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Sharp trapezoid and mid-point type inequalities on closed balls in $\mathbb{R}^3$

M. Rostamian Delavar[1*]

*Correspondence:
m.rostamian@ub.ac.ir
[1]Department of Mathematics,
Faculty of Basic Sciences, University
of Bojnord, Bojnord, Iran

Abstract

This paper deals with some trapezoid and mid-point type inequalities on closed balls in $\mathbb{R}^3$. Three kinds of functions are considered: convex, Lipschitz, and bounded functions. The spherical coordinates are used to obtain sharp inequalities. Also a reverse result is given for the right-hand side of Hermite–Hadamard's inequality obtained on closed balls in $\mathbb{R}^3$.

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1 Introduction and preliminaries

Consider the closed ball $\bar{\mathcal{B}}(\mathcal{C},R)$ in the space $\mathbb{R}^3$ with center $\mathcal{C} = (a,b,c) \in \mathbb{R}^3$ and radius $R > 0$ defined as

$$\bar{\mathcal{B}}(\mathcal{C},R) = \left\{ (x,y,z) \in \mathbb{R}^3 \,|\, (x-a)^2 + (y-b)^2 + (z-c)^2 \leq R^2 \right\}.$$

Also consider $\sigma(\mathcal{C},R)$ as the boundary (the surface) of $\bar{B}(\mathcal{C},R)$, i.e.,

$$\sigma(\mathcal{C},R) = \left\{ (x,y,z) \in \mathbb{R}^3 \,|\, (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \right\}.$$

The following result has been proved in [1], which is the Hermite–Hadamard's inequality for convex functions defined on closed ball $\bar{\mathcal{B}}(\mathcal{C},R)$.

Theorem 1.1 *Let $f : \bar{\mathcal{B}}(\mathcal{C},R) \to \mathbb{R}$ be a convex mapping on the ball $\bar{\mathcal{B}}(\mathcal{C},R)$. Then we have the inequality*

$$f(\mathcal{C}) \leq \frac{1}{v(\bar{\mathcal{B}}(\mathcal{C},R))} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\,dx\,dy\,dz \leq \frac{1}{\sigma(\bar{B}(\mathcal{C},R))} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\,d\sigma, \qquad (1)$$

where $v(\bar{\mathcal{B}}(\mathcal{C},R)) = \frac{4\pi R^3}{3}$ and $\sigma(\bar{B}(\mathcal{C},R)) = \frac{1}{4\pi R^2}$.

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The main purpose of this paper is estimating two bounds $\mathcal{B}_1$ and $\mathcal{B}_2$ such that

$$\left| \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{B}(\mathcal{C},R)} f(x,y,z)\, dV - f(\mathcal{C}) \right| \le \mathcal{B}_1, \tag{2}$$

and

$$\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\, d\sigma - \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{B}(\mathcal{C},R)} f(x,y,z)\, dV \right| \le \mathcal{B}_2. \tag{3}$$

Depending on the properties of the function f and the radius R, different values will be obtained for $\mathcal{B}_1$ and $\mathcal{B}_2$.

We call (2) a *mid-point type inequality* due to the following result obtained in [2] and interpretation of Fig. 1.

Theorem 1.2 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\left| \int_a^b f(x)\, dx - (b-a)f\left(\frac{a+b}{2}\right) \right| \le \frac{1}{8}(b-a)^2 \left(|f'(a)| + |f'(b)| \right). \tag{4}$$

According to (4), we have an estimate for the difference between the area under the graph of f, i.e., $\int_a^b f(x)\, dx$, and the area of rectangle *abcd*, i.e., $(b-a)f(\frac{a+b}{2})$ (see Fig. 1).

Also we call (3) a *trapezoid type inequality* due to the following result and Fig. 2.

Figure 1 Mid-point type inequality

Figure 2 Trapezoid type inequality
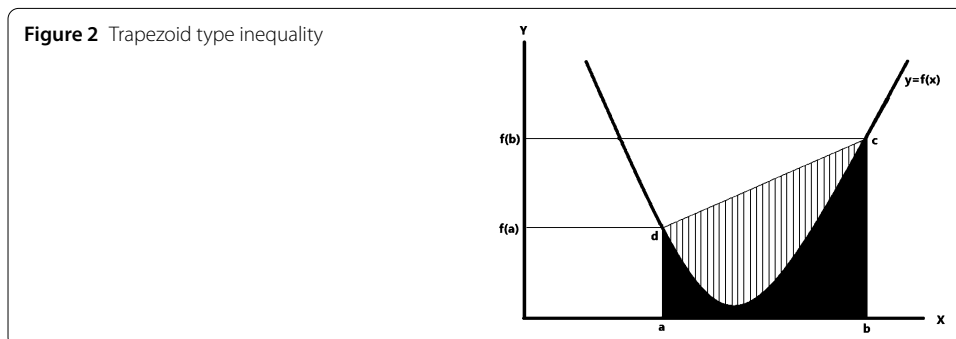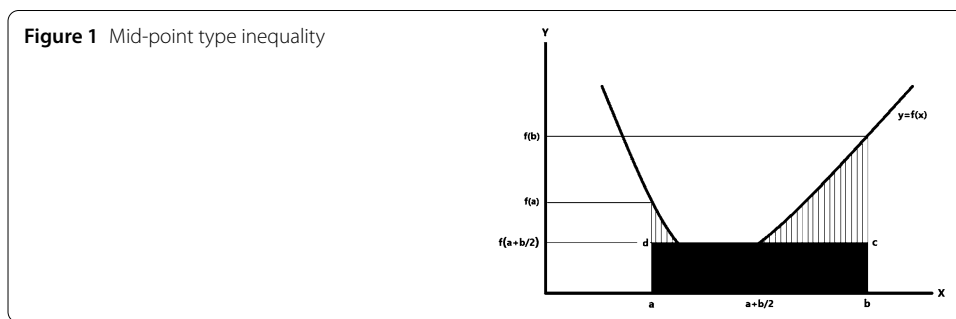
Theorem 1.3 ([3]) *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds*:

$$\left| \int_a^b f(x)\, dx - (b-a)\frac{f(a) + f(b)}{2} \right| \leq \frac{1}{8}(b-a)^2 \big(|f'(a)| + |f'(b)| \big). \tag{5}$$

According to (5), we can estimate the difference between the area of trapezoid *abcd*, i.e., $(b-a)\frac{f(a)+f(b)}{2}$, and the area under the graph of f (see Fig. 2).

Note that to obtain (4) and (5), the absolute values of the derivative of f at boundary points of interval $[a, b]$ play a fundamental role. For more results about Hermite–Hadamard's inequality, we refer an interested reader to [4–18] and the references therein.

Before presenting our main results, here we obtain a new representation of (1) and also give a reverse type theorem.

