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Asymptotically almost periodic dynamics on delayed Nicholson-type system involving patch structure

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Abstract

This paper explores a delayed Nicholson-type system involving patch structure. Applying differential inequality techniques and the fluctuation lemma, we establish a new sufficient condition which guarantees the existence of positive asymptotically almost periodic solutions for the addressed system. The results of this article are completely new and supplement the previous publications.

Keywords: Nicholson-type system; Global attractivity; Patch structure; Asymptotically almost periodic solution

1 Introduction

As we all know, periodicity is important in real surroundings and the world, but almost periodicity is always more accurate, more realistic, and more general than periodicity when adding varied environmental factors. In comparison with periodic effects, almost periodic effects are more frequent in lots of real world applications [1–4]. In particular, the existence and global stability of almost periodic solutions for the famous scalar Nicholson's blowflies equation

$$x'(t) = -a(t)x(t) + \sum_{j=1}^{m} \beta_j(t)x\big(t-\tau_j(t)\big)e^{-\gamma_j(t)x(t-\tau_j(t))} \tag{1.1}$$

and the Nicholson's blowflies systems with patch structure

$$\begin{aligned} x_i'(t) = {} & -a_{ii}(t)x_i(t) + \sum_{j=1,j\neq i}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{m} \beta_{ij}(t) \\ & \times x_i\big(t-\tau_{ij}(t)\big)e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))}, \quad i \in Q := \{1,2,\ldots,n\}, \end{aligned} \tag{1.2}$$

have been extensively investigated in previous studies [5, 6] and [7], respectively. It is easy to know that scalar Nicholson's blowflies Eq. (1.1) is a special case of Nicholson's blowflies system (1.2), where $x_i(t)$ denotes the density of the ith-population at time t, $a_{ij}(t)$ $(i \neq j)$ is the rate of the population moving from class j to class i at time t, $a_{ii}(t)$ is the coefficient of instantaneous loss (which integrates both the death rate and the dispersal rates of the population in class i moving to the other classes), $\beta_{ij}(t)x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))}$ is the birth function, $\beta_{ij}(t)$ is the birth rate for the species, $\frac{1}{\gamma_{ij}(t)}$ is the ith-population reproducing at its maximum rate, and $\tau_{ij}(t)$ is the generation time of the ith-population at time t. For the feedback function xe^{-x} and its derivative $\frac{1-x}{e^x}$, the author in [8] pointed out that there exist two fixed positive numbers κ and $\tilde{\kappa}$ such that

$$\begin{aligned} &\kappa \approx 0.7215355, \qquad \widetilde{\kappa} \approx 1.342276, \qquad \frac{1-\kappa}{e^{\kappa}} = \frac{1}{e^2}, \\ &\sup_{x\geq\kappa}\left|\frac{1-x}{e^x}\right| = \frac{1}{e^2}, \qquad \kappa e^{-\kappa} = \widetilde{\kappa}e^{-\widetilde{\kappa}}. \end{aligned} \tag{1.3}$$

It is worth noting that the global exponential stability of almost periodic solutions of (1.1) has been shown in [5, 6] under the restriction that the almost periodic solution exists in a small interval $[\kappa, \widetilde{\kappa}] \approx [0.7215355, 1.342276]$, and the global exponential stability of (1.2) has been established in [7] where the authors adopted the restraint that the almost periodic solution exists in a small domain

$$\underbrace{[\kappa,\widetilde{\kappa}] \times \cdots \times [\kappa,\widetilde{\kappa}]}_{n} = \underbrace{[0.7215355, 1.342276] \times \cdots \times [0.7215355, 1.342276]}_{n}. \tag{1.4}$$

Obviously, the above restriction and restraint do not accord with the biological significance of the population models.

On the other hand,

$$\gamma_{ij}(t) \geq 1 \quad \text{for all } t \in \mathbb{R}, i \in Q, j \in I := \{1, 2, \dots, m\}, \tag{1.5}$$

has been made as the crucial assumption in [5–7]. It should be mentioned that the stability of a class of delayed nonlinear density-dependent mortality Nicholson's blowflies model has been investigated in [9–12] without assumption (1.5), when the maximum reproducing rate is not limited (i.e. $\frac{1}{\gamma_{ij}(t)}$ maybe sufficiently large). However, there is no research work on the global exponential stability of almost periodic solutions for Nicholson's blowflies Eq. (1.1) without assumption (1.5) and avoiding $[\kappa, \widetilde{\kappa}]$ as the existence interval for almost periodic solutions. In particular, to the best of our knowledge, there has not yet been research work on the global stability of almost periodic solutions of Nicholson's blowflies systems with patch structure and nonlinear density-dependent mortality terms when the almost periodic solutions do not belong to the above domain (1.4).

