Observe here that $\lim_{\lambda\to 0}S_{2,\lambda}(n,l)=S_2(n,l)$, where $S_2(n,l)$ are the Stirling numbers of the second kind given by

$$\frac{1}{k!}\bigl(e^t-1\bigr)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}\quad (k\geq 0)\ (\text{see [1–33]}).$$

Roman [30] defined the Bernoulli polynomials of the second kind by

$$\frac{t}{\log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!}. \tag{18}$$

It is well known that $b_n(x)=B_n^{(n)}(x+1),(n\geq 0)$, where $B_n^{(r)}(x)$ are the Bernoulli polynomials of order r defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}.$$

In [12], the degenerate Bernoulli polynomials of the second kind are defined by

$$\frac{t}{\log_{\lambda}(1+t)}(1+t)^{x} = \sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^{n}}{n!}. \tag{19}$$

Note that $\lim_{\lambda\to 0} b_{n,\lambda}(x) = b_{n}(x), (n \geq 0)$.

As is well known, the Changhee polynomials of order r are defined by

$$\left(\frac{2}{2+t}\right)^{r}(1+t)^{x} = \sum_{n=0}^{\infty} \mathrm{Ch}_{n}^{(r)}(x)\frac{t^{n}}{n!} \quad \text{(see [7, 10, 28])}. \tag{20}$$

When $x = 0$, $\mathrm{Ch}_{n}^{(r)} = \mathrm{Ch}_{n}^{(r)}(0)$ are called the Changhee numbers of order r.

In particular, for $r = 1$, the Changhee numbers are given by

$$\frac{2}{t+2} = \sum_{n=0}^{\infty} \mathrm{Ch}_{n}\frac{t^{n}}{n!}. \tag{21}$$

2 Degenerate poly-Genocchi numbers and polynomials

In this section, we consider the modified degenerate polyexponential function given by

$$\mathrm{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{(n-1)!n^{k}}x^{n} \quad (\lambda \in \mathbb{R}). \tag{22}$$

Note that $\mathrm{Ei}_{1,\lambda}(x) = \sum_{n=1}^{\infty}(1)_{n,\lambda}\frac{x^{n}}{n!} = e_{\lambda}(x) - 1$.

In view of (7), we consider the degenerate poly-Genocchi polynomials given by

$$\frac{2\mathrm{Ei}_{k,\lambda}(\log_{\lambda}(1+t))}{e_{\lambda}(t)+1}e_{\lambda}^{x}(t) = \sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x)\frac{t^{n}}{n!}. \tag{23}$$

When $x = 0$, $G_{n,\lambda}^{(k)} = G_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Genocchi numbers.

From (6) and (23), we note that

$$\lim_{\lambda\to 0} G_{n,\lambda}^{(k)}(x) = G_{n,}^{(k)}(x) \quad (n \geq 0).$$

By (22) and (16), we get

$$\begin{aligned}
\mathrm{Ei}_{k,\lambda}\bigl(\log_{\lambda}(1+t)\bigr) &= \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}(\log_{\lambda}(1+t))^{n}}{(n-1)!n^{k}} \\
&= \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{n^{k-1}}\frac{1}{n!}\bigl(\log_{\lambda}(1+t)\bigr)^{n} \\
&= \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{n^{k-1}}\sum_{m=n}^{\infty} S_{1,\lambda}(m,n)\frac{t^{m}}{m!} \\
&= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{m} \frac{(1)_{n,\lambda}}{n^{k-1}}S_{1,\lambda}(m,n)\right)\frac{t^{m}}{m!}.
\end{aligned} \tag{24}$$

Thus, by (24), we have

$$\begin{aligned}\frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)+1}e_\lambda^x(t) &= \frac{2t}{e_\lambda(t)+1}e_\lambda^x(t)\frac{1}{t}\mathrm{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr)\\ &= \sum_{l=0}^\infty G_{l,\lambda}(x)\frac{t^l}{l!}\sum_{m=0}^\infty \frac{1}{m+1}\sum_{j=1}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)\frac{t^m}{m!}\\ &= \sum_{n=0}^\infty\left(\sum_{m=0}^n \frac{\binom{n}{m}}{m+1}\sum_{j=1}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)G_{n-m,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned}\tag{25}$$

Therefore, by (23) and (25), we obtain the following theorem.