If we consider a convex function $f : \bar{\mathcal{B}}(\mathcal{C}, R) \to \mathbb{R}$ and the change of coordinates

$$\begin{aligned}
&\mathcal{T} : \bar{D}\big((a,b), R\big) \times [0,1] \to \bar{\mathcal{B}}(\mathcal{C}, R), \\
&\mathcal{T}(x, y, \lambda) = \big(x, y, (2\lambda - 1)\sqrt{R^2 - x^2 - y^2}\big),
\end{aligned} \tag{6}$$

where $\bar{D}((a,b), R)$ is a closed disk centered at the point (a, b) having radius $R > 0$, then we obtain

$$\begin{aligned}
&\iiint_{\bar{\mathcal{B}}(\mathcal{C}, R)} f(x, y, z)\, dV \\
&= 2 \int_{-R}^{R} \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} \int_0^1 f\big((1 - \lambda)\big(x, y, -\sqrt{R^2 - x^2 - y^2}\big) + \lambda\big(x, y, \sqrt{R^2 - x^2 - y^2}\big)\big) \\
&\quad \times \sqrt{R^2 - x^2 - y^2}\, d\lambda\, dy\, dx \\
&\leq 2 \int_{-R}^{R} \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} \int_0^1 (1 - \lambda) f\big(x, y, -\sqrt{R^2 - x^2 - y^2}\big) \sqrt{R^2 - x^2 - y^2}\, d\lambda\, dy\, dx \\
&\quad + 2 \int_{-R}^{R} \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} \int_0^1 \lambda f\big(x, y, \sqrt{R^2 - x^2 - y^2}\big) \sqrt{R^2 - x^2 - y^2}\, d\lambda\, dy\, dx \\
&= \int_{-R}^{R} \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} f\big(x, y, -\sqrt{R^2 - x^2 - y^2}\big) \sqrt{R^2 - x^2 - y^2}\, dy\, dx \\
&\quad + \int_{-R}^{R} \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} f\big(x, y, \sqrt{R^2 - x^2 - y^2}\big) \sqrt{R^2 - x^2 - y^2}\, dy\, dx. \tag{7}
\end{aligned}$$

Choosing $z = \sqrt{R^2 - x^2 - y^2}$ in the latter integrals, the fact that $\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} = \frac{R}{\sqrt{R^2 - x^2 - y^2}} = \frac{R}{z}$, and using the surface integral formula for $\sigma(\mathcal{C}, R)$ imply that

$$\iiint_{\bar{\mathcal{B}}(\mathcal{C}, R)} f(x, y, z)\, dV \leq \frac{1}{R} \iint_{\sigma(\mathcal{C}, R)} f(x, y, z) z^2\, d\sigma. \tag{8}$$

Inequality (8) gives another representation for (1).

In a special case for a convex function $f : \bar{D}((a,b), R) \to \mathbb{R}$ we get

$$\iint_{\bar{D}((a,b), R)} f(x, y)\, dA \leq \frac{1}{R} \int_{\sigma((a,b), R)} f(x, y) y^2\, d\sigma.$$

Now for a reverse type result, consider a continuous function f defined on a convex subset $\mathcal{V} \subset \mathbb{R}^3$ such that (8) holds for all closed balls included in $\mathcal{V}$. Then f is convex on $\mathcal{V}$ because otherwise there would exist $\mathcal{X}, \mathcal{Y} \in \mathcal{V}$ and $\lambda \in (0,1)$ such that

$$f\big(\lambda \mathcal{X} + (1-\lambda)\mathcal{Y}\big) > \lambda f(\mathcal{X}) + (1-\lambda)f(\mathcal{Y}). \tag{9}$$

Since f is continuous on $\mathcal{V}$, we can find $R > 0$ and a point $\mathcal{Z} = (\bar{a}, \bar{b}, \bar{c})$ in a convex combination of $\mathcal{X}$ and $\mathcal{Y}$ such that (9) holds on the whole of $\bar{\mathcal{B}}((\bar{a}, \bar{b}, \bar{c}), R) \subset \mathcal{V}$. So by the change of coordinates (6) and structure presented in (7) for $\bar{D}((\bar{a}, \bar{b}), R)$ and $\bar{\mathcal{B}}((\bar{a}, \bar{b}, \bar{c}), R)$, we obtain that

$$\iiint_{\bar{\mathcal{B}}((\bar{a},\bar{b},\bar{c}),R)} f(x,y,z)\,dV > \frac{1}{R}\iint_{\sigma((\bar{a},\bar{b},\bar{c}),R)} f(x,y,z)z^2\,d\sigma,$$

which is a contradiction and this proves the convexity of f on $\mathcal{V}$.

In the following sections we consider convex, Lipschitz, and bounded functions to obtain some trapezoid and mid-point type inequalities on a closed ball. We use the spherical coordinates in calculating the integrals.

2 Convex functions

In this section we obtain trapezoid and mid-point type inequalities for the case that the partial derivative absolute values of a considered function with respect to the radius in spherical coordinates is convex. We need the following lemma.

Lemma 2.1 *For an integrable function $f : \bar{\mathcal{B}}(\mathcal{C}, R) \to \mathbb{R}$, we have*

$$\iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\,dV$$
$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} f(a + \rho\cos\theta\sin\varphi, b + \rho\sin\theta\sin\varphi, c + \rho\cos\varphi)$$
$$\times \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta, \tag{10}$$

and

$$\iint_{\sigma(\mathcal{C},R)} f(x,y,z)\,d\sigma$$
$$= \int_0^{2\pi} \int_0^{\pi} f(a + R\cos\theta\sin\varphi, b + R\sin\theta\sin\varphi, c + R\cos\varphi)R^2\sin\varphi\,d\varphi\,d\theta$$
$$= \frac{3}{R} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} f(a + R\cos\theta\sin\varphi, b + R\sin\theta\sin\varphi, c + R\cos\varphi)$$
$$\times \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta. \tag{11}$$

Proof Consider the spherical transformation

$$\begin{cases} x(\rho, \varphi, \theta) = a + \rho\cos\theta\sin\varphi, \\ y(\rho, \varphi, \theta) = b + \rho\sin\theta\sin\varphi, \\ z(\rho, \varphi, \theta) = c + \rho\cos\varphi, \end{cases} \quad \rho \in [0, R], \varphi \in [0, \pi], \theta \in [0, 2\pi].$$

It is obvious that the Jacobian of this transformation is $J = \rho^2 \sin\varphi$. So we have (10).

For (11), consider the curve $\eta : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ defined by

$$
\eta(\theta, \varphi): \quad
\begin{cases}
x(\varphi, \theta) = a + R\cos\theta\sin\varphi, \\
y(\varphi, \theta) = b + R\sin\theta\sin\varphi, \qquad \varphi \in [0, \pi], \theta \in [0, 2\pi]. \\
z(\varphi, \theta) = c + R\cos\varphi,
\end{cases}
$$

It is clear that $\eta([0, \pi] \times [0, 2\pi]) = \sigma(C, R)$ and then by integrating with respect to the surface (arc length) we get

$$
\iint_{\sigma(C,R)} f(x, y, z)\, d\sigma
$$

$$
= \iint_{\sigma(C,R)} f(\eta)\, dl(\eta)
$$

$$
= \int_0^{2\pi} \int_0^{\pi} f\big(x(\varphi,\theta), y(\varphi,\theta), z(\varphi,\theta)\big)
$$

$$
\times \left(\left[\frac{\partial x(\varphi,\theta)}{\partial \theta} \right]^2 + \left[\frac{\partial y(\varphi,\theta)}{\partial \theta} \right]^2 + \left[\frac{\partial z(\varphi,\theta)}{\partial \theta} \right]^2 \right)^{\frac{1}{2}}
$$

$$
\times \left(\left[\frac{\partial x(\varphi,\theta)}{\partial \varphi} \right]^2 + \left[\frac{\partial y(\varphi,\theta)}{\partial \varphi} \right]^2 + \left[\frac{\partial z(\varphi,\theta)}{\partial \varphi} \right]^2 \right)^{\frac{1}{2}} d\varphi\, d\theta
$$

$$
= \int_0^{2\pi} \int_0^{\pi} f(a + R\cos\theta\sin\varphi, b + R\sin\theta\sin\varphi, c + R\cos\varphi) R^2 \sin\varphi\, d\varphi\, d\theta
$$

$$
= \frac{3}{R} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} f(a + R\cos\theta\sin\varphi, b + R\sin\theta\sin\varphi, c + R\cos\varphi)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.
$$

This proves (11). □

The following is a sharp trapezoid type inequality related to (1), where we consider a function with convex partial derivative (with respect to the radius ρ) absolute values defined on $\bar{\mathcal{B}}(C, R)$.