Regarding the above discussions, in this paper, without adopting $\underbrace{[\kappa,\widetilde{\kappa}] \times \cdots \times [\kappa,\widetilde{\kappa}]}_{n}$ as the existence domain of almost periodic solutions, we establish the existence and global exponential stability of positive almost periodic solutions for Nicholson's blowflies systems involving patch structure and nonlinear density-dependent mortality terms. Our results improve and complement some existing ones in the recent publications [5–7, 12], and its effectiveness is demonstrated by some numerical examples.

This paper is organized as follows: In Sect. 2, some necessary definitions, lemmas, and assumptions are presented. In Sect. 3, the existence and global attractivity of positive asymptotically almost periodic solutions are demonstrated by virtue of some differential inequalities and analytic techniques. To verify our theoretical results, a numerical experiment is carried out in Sect. 4. Conclusions are drawn in Sect. 5.

2 Preliminary results

Throughout this paper, it will be assumed that

$$\sigma_i = \max_{j\in I} \sup_{t\in[t_0,+\infty)} \tau_{ij}(t) > 0, \qquad \liminf_{t\to+\infty}\left[a_{ii}(t) - \sum_{j=1,j\neq i}^{n} a_{ij}(t)\right] > 0, \tag{2.1}$$

$$\gamma^- = \min_{i\in Q} \liminf_{t\to+\infty} \gamma_{ij}(t) > 0, \qquad \limsup_{t\to+\infty} \gamma_{ij}(t) \le 1, \quad i\in Q, j\in I. \tag{2.2}$$

For $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, define

$$|x| = \bigl(|x_1|,\dots,|x_n|\bigr), \qquad \|x\| = \max_{i\in Q} |x_i|.$$

Let $\mathbb{R}^+ = [0,+\infty)$ and $C_+ = \prod_{i=1}^n C([-\sigma_i,0],\mathbb{R}^+)$. For $\mathbb{J},\mathbb{J}_1,\mathbb{J}_2 \subseteq \mathbb{R}$, denote

$$W_0\bigl(\mathbb{R}^+,\mathbb{J}\bigr) = \Bigl\{\nu : \nu \in C\bigl(\mathbb{R}^+,\mathbb{J}\bigr), \lim_{t\to+\infty} \nu(t) = 0\Bigr\},$$

and let $\mathrm{BC}(\mathbb{J}_1,\mathbb{J}_2)$ be the set of bounded and continuous functions from $\mathbb{J}_1$ to $\mathbb{J}_2$.

Definition 2.1 (see [1, 2]) A subset P of $\mathbb{R}$ is said to be relatively dense in $\mathbb{R}$ if there exists a constant $l > 0$ such that $[t,t+l]\cap P \neq \emptyset$ ($t\in\mathbb{R}$). $u \in \mathrm{BC}(\mathbb{R},\mathbb{J})$ is almost periodic on $\mathbb{R}$ if, for any $\epsilon > 0$, the set $T(u,\epsilon) = \{\delta : |u(t+\delta) - u(t)| < \epsilon, \forall t\in\mathbb{R}\}$ is relatively dense.

Definition 2.2 (see [1, 2]) $u \in C(\mathbb{R}^+,\mathbb{J})$ is asymptotically almost periodic if there exist an almost periodic function h and a continuous function $g \in W_0(\mathbb{R}^+,\mathbb{J})$ such that $u = h + g$.

For $\mathbb{J}\subseteq\mathbb{R}$, we denote the set of almost periodic functions from $\mathbb{R}$ to $\mathbb{J}$ by $\mathrm{AP}(\mathbb{R},\mathbb{J})$. The set of asymptotic almost periodic functions will be represented by $\mathrm{AAP}(\mathbb{R},\mathbb{J})$. In addition, $\mathrm{AP}(\mathbb{R},\mathbb{J})$ is a proper subspace of $\mathrm{AAP}(\mathbb{R},\mathbb{J})$ [1, 2].

Remark 2.1 (see [1, p. 64, Remark 5.16]) The decomposition given in Definition 2.2 is unique.