Theorem 1 *For $n\geq 0$, we have*

$$G_{n,\lambda}^{(k)}(x)=\sum_{m=0}^n\frac{\binom{n}{m}}{m+1}\sum_{j=1}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)G_{n-m,\lambda}(x).\tag{26}$$

From (13), we note that

$$\begin{aligned}\frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)+1}e_\lambda^x(t) &= \frac{2}{e_\lambda(t)+1}e_\lambda^x(t)\mathrm{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr)\\ &= \sum_{l=0}^\infty \mathcal{E}_{l,\lambda}(x)\frac{t^l}{l!}\sum_{m=1}^\infty\sum_{j=1}^{m}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m,j)\frac{t^m}{m!}\\ &= \sum_{n=1}^\infty\left(\sum_{m=1}^n \binom{n}{m}\sum_{j=1}^{m}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m,j)\mathcal{E}_{n-m,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned}\tag{27}$$

Therefore, by (23) and (27), we obtain the following theorem.

Theorem 2 *$G_{0,\lambda}^{(k)}(x)=0$, and, for $n\in\mathbb{N}$, we have*

$$G_{n,\lambda}^{(k)}(x)=\sum_{m=1}^n\binom{n}{m}\sum_{j=1}^{m}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m,j)\mathcal{E}_{n-m,\lambda}(x).\tag{28}$$

From (22), we note that

$$\frac{d}{dx}\mathrm{Ei}_{k,\lambda}(x)=\frac{d}{dx}\sum_{n=1}^\infty\frac{(1)_{n,\lambda}x^n}{(n-1)!n^k}=\frac{1}{x}\sum_{n=1}^\infty\frac{(1)_{n,\lambda}x^n}{(n-1)!n^{k-1}}=\frac{1}{x}\mathrm{Ei}_{k-1,\lambda}(x).\tag{29}$$

Thus, by (29), we get

$$\begin{aligned}\mathrm{Ei}_{k,\lambda}(x) &= \int_0^x\frac{1}{t}\mathrm{Ei}_{k-1,\lambda}(t)\,dt\\ &= \int_0^x\underbrace{\frac{1}{t}\int_0^t\cdots\frac{1}{t}\int_0^t}_{(k-2)\text{-times}}\frac{1}{t}\mathrm{Ei}_{1,\lambda}(t)\,dt\,dt\cdots dt\end{aligned}$$

$$= \int_0^x \underbrace{\frac{1}{t}\int_0^t \cdots \frac{1}{t}\int_0^t}_{(k-2)\text{-times}} \frac{e_\lambda(t)-1}{t}\,dt\,dt\cdots dt, \tag{30}$$

where k is a positive integer with $k \geq 2$.

From (23) and (30), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)} \frac{t^n}{n!} &= \frac{2}{e_\lambda(t)+1}\mathrm{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr) \\ &= \frac{2}{e_\lambda(t)+1}\int_0^t \underbrace{\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t \cdots \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}}_{(k-2)\text{-times}}\int_0^t \frac{t(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\,dt\cdots dt, \end{aligned}\tag{31}$$

where k is a positive integer with $k \geq 2$.

Thus, by (31), we get

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,\lambda}^{(2)} \frac{t^n}{n!} &= \frac{2}{e_\lambda(t)+1}\int_0^t \frac{t}{\log_\lambda(1+t)}(1+t)^{\lambda-1}\,dt \\ &= \frac{2t}{e_\lambda(t)+1}\sum_{m=0}^{\infty}\frac{b_{m,\lambda}(\lambda-1)}{m+1}\frac{t^m}{m!} \\ &= \sum_{l=0}^{\infty} G_{l,\lambda}\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{b_{m,\lambda}(\lambda-1)}{m+1}\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}G_{l,\lambda}\frac{b_{n-l,\lambda}(\lambda-1)}{n-l+1}\right)\frac{t^n}{n!}. \end{aligned}\tag{32}$$

Therefore, by comparing the coefficients on both sides of (32), we obtain the following theorem.