Theorem 2.2 *For $\mathcal{V} \subset \mathbb{R}^3$, suppose that $\bar{\mathcal{B}}(C, R) \subset \mathcal{V}^\circ$ where $\mathcal{V}^\circ$ is the interior of $\mathcal{V}$. Consider $f : \bar{\mathcal{B}}(C, R) \to \mathbb{R}$ which has continuous partial derivatives with respect to the variables ρ, φ, and θ on $\bar{\mathcal{B}}(C, R)$ in spherical coordinates. If $|\frac{\partial f}{\partial \rho}|$ is convex on $\bar{\mathcal{B}}(C, R)$, then*

$$
\left| \frac{1}{4\pi R^2} \iint_{\sigma(C,R)} f(x, y, z)\, d\sigma - \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(C,R)} f(x, y, z)\, dV \right|
$$

$$
\leq \frac{1}{16\pi R} \iint_{\sigma(C,R)} \left| \frac{\partial f}{\partial \rho} \right|(x, y, z)\, d\sigma. \tag{12}
$$

Furthermore, inequality (12) is sharp.

Proof For fixed $\varphi \in [0, \pi]$ and $\theta \in [0, 2\pi]$ and arbitrary $\rho \in [0, R]$, since

$$
\left(\left[\frac{\partial x(\rho, \varphi, \theta)}{\partial \rho} \right]^2 + \left[\frac{\partial y(\rho, \varphi, \theta)}{\partial \rho} \right]^2 + \left[\frac{\partial z(\rho, \varphi, \theta)}{\partial \rho} \right]^2 \right)^{\frac{1}{2}} = 1,
$$

by integration by parts we have

$$\int_0^R \frac{\partial f}{\partial \rho}(a + \rho \cos\theta \sin\varphi, b + \rho \sin\theta \sin\varphi, c + \rho \cos\varphi)\rho^3 \sin\varphi \, d\rho$$

$$= R^3 f(a + R\cos\theta \sin\varphi, b + R\sin\theta \sin\varphi, c + R\cos\varphi)$$

$$- 3\int_0^R f(a + \rho \cos\theta \sin\varphi, b + \rho \sin\theta \sin\varphi, c + \rho \cos\varphi)\rho^2 \sin\varphi \, d\rho. \tag{13}$$

So integrating with respect to $\varphi \in [0, \pi]$ and $\theta \in [0, 2\pi]$ in (13), along with (10) and (11) obtained in Lemma 2.1 and the convexity of $|\frac{\partial f}{\partial \rho}|$ on $\bar{\mathcal{B}}(\mathcal{C}, R)$, implies that

$$\left| R \iint_{\sigma(\mathcal{C},R)} f(x,y,z) \, d\sigma - 3 \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z) \, dV \right|$$

$$\leq \int_0^{2\pi} \int_0^\pi \int_0^R \left|\frac{\partial f}{\partial \rho}\right|(a + \rho \cos\theta \sin\varphi, b + \rho \sin\theta \sin\varphi, c + \rho \cos\varphi)\rho^3 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\leq \int_0^{2\pi} \int_0^\pi \int_0^R \left|\frac{\partial f}{\partial \rho}\right|\left(\left(1 - \frac{\rho}{R}\right)(a,b,c)\right.$$

$$\left. + \frac{\rho}{R}(a + R\cos\theta \sin\varphi, b + R\sin\theta \sin\varphi, c + R\cos\varphi)\right)$$

$$\times \rho^3 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\leq \int_0^{2\pi} \int_0^\pi \int_0^R \rho^3\left(1 - \frac{\rho}{R}\right)\left|\frac{\partial f}{\partial \rho}\right|(a,b,c)\sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$+ \int_0^{2\pi} \int_0^\pi \int_0^R \frac{\rho^4}{R}\left|\frac{\partial f}{\partial \rho}\right|(a + R\cos\theta \sin\varphi, b + R\sin\theta \sin\varphi, c + R\cos\varphi)$$

$$\times \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \frac{\pi R^4}{5}\left|\frac{\partial f}{\partial \rho}\right|(a,b,c) + \frac{R^2}{5}\int\int_{\sigma(\mathcal{C},R)}\left|\frac{\partial f}{\partial \rho}\right|(x,y,z) \, d\sigma. \tag{14}$$

By considering the left-hand side of (1) for $|\frac{\partial f}{\partial \rho}|$ and applying it in (14), we have

$$\left| R \iint_{\sigma(\mathcal{C},R)} f(x,y,z) \, d\sigma - 3 \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z) \, dV \right|$$

$$\leq \frac{R^2}{20}\iint_{\sigma(\mathcal{C},R)}\left|\frac{\partial f}{\partial \rho}\right|(x,y,z) \, d\sigma + \frac{R^2}{5}\iint_{\sigma(\mathcal{C},R)}\left|\frac{\partial f}{\partial \rho}\right|(x,y,z) \, d\sigma$$

$$= \frac{R^2}{4}\iint_{\sigma(\mathcal{C},R)}\left|\frac{\partial f}{\partial \rho}\right|(x,y,z) \, d\sigma. \tag{15}$$

By dividing (15) with $4\pi R^3$, we obtain the desired result (12).

To show the sharpness of (12), consider the function $f : \bar{\mathcal{B}}(\mathcal{C}, R) \to \mathbb{R}$ defined as

$$f(x,y,z) = R - \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

Using spherical coordinates, we have $f(\rho, \varphi, \theta) = R - \rho$, for $\rho \in [0, R]$, $\varphi \in [0, \pi]$ and $\theta \in [0, 2\pi]$. With some calculations we obtain that

$$\frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\,dV$$

$$= \frac{1}{\frac{4}{3}\pi R^3} \int_0^{2\pi} \int_0^{\pi} \int_0^R (R - \rho)\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{R}{4}, \tag{16}$$

and

$$\iint_{\sigma(\mathcal{C},R)} f(x,y,z)\,d\sigma = 0. \tag{17}$$

On the other hand, since $|\frac{\partial f}{\partial \rho}| = 1$,

$$\frac{1}{16\pi R} \iint_{\sigma(\mathcal{C},R)} \left|\frac{\partial f}{\partial \rho}\right|(x,y,z)\,d\sigma = \frac{R}{4}.$$

From (16) and (17) we have the sharpness of (12). □

Now we obtain the midpoint type inequality related to (1), where the partial derivative absolute value of considered function defined on $\bar{\mathcal{B}}(\mathcal{C},R)$ is convex.