Hereafter, let $a_{ii}, \gamma_{ij} \in \mathrm{AAP}(\mathbb{R},(0,+\infty))$, a_{ij} $(i\neq j)$, $\beta_{ij}, \tau_{ij} \in \mathrm{AAP}(\mathbb{R},\mathbb{R}^+)$, and

$$a_{ij} = a_{ij}^h + a_{ij}^g, \qquad \beta_{ij} = \beta_{ij}^h + \beta_{ij}^g, \qquad \gamma_{ij} = \gamma_{ij}^h + \gamma_{ij}^g, \qquad \tau_{ij} = \tau_{ij}^h + \tau_{ij}^g, \tag{2.3}$$

where $a_{ii}^h, \gamma_{ij}^h \in \mathrm{AP}(\mathbb{R},(0,+\infty))$, a_{ij}^h $(i\neq j)$, $\beta_{ij}^h, \tau_{ij}^h \in \mathrm{AP}(\mathbb{R},\mathbb{R}^+)$, $a_{ij}^g, \beta_{ij}^g, \gamma_{ij}^g, \tau_{ij}^g \in W_0(\mathbb{R}^+,\mathbb{R}^+)$, $\liminf_{t\to+\infty} \beta_{ij}(t) > 0$, and $i\in Q, j\in I$.

Then, we need to introduce a nonlinear almost periodic differential system

$$\begin{aligned} x_i'(t) &= -a_{ii}^h(t)x_i(t) \\ &\quad + \sum_{j=1,j\neq i}^{n} a_{ij}^h(t)x_j(t) + \sum_{j=1}^{m} \beta_{ij}^h(t)x_i\bigl(t-\tau_{ij}^h(t)\bigr)e^{-\gamma_{ij}^h(t)x_i(t-\tau_{ij}^h(t))}, \end{aligned} \tag{1.2^h}$$

with the following admissible initial conditions:

$$x_i(t_0+\theta)=\varphi_i(\theta),\quad \theta\in[-\sigma_i,0],\varphi=(\varphi_1,\dots,\varphi_n)\in C_+ \text{ and } \varphi_i(0)>0, \tag{2.4}$$

where $i\in Q$.

We set $x(t;t_0,\varphi)$ for a solution of (1.2) with initial value problem (2.4), and the maximal right-interval of existence of $x(t;t_0,\varphi)$ is marked by $[t_0,\eta(\varphi))$. Then, the existence and uniqueness of $x(t;t_0,\varphi)$ are easy to obtain from [8].

Lemma 2.1 (see [12, Lemma 2.1]) *Let A and δ be constants and satisfy that*

$$A>1,\qquad e<\frac{1}{\delta}\le e^2\quad \textit{and}\quad \delta=Ae^{-A}. \tag{2.5}$$

Then $\delta A>\frac{1}{e}$.

Lemma 2.2 *$x(t;t_0,\varphi)$ is positive and bounded on $[t_0,\eta(\varphi))$, and $\eta(\varphi)=+\infty$.*

Proof First, we state that

$$x_i(t)>0\quad\text{for all } t\in[t_0,\eta(\varphi)), i\in Q. \tag{2.6}$$

Otherwise, we can find $i_0\in Q$ and $\bar{t}_{i_0}\in(t_0,\eta(\varphi))$ to satisfy that

$$x_{i_0}(\bar{t}_{i_0})=0,\qquad x_j(t)>0\quad\text{for all } t\in[t_0,\bar{t}_{i_0}), j\in Q.$$

From the facts that

$$\begin{cases} x_{i_0}(t_0)=\varphi_{i_0}(0)>0,\\ x'_{i_0}(t)\ge -a_{i_0i_0}(t)x_{i_0}(t)+\sum_{j=1}^m\beta_{i_0j}(t)x_{i_0}(t-\tau_{i_0j}(t))e^{-\gamma_{i_0j}(t)x_{i_0}(t-\tau_{i_0j}(t))},\quad t\in[t_0,\bar{t}_{i_0}),\end{cases}$$

we obtain

$$\begin{aligned} 0 &= x_{i_0}(\bar{t}_{i_0})\\ &\ge e^{-\int_{t_0}^{\bar{t}_{i_0}}a_{i_0i_0}(u)\,du}x_{i_0}(t_0)+e^{-\int_{t_0}^{\bar{t}_{i_0}}a_{i_0i_0}(u)\,du}\\ &\quad\times\int_{t_0}^{\bar{t}_{i_0}}e^{\int_{t_0}^{s}a_{i_0i_0}(v)\,dv}\sum_{j=1}^m\beta_{i_0j}(s)x_{i_0}\big(s-\tau_{i_0j}(s)\big)e^{-\gamma_{i_0j}(s)x_{i_0}(s-\tau_{i_0j}(s))}\,ds\\ &> 0,\end{aligned}$$

which is a contradiction and completes the above statement.