Theorem 3 $G_{0,\lambda}^{(2)} = 0$, *and*, *for* $n \geq 1$, *we have*

$$G_{n,\lambda}^{(2)} = \sum_{l=0}^{n}\binom{n}{l}G_{l,\lambda}\frac{b_{n-l,\lambda}(\lambda-1)}{n-l+1}. \tag{33}$$

Indeed, by (31), we get the following theorem.

Theorem 4 *We have the following identity*:

$$\begin{aligned} G_{n,\lambda}^{(k)} &= \sum_{l=0}^{n}\binom{n}{l}G_{n-l,\lambda} \\ &\quad\times \sum_{l_1+\cdots+l_{k-1}=l}\binom{l}{l_1,\ldots,l_{k-1}}\frac{b_{l_1}(\lambda-1)}{l_1+1}\frac{b_{l_2}(\lambda-1)}{l_1+l_2+1}\cdots\frac{b_{l_{k-1}}(\lambda-1)}{l_1+\cdots+l_{k-1}+1}, \end{aligned}\tag{34}$$

where n *is a positive integer.*

Let $k \geq 1$ be an integer. For $s \in \mathbb{C}$, we consider the function $\zeta_{k,\lambda}(s)$ given by

$$\begin{aligned}\zeta_{k,\lambda}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e_\lambda(t)+1} \mathrm{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr)\,dt \\ &= \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{e_\lambda(t)+1} \mathrm{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr)\,dt \\ &\quad + \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{e_\lambda(t)+1} \mathrm{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr)\,dt. \end{aligned} \tag{35}$$

The second integral converges absolutely for any $s \in \mathbb{C}$ and hence the second term on the right hand side vanishes at nonpositive integers. That is,

$$\lim_{s\to -m} \left| \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{e_\lambda(t)+1} \mathrm{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr)\,dt \right| \leq \frac{1}{\Gamma(-m)} M = 0. \tag{36}$$

On the other hand, for $\mathrm{Re}(s) > 0$, the first integral in (35) can be written as

$$\frac{1}{\Gamma(s)} \sum_{l=0}^{\infty} \frac{G_{l,\lambda}^{(k)}}{l!} \frac{1}{s+l},$$

which defines an entire function of s.

Thus, we may conclude that $\zeta_{k,\lambda}(s)$ can be continued to an entire function of s.

By (35) and (36), we get

$$\begin{aligned}\zeta_{k,\lambda}(-m) &= \lim_{s\to -m} \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{e_\lambda(t)+1} \mathrm{Ei}_{k,\lambda}\bigl(\log_\lambda(1+t)\bigr)\,dt \\ &= \lim_{s\to -m} \frac{1}{\Gamma(s)} \sum_{l=0}^{\infty} \frac{G_{l,\lambda}^{(k)}}{s+l} \frac{1}{l!} \\ &= \cdots + 0 + \lim_{s\to -m} \frac{1}{\Gamma(s)} \frac{1}{s+m} \frac{G_{m,\lambda}^{(k)}}{m!} + 0 + \cdots. \end{aligned}$$

Now, by using the well-known Euler reflection formula, we have

$$\begin{aligned}\zeta_{k,\lambda}(-m) &= \lim_{s\to -m} \frac{\left(\frac{\Gamma(1-s)\sin(\pi s)}{\pi}\right)}{s+m} \frac{G_{m,\lambda}^{(k)}}{m!} \\ &= (-1)^m G_{m,\lambda}^{(k)}. \end{aligned}$$

By replacing t by $\log_\lambda(1+t)$ in (12), we get

$$\frac{2}{2+t} \log_\lambda(1+t) = \sum_{m=1}^{\infty} G_{m,\lambda} \frac{1}{m!} \bigl(\log_\lambda(1+t)\bigr)^m.$$

Thus, we have

$$\frac{2}{2+t} = \sum_{m=0}^{\infty} \frac{G_{m+1,\lambda}}{m+1} \frac{1}{m!} \bigl(\log_\lambda(1+t)\bigr)^m$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{G_{m+1,\lambda}}{m+1} \sum_{n=m}^{\infty} S_{1,\lambda}(n,m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{G_{m+1,\lambda}}{m+1} S_{1,\lambda}(n,m) \right) \frac{t^n}{n!}. \qquad (37)
\end{aligned}
$$

Therefore, by (21) and (37), we obtain the following theorem.