Theorem 2.3 *Suppose that $\bar{\mathcal{B}}(\mathcal{C},R) \subset \mathcal{V}^{\circ}$, where $\mathcal{V} \subset \mathbb{R}^3$. Consider $f : \bar{\mathcal{B}}(\mathcal{C},R) \to \mathbb{R}$ which has continuous partial derivatives with respect to the variables ρ, φ, and θ on $\bar{\mathcal{B}}(\mathcal{C},R)$ in spherical coordinates. If $|\frac{\partial f}{\partial \rho}|$ is convex on $\bar{\mathcal{B}}(\mathcal{C},R)$, then*

$$\left| \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\,dV - f(\mathcal{C}) \right| \leq \frac{5}{16\pi R} \iint_{\sigma(\mathcal{C},R)} \left|\frac{\partial f}{\partial \rho}\right|(x,y,z)\,d\sigma. \tag{18}$$

Proof Similar to the proof of Theorem 2.2, for fixed $\varphi \in [0, \pi]$ and $\theta \in [0, 2\pi]$, we have

$$\int_0^R \frac{\partial f}{\partial \rho}(a + \rho\cos\theta\sin\varphi, b + \rho\sin\theta\cos\varphi, c + \rho\cos\varphi)\sin\varphi\,d\rho$$

$$= f(a + R\cos\theta\sin\varphi, b + R\sin\theta\sin\varphi, c + R\cos\varphi)\sin\varphi - f(\mathcal{C})\sin\varphi. \tag{19}$$

Integration with respect to the variables $\varphi \in [0, \pi]$ and $\theta \in [0, 2\pi]$ in (19) implies that

$$\int_0^{2\pi} \int_0^{\pi} \int_0^R \frac{\partial f}{\partial \rho}(a + \rho\cos\theta\sin\varphi, b + \rho\sin\theta\cos\varphi, c + \rho\cos\varphi)\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \frac{1}{R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\,d\sigma - 4\pi f(\mathcal{C}).$$

So from the convexity of $|\frac{\partial f}{\partial \rho}|$ we get

$$\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\,d\sigma - f(\mathcal{C}) \right|$$

$$\leq \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^R \left| \frac{\partial f}{\partial \rho} \right| (a + \rho \cos\theta \sin\varphi, b + \rho \sin\theta \sin\varphi, c + \rho \cos\varphi) \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\leq \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^R \left(1 - \frac{\rho}{R}\right) \left| \frac{\partial f}{\partial \rho} \right| (\mathcal{C}) \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$+ \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^R \frac{\rho}{R} \left| \frac{\partial f}{\partial \rho} \right| (a + R\cos\theta \sin\varphi, b + R\sin\theta \sin\varphi, c + R\cos\varphi)$$

$$\times \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \frac{R}{2} \left| \frac{\partial f}{\partial \rho} \right| (\mathcal{C}) + \frac{1}{8\pi R} \iint_{\sigma(\mathcal{C},R)} \left| \frac{\partial f}{\partial \rho} \right| (x,y,z) \, d\sigma. \tag{20}$$

It follows from triangle inequality, (20), (12) and (1)(for $|\frac{\partial f}{\partial \rho}|$) that

$$\left| \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z) \, dV - f(\mathcal{C}) \right|$$

$$\leq \frac{1}{16\pi R} \iint_{\sigma(\mathcal{C},R)} \left| \frac{\partial f}{\partial \rho} \right| (x,y,z) \, d\sigma + \frac{R}{2} \left| \frac{\partial f}{\partial \rho} \right| (\mathcal{C}) + \frac{1}{8\pi R} \iint_{\sigma(\mathcal{C},R)} \left| \frac{\partial f}{\partial \rho} \right| (x,y,z) \, d\sigma$$

$$\leq \frac{3}{16\pi R} \iint_{\sigma(\mathcal{C},R)} \left| \frac{\partial f}{\partial \rho} \right| (x,y,z) \, d\sigma + \frac{1}{8\pi R} \iint_{\sigma(\mathcal{C},R)} \left| \frac{\partial f}{\partial \rho} \right| (x,y,z) \, d\sigma$$

$$= \frac{5}{16\pi R} \iint_{\sigma(\mathcal{C},R)} \left| \frac{\partial f}{\partial \rho} \right| (x,y,z) \, d\sigma,$$

which implies the desired result. □

Corollary 2.4 ([17]) *Consider a set $I \subset \mathbb{R}^2$ with $D(C,R) \subset I^{\circ}$. Suppose that the mapping $f : D(C,R) \to \mathbb{R}$ has continuous partial derivatives in the disk $D(C,R)$ with respect to the variables r and θ in polar coordinates. If for any constant $\theta \in [0,2\pi]$, the function $|\frac{\partial f}{\partial r}|$ is convex with respect to the variable r on $[0,R]$ then*

$$\left| \frac{1}{2\pi R} \int_{\partial(C,R)} f(\gamma) \, dl(\gamma) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y) \, dx \, dy \right| \leq \frac{1}{6\pi} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right| (\gamma) \, dl(\gamma),$$

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y) \, dx \, dy - f(C) \right| \leq \frac{2}{3\pi} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right| (\gamma) \, dl(\gamma).$$

Remark 2.5 In the proof of Theorem 2.3, we can find the following inequality:

$$\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z) \, d\sigma - f(\mathcal{C}) \right|$$

$$\leq \frac{R}{2} \left| \frac{\partial f}{\partial \rho} \right| (\mathcal{C}) + \frac{1}{8\pi R} \iint_{\sigma(\mathcal{C},R)} \left| \frac{\partial f}{\partial \rho} \right| (x,y,z) \, d\sigma. \tag{21}$$

Although (18) is not sharp, if we consider $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ for $x,y,z \in \bar{\mathcal{B}}(\mathcal{C},R)$, we will find that inequality (21) is sharp.

Remark 2.6 If we drop out the convexity condition of $|\frac{\partial f}{\partial \rho}|$ in Theorems 2.2, 2.3, and consider the condition

$$\left\| \frac{\partial f}{\partial \rho} \right\|_{\infty_{\bar{B}(\mathcal{C},R)}} = \sup_{w \in \bar{B}(\mathcal{C},R)} \left| f(w) \right| < \infty,$$

instead of that, then we get the following Ostrowski type inequalities (see [19, 20]) on a closed ball:

$$\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\, d\sigma - \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\, dV \right| \le \frac{R \| \frac{\partial f}{\partial \rho} \|_{\infty_{\bar{B}(\mathcal{C},R)}}}{4},$$

and

$$\left| \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\, dV - f(\mathcal{C}) \right| \le R \left\| \frac{\partial f}{\partial \rho} \right\|_{\infty_{\bar{B}(\mathcal{C},R)}}.$$

3 Lipschitz functions

In this section we consider Lipschitz functions with respect to the Euclidian norm to obtain some trapezoid and mid-point type inequalities on $\bar{\mathcal{B}}(\mathcal{C},R)$.

Definition 3.1 ([21]) A function $f : \mathcal{V} \subset \mathbb{R}^3 \to \mathbb{R}$ is said to satisfy a Lipschitz condition (briefly, f is $\mathcal{L}$-Lipschitz) on $\mathcal{V}$ with respect to a norm $\| \cdot \|$, if there exists a constant $\mathcal{L} > 0$ such that

$$\left| f(x) - f(y) \right| \le \mathcal{L} \| x - y \|,$$

for any $x, y \in \mathcal{V}$.