Now we evidence that $\eta(\varphi)=+\infty$. For all $t\in[t_0,\eta(\varphi))$, $i\in Q$, we define $y_i(t)=\max_{t_0-\sigma_i\le s\le t}x_i(s)$ and $y(t)=\max_{i\in Q}y_i(t)$, we gain

$$\begin{aligned} x_i'(t) &\le \sum_{j=1,j\ne i}^n\alpha_{ij}(t)x_j(t)+\sum_{j=1}^m\beta_{ij}(t)x_i\big(t-\tau_{ij}(t)\big)\\ &\le\left[\sum_{j=1,j\ne i}^n\alpha_{ij}(t)+\sum_{j=1}^m\beta_{ij}(t)\right]y(t)\end{aligned}$$

and

$$x_i(t) \le x_i(t_0) + \int_{t_0}^{t} \left[\sum_{j=1,j\neq i}^{n} \alpha_{ij}(v) + \sum_{j=1}^{m} \beta_{ij}(v) \right] y(v)\,dv,$$
$$\le \|\varphi\| + \int_{t_0}^{t} \left[\sum_{j=1,j\neq i}^{n} \alpha_{ij}(v) + \sum_{j=1}^{m} \beta_{ij}(v) \right] y(v)\,dv,$$

which suggests that

$$y(t) \le \|\varphi\| + \int_{t_0}^{t} \left[\sum_{j=1,j\neq i}^{n} \alpha_{ij}(v) + \sum_{j=1}^{m} \beta_{ij}(v) \right] y(v)\,dv.$$

Hence, by the Gronwall–Bellman inequality, we obtain

$$x_i(t) \le y_i(t) \le y(t) \le \|\varphi\| e^{\int_{t_0}^{t} [\sum_{j=1,j\neq i}^{n} \alpha_{ij}(v) + \sum_{j=1}^{m} \beta_{ij}(v)]\,dv}, \quad \forall t \in [t_0, \eta(\varphi)), i \in Q.$$

It follows from Theorem 2.3.1 in [13] that $\eta(\varphi) = +\infty$, and then $x_i(t) > 0$ for all $t \in [t_0, +\infty)$.

Next, we demonstrate that $x(t)$ is bounded on $[t_0, +\infty)$. For $t \in [t_0 - \sigma_i, +\infty)$ and $i \in Q$, we define

$$M_i(t) = \max\Bigl\{ \xi : \xi \le t, x_i(\xi) = \max_{t_0 - \sigma_i \le s \le t} x_i(s) \Bigr\}.$$

Suppose that $x(t)$ is unbounded on $[t_0, +\infty)$. Then we can choose $i^* \in Q$ and a strictly monotone increasing sequence $\{\zeta_n\}_{n=1}^{+\infty}$ such that $\lim_{n\to+\infty} \zeta_n = +\infty$,

$$x_{i^*}\bigl(M_{i^*}(\zeta_n)\bigr) = \max_{j\in Q}\bigl\{ x_j\bigl(M_j(\zeta_n)\bigr)\bigr\}, \qquad \lim_{n\to+\infty} x_{i^*}\bigl(M_{i^*}(\zeta_n)\bigr) = +\infty, \tag{2.7}$$

and then

$$\lim_{n\to+\infty} M_{i^*}(\zeta_n) = +\infty. \tag{2.8}$$

It follows that there exists $n^* > 0$ satisfying

$$M_{i^*}(\zeta_n) > t_0, \qquad x_{i^*}\bigl(M_{i^*}(\zeta_n)\bigr) > \sup_{t\in[t_0,+\infty)} \frac{\sum_{j=1}^{m} \frac{\beta_{i^*j}(t)}{\gamma_{i^*j}(t)} \frac{1}{e}}{[a_{i^*i^*}(t) - \sum_{j=1,j\neq i^*}^{n} a_{i^*j}(t)]} \tag{2.9}$$

for all $n > n^*$.

According to the fact $\sup_{u\ge 0} ue^{-u} = \frac{1}{e}$, it follows from (1.2) and (2.1) that, for all $n > n^*$,

$$\begin{aligned}
0 &\le x'_{i^*}\bigl(M_{i^*}(\zeta_n)\bigr) \\
&= -a_{i^*i^*}\bigl(M_{i^*}(\zeta_n)\bigr) x_{i^*}\bigl(M_{i^*}(\zeta_n)\bigr) + \sum_{j=1,j\neq i}^{n} a_{i^*j}\bigl(M_{i^*}(\zeta_n)\bigr) x_j\bigl(M_{i^*}(\zeta_n)\bigr) \\
&\quad + \sum_{j=1}^{m} \frac{\beta_{i^*j}(M_{i^*}(\zeta_n))}{\gamma_{i^*j}(M_{i^*}(\zeta_n))} \gamma_{i^*j}\bigl(M_{i^*}(\zeta_n)\bigr) x_{i^*}\bigl(M_{i^*}(\zeta_n) - \tau_{i^*j}\bigl(M_{i^*}(\zeta_n)\bigr)\bigr)
\end{aligned}$$