Theorem 5 *For $n \geq 0$, we have*

$$
\mathrm{Ch}_n = \sum_{m=0}^{n} \frac{G_{m+1,\lambda}}{m+1} S_{1,\lambda}(n,m).
$$

Remark 6 We note here that Theorem 5 is identical to Theorem 2.5 with $x = 0$ in [10] if we let $\lambda \to 0$. Namely, we get

$$
\mathrm{Ch}_n = \sum_{m=0}^{n} E_m S_1(n,m) \quad (n \geq 0).
$$

By replacing t by $e_\lambda(t) - 1$ in (21), we get

$$
\begin{aligned}
\frac{2}{e_\lambda(t)+1} &= \sum_{m=0}^{\infty} \mathrm{Ch}_m \frac{1}{m!} \left(e_\lambda(t) - 1\right)^m \\
&= \sum_{m=0}^{\infty} \mathrm{Ch}_m \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \mathrm{Ch}_m S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} G_{n,\lambda} \frac{t^n}{n!} = \frac{2t}{e_\lambda(t)+1} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \mathrm{Ch}_m S_{2,\lambda}(n,m) \right) \frac{t^{n+1}}{n!} \\
&= \sum_{n=1}^{\infty} \left(n \sum_{m=0}^{n-1} \mathrm{Ch}_m S_{2,\lambda}(n-1,m) \right) \frac{t^n}{n!}. \qquad (38)
\end{aligned}
$$

Therefore, by comparing the coefficients on both sides of (38), we obtain the following theorem.

Theorem 7 *For $n \geq 1$, we have*

$$
\frac{G_{n,\lambda}}{n} = \sum_{m=0}^{n-1} \mathrm{Ch}_m S_{2,\lambda}(n-1,m).
$$

Remark 8 We note here that Theorem 7 reduces to Theorem 2.4 in [10] if we let $\lambda \to 0$. Namely, we have

$$E_n = \sum_{m=0}^{n} \mathrm{Ch}_m S_2(n,m) \quad (n \geq 0).$$

By replacing t by $\log_\lambda(1+t)$ in (11), we get

$$\begin{aligned} r!\left(\frac{2}{2+t}\right)^r (1+t)^x \frac{1}{r!}\left(\log_\lambda(1+t)\right)^r &= \sum_{m=0}^{\infty} \widehat{G}_{m,\lambda}^{(r)}(x) \frac{1}{m!}\left(\log_\lambda(1+t)\right)^m \\ &= \sum_{m=0}^{\infty} \widehat{G}_{m,\lambda}^{(r)}(x) \sum_{n=m}^{\infty} S_{1,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=r}^{\infty}\left(\sum_{m=r}^{n} \widehat{G}_{m,\lambda}^{(r)}(x) S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{39}$$

On the other hand,

$$\begin{aligned} r!\left(\frac{2}{2+t}\right)^r (1+t)^x \frac{1}{r!}\left(\log_\lambda(1+t)\right)^r &= r!\sum_{l=0}^{\infty} \mathrm{Ch}_l^{(r)}(x)\frac{t^l}{l!}\sum_{m=r}^{\infty} S_{1,\lambda}(m,r)\frac{t^m}{m!} \\ &= r!\sum_{n=r}^{\infty}\sum_{m=r}^{n}\binom{n}{m}\mathrm{Ch}_{n-m}^{(r)}(x) S_{1,\lambda}(m,r)\frac{t^n}{n!}. \end{aligned} \tag{40}$$

Therefore, by (39) and (40), we obtain the following theorem.