If $f : \bar{\mathcal{B}}(\mathcal{C},R)$ is Lipschitz with respect to the Euclidian norm with the constant $\mathcal{L} > 0$, then for any $x = (a + \rho_1 \cos\theta_1 \sin\varphi_1, b + \rho_1 \sin\theta_1 \sin\varphi_1, c + \rho_1 \cos\varphi_1)$ and $y = (a + \rho_2 \cos\theta_2 \sin\varphi_2, b + \rho_2 \sin\theta_2 \sin\varphi_2, c + \rho_2 \cos\varphi_2)$, with some calculations we obtain that

$$\left| f(x) - f(y) \right| \le \mathcal{L} \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2 \mathcal{M}(\varphi_1, \varphi_2, \theta_1, \theta_2)},$$

where $M(\varphi_1, \varphi_2, \theta_1, \theta_2) = [\sin\varphi_1 \sin\varphi_2 \cos(\theta_1 - \theta_2) + \cos\varphi_1 \cos\varphi_2]$, $\rho_1, \rho_2 \in [0, R]$, $\theta_1, \theta_2 \in [0, 2\pi]$ and $\varphi_1, \varphi_2 \in [0, \pi]$. Also it is obvious that if $f : \mathcal{V} \subseteq \mathbb{R}^3 \to \mathbb{R}$ is Lipschitz with a constant $\mathcal{L} > 0$ on $\mathcal{V}$, then it is continuous and so integrable on $\mathcal{V}$. We need the following result.

Lemma 3.2 *For any* $\varphi_i \in [0, \pi]$ *and* $\theta_i \in [0, 2\pi]$ $(i \in \{1, 2\})$ *we have*

$$-1 \le \cos(\varphi_1 + \varphi_2) \le \mathcal{M}(\varphi_1, \varphi_2, \theta_1, \theta_2) \le \cos(\varphi_1 - \varphi_2) \le 1.$$

Proof For any $\theta_1, \theta_2 \in [0, 2\pi]$ it is obvious that $\cos(\theta_1 - \theta_2) \le 1$. On the other hand, since for any $\varphi_1, \varphi_2 \in [0, \pi]$, $\sin\varphi_1 \sin\varphi_2$ is nonnegative,

$$\sin\varphi_1 \sin\varphi_2 \cos(\theta_1 - \theta_2) \le \sin\varphi_1 \sin\varphi_1.$$

So

$$\mathcal{M}(\varphi_1, \varphi_2, \theta_1, \theta_2) = \sin\varphi_1 \sin\varphi_2 \cos(\theta_1 - \theta_2) + \cos\varphi_1 \cos\varphi_2$$

$$\leq \sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 = \cos(\varphi_1 - \varphi_2) \leq 1.$$

Similarly, we can prove that $\mathcal{M}(\varphi_1, \varphi_2, \theta_1, \theta_2) \geq \cos(\varphi_1 + \varphi_2) \geq -1$. □

The following trapezoid type inequality related to (1) for $\mathcal{L}$-Lipschitz functions on $\bar{\mathcal{B}}(\mathcal{C}, R)$ holds.

Theorem 3.3 *Let $f : \bar{\mathcal{B}}(\mathcal{C}, R) \to \mathbb{R}$ be an $\mathcal{L}$-Lipschitz function. Then*

$$\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C}, R)} f(x, y, z)\, d\sigma - \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C}, R)} f(x, y, z)\, dV \right| \leq \frac{\mathcal{L}R}{4}. \tag{22}$$

Inequality (22) *is sharp.*

Proof Since f is Lipschitz with constant $\mathcal{L} > 0$ on $\bar{\mathcal{B}}(\mathcal{C}, R)$, we get

$$\left| \int_0^{2\pi} \int_0^{\pi} \int_0^R f(a + \rho\cos\theta\sin\varphi, b + \rho\sin\theta\cos\varphi, c + \rho\cos\varphi)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \right.$$

$$\left. - \int_0^{2\pi} \int_0^{\pi} \int_0^R f(a + R\cos\theta\sin\varphi, b + R\sin\theta\cos\varphi, c + R\cos\varphi)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \right|$$

$$\leq \mathcal{L} \int_0^{2\pi} \int_0^{\pi} \int_0^R \left\| \big((\rho - R)\cos\theta\sin\varphi, (\rho - R)\sin\theta\sin\varphi, (\rho - R)\cos\varphi \big) \right\|$$

$$\times \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \mathcal{L} \int_0^{2\pi} \int_0^{\pi} \int_0^R (R - \rho)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \frac{\mathcal{L}\pi R^4}{3}. \tag{23}$$

Now by replacing (10) and (11) in (23) and then dividing the result by $\frac{4}{3}\pi R^3$, we deduce the desired result.

To prove the sharpness of (22), consider the function $f : \bar{\mathcal{B}}(\mathcal{C}, R) \to \mathbb{R}$ defined by

$$f(a + \rho\cos\theta\sin\varphi, b + \rho\sin\theta\cos\varphi, c + \rho\cos\varphi) = \mathcal{L}(R - \rho),$$

for $\mathcal{L} > 0$, $\rho \in [0, R]$, $\varphi \in [0, \pi]$, and $\theta \in [0, 2\pi]$. The function f is Lipschitz with constant $\mathcal{L}$. Consider $x = (a + \rho_1\cos\theta_1\sin\varphi_1, b + \rho_1\sin\theta_1\sin\varphi_1, c + \rho_1\cos\varphi_1)$ and $y = (a + \rho_2\cos\theta_2\sin\varphi_2, b + \rho_2\sin\theta_2\sin\varphi_2, c + \rho_2\cos\varphi_2)$, for $\rho_1, \rho_2 \in [0, R]$, $\varphi_1, \varphi_2 \in [0, \pi]$, $\theta_1, \theta_2 \in [0, 2\pi]$. Then by Lemma 3.2 we have

$$\left| f(x) - f(y) \right|$$

$$= \left| f(a + \rho_1\cos\theta_1\sin\varphi_1, b + \rho_1\sin\theta_1\cos\varphi_1, c + \rho_1\cos\varphi_1) \right.$$

$$\left. - f(a + \rho_2\cos\theta_2\sin\varphi_2, b + \rho_2\sin\theta_2\cos\varphi_2, c + \rho_2\cos\varphi_2) \right|$$

$$= \mathcal{L}|\rho_2 - \rho_1| = \mathcal{L}\sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2} \leq \mathcal{L}\sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \mathcal{M}(\varphi_1, \varphi_2, \theta_1, \theta_2)}$$

$$= \mathcal{L} \big\| (a + \rho_1 \cos\theta_1 \sin\varphi_1, b + \rho_1 \sin\theta_1 \cos\varphi_1, c + \rho_1 \cos\varphi_1)$$

$$- (a + \rho_2 \cos\theta_2 \sin\varphi_2, b + \rho_2 \sin\theta_2 \cos\varphi_2, c + \rho_2 \cos\varphi_2) \big\| = \mathcal{L} \|x - y\|.$$

It is not hard to see that $f(a + \rho \cos\theta \sin\varphi, b + \rho \sin\theta \cos\varphi, c + \rho \cos\varphi) \geq 0$ for all $0 \leq \rho \leq R$, $0 \leq \varphi \leq \pi$, and $0 \leq \theta \leq 2\pi$. Also for the case $\rho = R$, we have $f(a + R \cos\theta \sin\varphi, b + R \sin\theta \cos\varphi, c + R \cos\varphi) = 0$. So we have

$$\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z) \, d\sigma - \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z) \, dV \right|$$

$$= \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z) \, dV$$

$$= \frac{\mathcal{L}}{\frac{4}{3}\pi R^3} \int_0^{2\pi} \int_0^{\pi} \int_0^R (R - \rho)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \frac{\mathcal{L}R}{4}. \qquad \square$$

For $\mathcal{L}$-Lipschitz functions we can obtain a mid-point type inequality as follows:

Theorem 3.4 *Let $f : \bar{\mathcal{B}}(\mathcal{C}, R) \to \mathbb{R}$ be an $\mathcal{L}$-Lipschitz function. Then*

$$\left| \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z) \, dV - f(\mathcal{C}) \right| \leq \frac{3\mathcal{L}R}{4}. \tag{24}$$