$$\times e^{-\gamma_{i^*j}(M_{i^*}(\zeta_n))x_{i^*}(M_{i^*}(\zeta_n)-\tau_{i^*j}(M_{i^*}(\zeta_n)))}$$
$$\leq \left[-a_{i^*i^*}\big(M_{i^*}(\zeta_n)\big) + \sum_{j=1,j\neq i}^{n} a_{i^*j}\big(M_{i^*}(\zeta_n)\big)\right]x_{i^*}\big(M_{i^*}(\zeta_n)\big) + \sum_{j=1}^{m} \frac{\beta_{i^*j}(M_{i^*}(\zeta_n))}{\gamma_{i^*j}(M_{i^*}(\zeta_n))}\frac{1}{e}$$

and

$$x_{i^*}\big(M_{i^*}(\zeta_n)\big) \leq \frac{\sum_{j=1}^{m} \frac{\beta_{i^*j}(M_{i^*}(\zeta_n))}{\gamma_{i^*j}(M_{i^*}(\zeta_n))}\frac{1}{e}}{a_{i^*i^*}(M_{i^*}(\zeta_n)) - \sum_{j=1,j\neq i^*}^{n} a_{i^*j}(M_{i^*}(\zeta_n))},$$

which contradicts (2.9) and suggests that $x(t)$ is bounded on $[t_0, +\infty)$. □

Lemma 2.3 *Assume that*

$$\liminf_{t\to+\infty}\left[\sum_{j=1,j\neq i}^{n}\frac{a_{ij}(t)}{a_{ii}(t)} + \sum_{j=1}^{m}\frac{\beta_{ij}(t)}{a_{ii}(t)}\right] > 1, \tag{2.10}$$

$$e < \liminf_{t\to+\infty}\left[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}}{a_{ii}(t) - \sum_{j=1,j\neq i}^{n} a_{ij}(t)}\right] \leq \limsup_{t\to+\infty}\left[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}}{a_{ii}(t) - \sum_{j=1,j\neq i}^{n} a_{ij}(t)}\right] < e^2, \tag{2.11}$$

and

$$\left.\begin{array}{l} \liminf_{t\to+\infty} \ln\left(\frac{\sum_{j=1}^{m}\beta_{ij}(t)}{a_{ii}(t)-\sum_{j=1,j\neq i}^{n}a_{ij}(t)}\right) > \frac{\kappa}{\gamma^-} \\ \frac{\liminf_{t\to+\infty}\left(\frac{\sum_{j=1}^{m}\beta_{ij}(t)}{a_{ii}(t)-\sum_{j=1,j\neq i}^{n}a_{ij}(t)}\right)}{\max_{1\leq i\leq n}\limsup_{t\to+\infty}\left[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}}{a_{ii}(t)-\sum_{j=1,j\neq i}^{n}a_{ij}(t)}\right]} > \frac{\kappa}{\gamma^-} \end{array}\right\}, \quad i \in Q, \tag{2.12}$$

hold. Then

$$\frac{\kappa}{\gamma^-} < l := \liminf_{t\to+\infty} x_i(t;t_0,\varphi) \leq L := \limsup_{t\to+\infty} x_i(t;t_0,\varphi) < A, \quad i \in Q, \tag{2.13}$$

where κ *is defined in* (1.3),

$$\delta = \frac{1}{\max_{1\leq i\leq n}\limsup_{t\to+\infty}\left[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}}{a_{ii}(t)-\sum_{j=1,j\neq i}^{n}a_{ij}(t)}\right]}, \qquad A > 1, \quad \textit{and} \quad \delta = Ae^{-A}.$$

Proof From Lemmas 2.1 and 2.2, we can designate $i^l, i^L \in Q$ such that

$$\begin{aligned} 0 &\leq l = \liminf_{t\to+\infty} x_{i^l}(t) = \min_{i\in Q}\liminf_{t\to+\infty} x_i(t) \\ &\leq L = \limsup_{t\to+\infty} x_{i^L}(t) = \max_{i\in Q}\limsup_{t\to+\infty} x_i(t) < +\infty. \end{aligned}$$