Theorem 9 *For $r \geq 1$, and $n \geq r$, we have*

$$\sum_{m=r}^{n} \widehat{G}_{m,\lambda}^{(r)}(x) S_{1,\lambda}(n,m) = r!\sum_{m=r}^{n}\binom{n}{m}\mathrm{Ch}_{n-m}^{(r)}(x) S_{1,\lambda}(m,r).$$

From (20), we can derive the following equation:

$$\begin{aligned} \frac{1}{t^r}\left(\frac{2t}{e_\lambda(t)+1}\right)^r e_\lambda^x(t) &= \sum_{m=0}^{\infty} \mathrm{Ch}_m^{(r)}(x)\frac{1}{m!}\left(e_\lambda(t)-1\right)^m \\ &= \sum_{m=0}^{\infty} \mathrm{Ch}_m^{(r)}(x)\sum_{n=m}^{\infty} S_{2,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \mathrm{Ch}_m^{(r)}(x) S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{41}$$

On the other hand,

$$\begin{aligned} \frac{1}{t^r}\left(\frac{2t}{e_\lambda(t)+1}\right)^r e_\lambda^x(t) &= \frac{1}{t^r}\sum_{n=r}^{\infty} \widehat{G}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} \\ &= \frac{1}{r!}\sum_{n=0}^{\infty}\frac{\widehat{G}_{n+r,\lambda}^{(r)}(x)}{\binom{n+r}{n}}\frac{t^n}{n!}. \end{aligned} \tag{42}$$

Therefore, by (41) and (42), we obtain the following theorem.

Theorem 10 *For $n \geq 0$, and $r \geq 1$, we have*

$$\widehat{G}^{(r)}_{n+r,\lambda}(x) = r!\binom{n+r}{n}\sum_{m=0}^{n} \mathrm{Ch}^{(r)}_{m}(x)S_{2,\lambda}(n,m).$$

Remark 11 We note here that Theorem 10 reduces to Theorem 2.4 in [10] if we let $\lambda \to 0$, and let $r = 1, x = 0$. Namely, we get

$$E_n = \frac{G_{n+1}}{n+1} = \sum_{m=0}^{n} \mathrm{Ch}_m S_2(n,m) \quad (n \geq 0).$$

3 Conclusion

In this paper, the degenerate poly-Genocchi polynomials were introduced by means of the modified degenerate polyexponential function. Those polynomials were represented in terms of the degenerate Genocchi polynomials and the degenerate Stirling numbers of the first kind and also of the degenerate Euler polynomials and the Stirling numbers of the first kind. For the degenerate poly-Genocchi numbers, the generating function was expressed by iterated integrals. It was shown that a zeta function connected with the degenerate poly-Genocchi numbers and originally defined for $\mathrm{Re}(s) > 0$ can be continued to an entire function. Also, its special values at nonpositive integers were determined. Some identities, relating the degenerate Genocchi numbers and Changhee numbers, were obtained. In addition, certain identities, involving the higher-order degenerate Genocchi polynomials and the higher-order Changhee polynomials, were deduced.

The study of degenerate versions has applications to differential equations, identities of symmetry and probability theory as well as to number theory and combinatorics. Indeed, infinitely many families of linear and non-linear ordinary differential equations, satisfied by the generating functions of some degenerate special polynomials, were found with the purpose of discovering some new combinatorial identities for those polynomials (see [8]). As to identities of symmetry, for various degenerate versions of many special polynomials, abundant identities of symmetry have been investigated by using p-adic integrals (see [13]). For probability theory, by using the generating functions of the moments of certain random variables some identities connecting some special numbers and moments of random variables were derived (see [19]). As another potential applications in probability, we let the reader refer to [23] where there exists a new family of special numbers and polynomials of higher-order with their generating functions in the spirit of probabilistic distributions and approach to negative hypergeometric distribution.

As one of our future projects, we would like to continue to study degenerate versions of certain special polynomials and numbers and their applications to physics, science and engineering as well as mathematics.

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TK and DSK conceived of the framework and structured the whole paper; DSK and TK wrote the paper; JK and HYK checked the results of the paper and typed the paper; DSK and TK completed the revision of the article. All authors have read and agreed to the final version of the manuscript.

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