Inequality (24) *is sharp.*

Proof Since the function f is $\mathcal{L}$-Lipschitz on $\bar{\mathcal{B}}(\mathcal{C}, R)$, we have

$$\big| f(a + \rho \cos\theta \sin\varphi, b + \rho \sin\theta \cos\varphi, c + \rho \cos\varphi) - f(\mathcal{C}) \big|$$

$$\leq \mathcal{L} \big\| (\rho \cos\theta \sin\varphi, \rho \sin\theta \cos\varphi, \rho \cos\varphi) \big\| = \mathcal{L}\rho,$$

for all $\rho \in [0, R]$, $\varphi \in [0, \pi]$, and $\theta \in [0, 2\pi]$. It follows that

$$\left| \int_0^{2\pi} \int_0^{\pi} \int_0^R f(a + \rho \cos\theta \sin\varphi, b + \rho \sin\theta \cos\varphi, c + \rho \cos\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \right.$$

$$\left. - \int_0^{2\pi} \int_0^{\pi} \int_0^R f(a,b,c)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \right|$$

$$\leq \int_0^{2\pi} \int_0^{\pi} \int_0^R \big| f(a + \rho \cos\theta \sin\varphi, b + \rho \sin\theta \cos\varphi, c + \rho \cos\varphi) - f(\mathcal{C}) \big|$$

$$\times \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\leq \mathcal{L} \int_0^{2\pi} \int_0^{\pi} \int_0^R \rho^3 \sin\varphi \, d\rho \, d\varphi \, d\theta = \mathcal{L}\pi R^4.$$

So we obtain that

$$\left| \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z) \, dV - \frac{4}{3}\pi R^3 f(\mathcal{C}) \right| \leq \mathcal{L}\pi R^4,$$

which implies the desired result.

Now consider the function $f : \bar{\mathcal{B}}(\mathcal{C}, R) \to \mathbb{R}$ defined by

$$f(a + \rho \cos\theta \sin\varphi, b + \rho \sin\theta \cos\varphi, c + \rho \cos\varphi) = \mathcal{L}\rho,$$

for $\mathcal{L} > 0$, $0 \leq \rho \leq R$, $0 \leq \varphi \leq \pi$, and $0 \leq \theta \leq 2\pi$. It is obvious that $f(\mathcal{C}) = 0$. By a similar method used in the proof of Theorem 3.3, the function f is $\mathcal{L}$-Lipschitz. So we have

$$\left| \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x, y, z) \, dV - f(\mathcal{C}) \right| = \frac{\mathcal{L}}{\frac{4}{3}\pi R^3} \int_0^{2\pi} \int_0^{\pi} \int_0^R \rho^3 \sin\varphi \, d\rho \, d\varphi \, d\theta = \frac{3\mathcal{L}R}{4},$$

showing that inequality (24) is sharp.   □

Remark 3.5 Consider an open set $\mathcal{V} \subset \mathbb{R}^3$ including $\bar{\mathcal{B}}(\mathcal{C}, R)$. For convex function f defined on $\mathcal{V}$, from Theorem D of Sect. 41 in [21] we have that f is $\mathcal{L}$-Lipschitz on $\bar{\mathcal{B}}(\mathcal{C}, R)$ and so from inequalities (22) and (24), along with inequality (1), we get the following results:

$$0 \leq \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x, y, z) \, d\sigma - \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x, y, z) \, dV \leq \frac{\mathcal{L}R}{3},$$

and

$$0 \leq \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x, y, z) \, dV - f(\mathcal{C}) \leq \frac{2\mathcal{L}R}{3}.$$

In the following, as an example we obtain a Lipschitz constant $\mathcal{L}$ for a real-valued function defined on a closed ball in $\mathbb{R}^3$.

Example 3.6 Consider $W = f(x, y, z) = (x - a)^n + (y - b)^n + (z - c)^n$, $n \in \mathbb{N}$, $(x, y, z) \in \bar{\mathcal{B}}(\mathcal{C}, R)$. To find a Lipschitz constant for f, we will do some calculations as follows. For $A, B \in \bar{\mathcal{B}}(\mathcal{C}, R)$, consider the path $\psi : [0, 1] \to \bar{\mathcal{B}}(\mathcal{C}, R)$ from B to A in $\bar{\mathcal{B}}(\mathcal{C}, R)$ as

$$\psi(t) = tA + (1 - t)B,$$

for $t \in [0, 1]$. Now using the fundamental theorem of calculus, we obtain that

$$\left| f(A) - f(B) \right| = \left| f(\psi(1)) - f(\psi(0)) \right| = \left| \int_0^1 \frac{df(\psi(t))}{dt} \, dt \right|.$$

On the other hand, from the chain rule for differentiation, we get

$$\frac{df(\psi(t))}{dt} = \nabla f(\psi(t)) \cdot \frac{d\psi}{dt} = \nabla f(\psi(t))(A - B),$$

where ∇f is the gradient vector of f. So using the Euclidean norm $\| \cdot \|$, we obtain

$$\left| \int_0^1 \frac{df(\psi(t))}{dt} \, dt \right| = \left| \int_0^1 \nabla f(\psi(t))(A - B) \, dt \right| \leq \|A - B\| \int_0^1 \left\| \nabla f(\psi(t)) \right\| dt$$

$$\leq \|A - B\| \sup_{u \in \bar{\mathcal{B}}(\mathcal{C},R)} \left\| \nabla f(u) \right\|,$$

which implies

$$\left| f(A) - f(B) \right| \leq \|A - B\| \sup_{u \in \bar{\mathcal{B}}(\mathcal{C}, R)} \left\| \nabla f(u) \right\|.$$

This shows that $\mathcal{L} = \sup_{u \in \bar{\mathcal{B}}(\mathcal{C}, R)} \|\nabla f(u)\|$ (if it exists) is a Lipschitz constant for f. Now for any $w = (x, y, z) \in \bar{\mathcal{B}}(\mathcal{C}, R)$, we have

$$\nabla f(w) = n\left((x - a)^{n-1}, (y - b)^{n-1}, (z - c)^{n-1}\right),$$

and then

$$\left\| \nabla f(u) \right\| = n\sqrt{\left((x-a)^{n-1}\right)^2 + \left((y-b)^{n-1}\right)^2 + \left((z-c)^{n-1}\right)^2}$$
$$\leq n\sqrt{\left((x-a)^2 + (y-b)^2 + (z-c)^2\right)^{n-1}} = nR^{n-1}.$$

So we can choose $\mathcal{L} = \sup_{u \in \bar{\mathcal{B}}(\mathcal{C}, R)} \|\nabla f(u)\| = nR^{n-1}$ as a Lipschitz constant for f on $\bar{\mathcal{B}}(\mathcal{C}, R)$.