By the fluctuation lemma [14, Lemma A.1], one can select a sequence $\{t_k^*\}_{k=1}^{+\infty}$ satisfying

$$\lim_{k\to+\infty} t_k^* = +\infty, \qquad \lim_{k\to+\infty} x_{i^L}\big(t_k^*\big) = L = \limsup_{t\to+\infty} x_{i^L}(t), \qquad \lim_{k\to+\infty} x'_{i^L}\big(t_k^*\big) = 0. \tag{2.14}$$

Now, we show that $l > 0$. By way of contradiction, we assume that

$$\liminf_{t\to+\infty} x_{i^l}(t) = \min_{i\in Q}\liminf_{t\to+\infty} x_i(t) = 0. \tag{2.15}$$

Let

$$\omega_i(t) = \max\Big\{\xi : \xi \le t, x_i(\xi) = \min_{t_0\le s\le t} x_i(s)\Big\}$$

for each $t \ge t_0$. From (2.15), we can choose $i^{**} \in Q$ and a strictly monotone increasing sequence $\{\xi_n\}_{n=1}^{+\infty}$ such that $\lim_{n\to+\infty}\xi_n = +\infty$,

$$x_{i^{**}}\big(\omega_{i^{**}}(\xi_n)\big) = \min_{j\in Q}\big\{x_j\big(\omega_j(\xi_n)\big)\big\}, \qquad \lim_{n\to+\infty} x_{i^{**}}\big(\omega_{i^{**}}(\xi_n)\big) = 0, \tag{2.16}$$

and then

$$\lim_{n\to+\infty}\omega_{i^{**}}(\xi_n) = +\infty. \tag{2.17}$$

According to (2.17), one can find that there exists $n^{**} > 0$ such that, for $n > n^{**}$ and $j \in I$,

$$\begin{aligned}
&\omega_{i^{**}}(\xi_n) > t_0 + \sigma_{i^{**}},\\
&0 \ge x'_{i^{**}}\big(\omega_{i^{**}}(\xi_n)\big)\\
&\quad = -a_{i^{**}i^{**}}\big(\omega_{i^{**}}(\xi_n)\big)x_{i^{**}}\big(\omega_{i^{**}}(\xi_n)\big) + \sum_{j=1,j\ne i^{**}}^{n} a_{i^{**}j}\big(\omega_{i^{**}}(\xi_n)\big)x_j\big(\omega_{i^{**}}(\xi_n)\big)\\
&\qquad + \sum_{j=1}^{m}\beta_{i^{**}j}\big(\omega_{i^{**}}(\xi_n)\big)x_{i^{**}}\big(\omega_{i^{**}}(\xi_n) - \tau_{i^{**}j}\big(\omega_{i^{**}}(\xi_n)\big)\big)\\
&\qquad \times e^{-\gamma_{i^{**}j}(\omega_{i^{**}}(\xi_n))x_{i^{**}}(\omega_{i^{**}}(\xi_n)-\tau_{i^{**}j}(\omega_{i^{**}}(\xi_n)))}
\end{aligned}$$

and

$$\begin{aligned}
&a_{i^{**}i^{**}}\big(\omega_{i^{**}}(\xi_n)\big)x_{i^{**}}\big(\omega_{i^{**}}(\xi_n)\big)\\
&\quad \ge \sum_{j=1}^{m}\beta_{i^{**}j}\big(\omega_{i^{**}}(\xi_n)\big)x_{i^{**}}\big(\omega_{i^{**}}(\xi_n) - \tau_{i^{**}j}\big(\omega_{i^{**}}(\xi_n)\big)\big)\\
&\qquad \times e^{-\gamma_{i^{**}j}(\omega_{i^{**}}(\xi_n))x_{i^{**}}(\omega_{i^{**}}(\xi_n)-\tau_{i^{**}j}(\omega_{i^{**}}(\xi_n)))}, \quad n > n^{**},
\end{aligned}$$

which together with (2.16) and the fact that $\liminf_{t\to+\infty}\beta_{i^{**}j}(t) > 0$ gives

$$\lim_{n\to+\infty} x_{i^{**}}\big(\omega_{i^{**}}(\xi_n) - \tau_{i^{**}j}\big(\omega_{i^{**}}(\xi_n)\big)\big) = 0, \quad j \in Q. \tag{2.18}$$

Note that

$$1 \ge \sum_{j=1,j\ne i^{**}}^{n}\frac{a_{i^{**}j}(\omega_{i^{**}}(\xi_n))x_j(\omega_{i^{**}}(\xi_n))}{a_{i^{**}i^{**}}(\omega_{i^{**}}(\xi_n))x_{i^{**}}(\omega_{i^{**}}(\xi_n))}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\frac{\beta_{i^{**}j}(\omega_{i^{**}}(\xi_n))x_{i^{**}}(\omega_{i^{**}}(\xi_n)-\tau_{i^{**}j}(\omega_{i^{**}}(\xi_n)))}{a_{i^{**}i^{**}}(\omega_{i^{**}}(\xi_n))x_{i^{**}}(\omega_{i^{**}}(\xi_n))}\\
&\times e^{-\gamma_{i^{**}j}(\omega_{i^{**}}(\xi_n))x_{i^{**}}(\omega_{i^{**}}(\xi_n)-\tau_{i^{**}j}(\omega_{i^{**}}(\xi_n)))}\\
\geq{}& \sum_{j=1,j\neq i^{**}}^{n}\frac{a_{i^{**}j}(\omega_{i^{**}}(\xi_n))}{a_{i^{**}i^{**}}(\omega_{i^{**}}(\xi_n))}+\sum_{j=1}^{m}\frac{\beta_{i^{**}j}(\omega_{i^{**}}(\xi_n))}{a_{i^{**}i^{**}}(\omega_{i^{**}}(\xi_n))}\\
&\times e^{-\gamma_{i^{**}j}(\omega_{i^{**}}(\xi_n))x_{i^{**}}(\omega_{i^{**}}(\xi_n)-\tau_{i^{**}j}(\omega_{i^{**}}(\xi_n)))},\quad n>n^{**}.
\end{aligned}
$$

Letting $n\to+\infty$, it follows from (2.10) and (2.18) that

$$
\begin{aligned}
1&\geq \lim_{n\to+\infty}\left[\sum_{j=1,j\neq i^{**}}^{n}\frac{a_{i^{**}j}(\omega_{i^{**}}(\xi_n))}{a_{i^{**}i^{**}}(\omega_{i^{**}}(\xi_n))}+\sum_{j=1}^{m}\frac{\beta_{i^{**}j}(\omega_{i^{**}}(\xi_n))}{a_{i^{**}i^{**}}(\omega_{i^{**}}(\xi_n))}\right]\\
&\geq \liminf_{t\to+\infty}\left[\sum_{j=1,j\neq i^{**}}^{n}\frac{a_{i^{**}j}(t)}{a_{i^{**}i^{**}}(t)}+\sum_{j=1}^{m}\frac{\beta_{i^{**}j}(t)}{a_{i^{**}i^{**}}(t)}\right]\\
&>1,
\end{aligned}
$$

which is a contradiction. Hence, $l>0$.

Furthermore, from the asymptotically almost periodicity of (1.2), we can select a subsequence of $\{k\}_{k\geq1}$ such that $\lim_{k\to+\infty}a_{i^Lj}(t_k^*)$, $\lim_{k\to+\infty}\beta_{i^Lq}(t_k^*)$, $\lim_{k\to+\infty}\gamma_{i^Lq}(t_k^*)$, $\lim_{k\to+\infty}x_j(t_k^*)$, and $\lim_{k\to+\infty}x_{i^L}(t_k^*-\tau_{i^Lq}(t_k^*))$ exist for all $j\in Q$, $q\in I$. In addition, from (1.2) and (2.14), we have

$$
\begin{aligned}
0={}&\lim_{k\to+\infty}x'_{i^L}(t_k^*)\\
={}&-\lim_{k\to+\infty}a_{i^Li^L}(t_k^*)L+\sum_{j=1,j\neq i^L}^{n}\lim_{k\to+\infty}a_{i^Lj}(t_k^*)\lim_{k\to+\infty}x_j(t_k^*)\\
&+\sum_{j=1}^{m}\lim_{k\to+\infty}\frac{\beta_{i^Lj}(t_k^*)}{\gamma_{i^Lj}(t_k^*)}\lim_{k\to+\infty}\gamma_{i^Lj}(t_k^*)x_{i^L}(t_k^*-\tau_{i^Lj}(t_k^*))\\
&\times e^{-\lim_{k\to+\infty}\gamma_{i^Lj}(t_k^*)\lim_{k\to+\infty}x_{i^L}(t_k^*-\tau_{i^Lj}(t_k^*))}\\
\leq{}&-\lim_{k\to+\infty}a_{i^Li^L}(t_k^*)L+\sum_{j=1,j\neq i^L}^{n}\lim_{k\to+\infty}a_{i^Lj}(t_k^*)L+\sum_{j=1}^{m}\lim_{k\to+\infty}\frac{\beta_{i^Lj}(t_k^*)}{\gamma_{i^Lj}(t_k^*)}\frac{1}{e},
\end{aligned}
$$