Using the above example, we have the following result:

Example 3.7 For $n \in \mathbb{N} \setminus \{1\}$, consider the function $f(\rho, \varphi, \theta) = (x_0 - \rho)^n + (y_0 - \rho)^n + (z_0 - \rho)^n$ defined on $\bar{\mathcal{B}}((x_0, y_0, z_0), R)$ such that $x_0, y_0, z_0 > 0$, $0 < R \leq \min\{x_0, y_0, z_0\}$ and $0 \leq \rho \leq R$. It follows that

$$\nabla\left(\frac{\partial f}{\partial \rho}\right)(\rho, \varphi, \theta) = n(n-1)\left((x_0 - \rho)^{n-2} + (y_0 - \rho)^{n-2} + (z_0 - \rho)^{n-2}, 0, 0\right),$$

and then

$$\mathcal{L} = n(n-1)\left(x_0^{n-2} + y_0^{n-2} + z_0^{n-2}\right),$$

is a Lipschitz constant for $\nabla(\frac{\partial f}{\partial \rho})$. On the other hand, it is not hard to prove that

$$\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C}, R)} f(x, y, z) \, d\sigma - f(\mathcal{C}) \right| \leq \frac{\mathcal{L}R^2}{2}. \tag{25}$$

So by (25), we have the following numerical inequality:

$$\left| \left((x_0 - R)^n + (y_0 - R)^n + (z_0 - R)^n\right) - \left(x_0^n + y_0^n + z_0^n\right) \right|$$
$$\leq \frac{n(n-1)(x_0^{n-2} + y_0^{n-2} + z_0^{n-2})R^2}{2}.$$

Remark 3.8 For any function $f : \bar{\mathcal{B}}(\mathcal{C}, R) \to \mathbb{R}$, we can apply the structure mentioned in the above example to obtain a Lipschitz constant $\mathcal{L} = \sup_{z \in \bar{\mathcal{B}}(\mathcal{C}, R)} \|\nabla f(z)\|$ with respect to the Euclidian norm $\|\cdot\|$, provided that the gradient vector of f exists everywhere in $\bar{\mathcal{B}}(\mathcal{C}, R)$ and also $\mathcal{L} < \infty$.

Remark 3.9 In Theorems 3.3 and 3.4, if we consider that $\frac{\partial f}{\partial \rho} : \bar{\mathcal{B}}(\mathcal{C}, R) \to \mathbb{R}$ is $\mathcal{L}$-Lipschitz and $f : \bar{\mathcal{B}}(\mathcal{C}, R) \to \mathbb{R}$ is integrable, then by (13) and (19) we can obtain (the details are

omitted)

$$\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\,d\sigma - \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\,dV \right| \le \frac{1}{5}\mathcal{L}R^2,$$

and

$$\left| \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\,dV - f(\mathcal{C}) \right| \le \frac{7}{10}\mathcal{L}R^2.$$

4 Bounded functions

In the last section we investigate trapezoid and mid-point type inequalities where considered functions are bounded.

Theorem 4.1 *Suppose that $\mathcal{V} \subset \mathbb{R}^3$, $\bar{\mathcal{B}}(\mathcal{C},R) \subset \mathcal{V}^\circ$ and $f : \bar{\mathcal{B}}(\mathcal{C},R) \to \mathbb{R}$ has continuous partial derivatives with respect to the variables ρ, φ, and θ on $\bar{\mathcal{B}}(\mathcal{C},R)$ in spherical coordinates. If $\frac{\partial f}{\partial \rho}$ is bounded on $\bar{\mathcal{B}}(\mathcal{C},R)$, then*

$$\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\,d\sigma - \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\,dV \right|$$
$$\le \left(\frac{\mathcal{U}_B - \mathcal{L}_B + |\mathcal{L}_B + \mathcal{U}_B|}{8} \right) R, \tag{26}$$

where $\mathcal{L}_B$ and $\mathcal{U}_B$ are lower and upper bounds of $\frac{\partial f}{\partial \rho}$ on $\bar{\mathcal{B}}(\mathcal{C},R)$, respectively.

Proof Consider $\mathcal{U}_B$ and $\mathcal{L}_B$ as the upper and lower bounds of an arbitrary function g defined on a set $\mathcal{V} \subset \mathbb{R}^3$, respectively. Then for all $x,y,z \in \mathcal{V}$, we have

$$\mathcal{L}_B - \frac{\mathcal{L}_B + \mathcal{U}_B}{2} \le g(x,y,z) - \frac{\mathcal{L}_B + \mathcal{U}_B}{2} \le \mathcal{U}_B - \frac{\mathcal{L}_B + \mathcal{U}_B}{2},$$

which implies that

$$\left| g(x,y,z) - \frac{\mathcal{L}_B + \mathcal{U}_B}{2} \right| \le \frac{\mathcal{U}_B - \mathcal{L}_B}{2}, \tag{27}$$

for all $x,y,z \in \mathcal{V}$. On the other hand, from (13) we get

$$\int_0^{2\pi} \int_0^{\pi} \int_0^R \frac{\partial f}{\partial \rho}(a + \rho \cos\theta \sin\varphi, b + \rho \sin\theta \sin\varphi, c + \rho \cos\varphi)\rho^3 \sin\varphi\,d\rho\,d\varphi\,d\theta$$
$$- \int_0^{2\pi} \int_0^{\pi} \int_0^R \frac{\mathcal{L}_B + \mathcal{U}_B}{2}\rho^3 \sin\varphi\,d\rho\,d\varphi\,d\theta$$
$$= \int_0^{2\pi} \int_0^{\pi} R^3 f(a + R \cos\theta \sin\varphi, b + R \sin\theta \sin\varphi, c + R \cos\varphi)\,d\varphi\,d\theta$$
$$- 3 \int_0^{2\pi} \int_0^{\pi} \int_0^R f(a + \rho \cos\theta \sin\varphi, b + \rho \sin\theta \sin\varphi, c + \rho \cos\varphi)\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$$
$$- \pi R^4 \frac{\mathcal{L}_B + \mathcal{U}_B}{2}.$$

Now if in (27) we consider $g = \frac{\partial f}{\partial \rho}$, $\mathcal{V} = \bar{\mathcal{B}}(\mathcal{C}, R)$, and utilize Lemma 2.1, then we obtain that

$$
\left| R \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\,d\sigma - 3 \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\,dV - \pi R^4 \frac{\mathcal{L}_B + \mathcal{U}_B}{2} \right|
$$

$$
\leq \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \left| \frac{\partial f}{\partial \rho}(a + \rho\cos\theta\sin\varphi, b + \rho\sin\theta\sin\varphi, c + \rho\cos\varphi) - \frac{\mathcal{L}_B + \mathcal{U}_B}{2} \right|
$$

$$
\times \rho^3 \sin\varphi\,d\rho\,d\varphi\,d\theta
$$

$$
\leq \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \left| \frac{\mathcal{U}_B - \mathcal{L}_B}{2} \right| \rho^3 \sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{\mathcal{U}_B - \mathcal{L}_B}{2}\pi R^4.
$$

Finally, by the use of the triangle inequality and dividing the result by $4\pi R^3$, we obtain inequality (26). $\qquad\qquad\square$

Theorem 4.2 *Suppose that $\mathcal{V} \subset \mathbb{R}^3$, $\bar{\mathcal{B}}(\mathcal{C}, R) \subset \mathcal{V}^{\circ}$ and $f : \bar{\mathcal{B}}(\mathcal{C}, R) \to \mathbb{R}$ has continuous partial derivatives with respect to the variables ρ, φ, and θ on $\bar{\mathcal{B}}(\mathcal{C}, R)$ in spherical coordinates. If $\frac{\partial f}{\partial \rho}$ is bounded on $\bar{\mathcal{B}}(\mathcal{C}, R)$, then*

$$
\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\,d\sigma - f(\mathcal{C}) \right| \leq \left(\frac{\mathcal{U}_B - \mathcal{L}_B + |\mathcal{L}_B + \mathcal{U}_B|}{2} \right) R, \tag{28}
$$

where $\mathcal{L}_B$ and $\mathcal{U}_B$ are lower and upper bounds of $\frac{\partial f}{\partial \rho}$ on $\bar{\mathcal{B}}(\mathcal{C}, R)$, respectively.

Proof Consider $\mathcal{L}_B$ and $\mathcal{U}_B$ as the upper and lower bounds of $\frac{\partial f}{\partial \rho}$. By (19), the following relations hold:

$$
\int_0^{2\pi} \int_0^{\pi} \int_0^{R} \left[\frac{\partial f}{\partial \rho}(a + \rho\cos\theta\sin\varphi, b + \rho\sin\theta\cos\varphi, c + \rho\cos\varphi) - \frac{\mathcal{L}_B + \mathcal{U}_B}{2} \right]
$$

$$
\times \sin\varphi\,d\rho\,d\varphi\,d\theta
$$

$$
= \int_0^{2\pi} \int_0^{\pi} f(a + R\cos\theta\sin\varphi, b + R\sin\theta\sin\varphi, c + R\cos\varphi)\sin\varphi\,d\varphi\,d\theta
$$

$$
- \int_0^{2\pi} \int_0^{\pi} f(\mathcal{C})\sin\varphi\,d\varphi\,d\theta - \frac{\mathcal{L}_B + \mathcal{U}_B}{2}R \int_0^{2\pi} \int_0^{\pi} \sin\varphi\,d\varphi\,d\theta
$$

$$
= \frac{1}{R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\,d\sigma - 4\pi f(\mathcal{C}) - 2\pi(\mathcal{L}_B + \mathcal{U}_B)R.
$$

This implies that

$$
\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\,d\sigma - f(\mathcal{C}) - \frac{\mathcal{L}_B + \mathcal{U}_B}{2}R \right|
$$

$$
\leq \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \left| \frac{\partial f}{\partial \rho}(a + \rho\cos\theta\sin\varphi, b + \rho\sin\theta\sin\varphi, c + \rho\cos\varphi) - \frac{\mathcal{L}_B + \mathcal{U}_B}{2} \right|
$$

$$
\times \sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{\mathcal{U}_B - \mathcal{L}_B}{2}R.
$$

Finally, by using the triangle inequality, we get

$$\left| \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\, d\sigma - f(\mathcal{C}) \right| \le \left(\frac{\mathcal{U}_B - \mathcal{L}_B + |\mathcal{L}_B + \mathcal{U}_B|}{2} \right) R. \qquad \square$$

Remark 4.3 If $f : \bar{\mathcal{B}}(\mathcal{C},R) \to \mathbb{R}$ is a convex function and bounded from above on $\bar{\mathcal{B}}(\mathcal{C},R)$ ($\mathcal{U}_B$ exists), then f is bounded on $\bar{\mathcal{B}}(\mathcal{C},R)$ because for an arbitrary $X \in \bar{\mathcal{B}}(0,R)$ and $\mathcal{C} = \frac{1}{2}(X + \mathcal{C}) + \frac{1}{2}(-X + \mathcal{C})$, from the convexity of f we have $2f(\mathcal{C}) - f(-X + \mathcal{C}) \le f(X + \mathcal{C})$. This implies that $2f(\mathcal{C}) - \mathcal{U}_B \le f(X + \mathcal{C})$ where $X + \mathcal{C}$ and $-X + \mathcal{C}$ belong to $\bar{\mathcal{B}}(\mathcal{C},R)$. Now it is enough to set $\mathcal{L}_B = 2f(\mathcal{C}) - \mathcal{U}_B$.

So if $\frac{\partial f}{\partial \rho} : \bar{\mathcal{B}}(\mathcal{C},R) \to \mathbb{R}$ is convex and bounded from above, then by (26), (28), and (1), the following inequalities hold:

$$0 \le \frac{1}{4\pi R^2} \iint_{\sigma(\mathcal{C},R)} f(x,y,z)\, d\sigma - \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\, dV$$
$$\le \left(\frac{\mathcal{U}_B - \mathcal{L}_B + |\mathcal{L}_B + \mathcal{U}_B|}{8} \right) R,$$

and

$$0 \le \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{\mathcal{B}}(\mathcal{C},R)} f(x,y,z)\, dV - f(\mathcal{C}) \le \left(\frac{\mathcal{U}_B - \mathcal{L}_B + |\mathcal{L}_B + \mathcal{U}_B|}{2} \right) R,$$

where $\mathcal{L}_B$ and $\mathcal{U}_B$ are lower and upper bounds of $\frac{\partial f}{\partial \rho}$ on $\bar{\mathcal{B}}(\mathcal{C},R)$, respectively.

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References
1. Dragomir, S.S.: On Hadamard's inequality for the convex mappings defined on a ball in the space and applications. Math. Inequal. Appl. **3**(2), 177–187 (2000)
2. Kirmaci, U.S.: Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. Appl. Math. Comput. **147**(1), 137–146 (2004)
3. Dragomir, S.S., Agarwal, R.P.: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**, 91–95 (1998)
4. Guessab, A., Schmeisser, G.: Convexity results and sharp error estimates in approximate multivariate integration. Math. Comput. **73**, 1365–1384 (2004)
5. Guessab, A., Schmeisser, G.: Sharp integral inequalities of the Hermite–Hadamard type. J. Approx. Theory **115**, 260–288 (2002)
6. Bessenyei, M.: The Hermite–Hadamard inequality on simplices. Am. Math. Mon. **115**(4), 339–345 (2008)
7. Chen, Y.: Multi-dimensional Hadamard's inequalities. Tamkang J. Math. **43**(1), 1–10 (2012)

8. Dragomir, S.S.: On Hadamard's inequality on a disk. J. Inequal. Pure Appl. Math. **1**(1), Article ID 2 (2000)
9. Hwang, D.-Y.: Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables. Appl. Math. Comput. **217**, 9598–9605 (2011)
10. Matłoka, M.: On Hadamard's inequality for *h*-convex function on a disk. Appl. Math. Comput. **235**, 118–123 (2014)
11. Neuman, E.: Inequalities involving multivariate convex functions II. Proc. Am. Math. Soc. **109**(4), 965–974 (1990)
12. Niculescu, C.P., Persson, L.E.: Convex Functions and Their Applications: A Contemporary Approach. CMS Books in Mathematics. Springer, Berlin (2006)
13. Nowicka, M., Witkowski, A.: Applications of the Hermite–Hadamard inequality. Math. Inequal. Appl. **19**(4), 1319–1334 (2016)
14. Pečarić, J., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings, and Statistical Applications. Academic Press, San Diego (1992)
15. Rostamian Delavar, M., Dragomir, S.S.: Weighted trapezoidal inequalities related to the area balance of a function with applications. Appl. Math. Comput. **340**, 5–14 (2019)
16. Rostamian Delavar, M., Dragomir, S.S.: On η-convexity. Math. Inequal. Appl. **20**, 203–216 (2017)
17. Rostamian Delavar, M., Dragomir, S.S., De La Sen, M.: Hermite–Hadamard's trapezoid and mid-point type inequalities on a disk. J. Inequal. Appl. **2019**, 105 (2019)
18. Varošanec, S.: On *h*-convexity. J. Math. Anal. Appl. **326**, 303–311 (2007)
19. Dragomir, S.S., Rassias, Th.M.: Ostrowski Type Inequalities and Applications in Numerical Integration. Kluwer Academic, Dordrecht (2002)
20. Ostrowski, A.M.: Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmitelwert. Comment. Math. Helv. **10**, 226–227 (1938)
21. Robert, A.W., Varbeg, D.E.: Convex Functions. Academic Press, New York (1973)