which, together with the definitions of δ and A, entails that

$$
\begin{aligned}
L&\leq \lim_{k\to+\infty}\left[\frac{\sum_{j=1}^{m}\frac{\beta_{i^Lj}(t_k^*)}{\gamma_{i^Lj}(t_k^*)}\frac{1}{e}}{a_{i^Li^L}(t_k^*)-\sum_{j=1,j\neq i^L}^{n}a_{i^Lj}(t_k^*)}\right]\\
&\leq\frac{1}{e}\max_{1\leq i\leq n}\limsup_{t\to+\infty}\left[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}}{a_{ii}(t)-\sum_{j=1,j\neq i}^{n}a_{ij}(t)}\right]\\
&<A.
\end{aligned}
\tag{2.19}
$$

Finally, we show that $l>\frac{\kappa}{\gamma^-}$. Again from the fluctuation lemma [14, Lemma A.1] and the asymptotically almost periodicity of (1.2), we can pick a sequence $\{t_k^{**}\}_{k=1}^{+\infty}$ such that

$\lim_{k\to+\infty} t_k^{**} = +\infty$,

$$\lim_{k\to+\infty} x_{i^l}\left(t_k^{**}\right) = l = \liminf_{t\to+\infty} x_{i^l}(t) \quad \text{and} \quad \lim_{k\to+\infty} x_{i^l}'\left(t_k^{**}\right) = 0, \tag{2.20}$$

and $\lim_{k\to+\infty} a_{i^l j}(t_k^{**})$, $\lim_{k\to+\infty} \beta_{i^l q}(t_k^{**})$, $\lim_{k\to+\infty} \gamma_{i^l q}(t_k^{**})$, $\lim_{k\to+\infty} x_j(t_k^{**})$, $\lim_{k\to+\infty} x_{i^l}(t_k^{**} - \tau_{i^l q}(t_k^{**}))$ exist for all $j \in Q$, $q \in I$.

From the fact that

$$\lim_{k\to+\infty} \gamma_{i^l j}\left(t_k^{**}\right) \le 1 \quad \text{and} \quad \min_{[a,b]\subseteq[0,+\infty)} ue^{-u} = \min\left\{ae^{-a}, be^{-b}\right\},$$

one can see

$$\begin{aligned} &\lim_{k\to+\infty} x_{i^l}\left(t_k^{**} - \tau_{i^l j}\left(t_k^{**}\right)\right) e^{-\lim_{k\to+\infty} \gamma_{i^l j}(t_k^{**}) x_{i^l}(t_k^{**} - \tau_{i^l j}(t_k^{**}))} \\ &\quad \ge \lim_{k\to+\infty} x_{i^l}\left(t_k^{**} - \tau_{i^l j}\left(t_k^{**}\right)\right) e^{-\lim_{k\to+\infty} x_{i^l}(t_k^{**} - \tau_{i^l j}(t_k^{**}))} \\ &\quad \ge \min\left\{le^{-l}, Le^{-L}\right\}. \end{aligned} \tag{2.21}$$

Consequently, according to (2.20) and (2.21), we gain

$$\begin{aligned} 0 &= \lim_{k\to+\infty} x_{i^l}'\left(t_k^{**}\right) \\ &\ge -\lim_{k\to+\infty} a_{i^l i^l}\left(t_k^{**}\right) l + \sum_{j=1, j\ne i^l}^{n} \lim_{k\to+\infty} a_{i^l j}\left(t_k^{**}\right) l \\ &\quad + \sum_{j=1}^{m} \lim_{k\to+\infty} \beta_{i^l j}\left(t_k^{**}\right) x_{i^l}\left(t_k^{**} - \tau_{i^l j}\left(t_k^{**}\right)\right) e^{-x_{i^l}(t_k^{**} - \tau_{i^l j}(t_k^{**}))} \\ &\ge -\lim_{k\to+\infty} a_{i^l i^l}\left(t_k^{**}\right) l + \sum_{j=1, j\ne i^l}^{n} \lim_{k\to+\infty} a_{i^l j}\left(t_k^{**}\right) l \\ &\quad + \min\left\{le^{-l}, Le^{-L}\right\} \sum_{j=1}^{m} \lim_{k\to+\infty} \beta_{i^l j}\left(t_k^{**}\right). \end{aligned} \tag{2.22}$$

If $le^{-l} = \min\{le^{-l}, Le^{-L}\}$, (2.12) and (2.22) yield

$$\begin{aligned} l &\ge \ln\left(\lim_{k\to+\infty} \frac{\sum_{j=1}^{m} \beta_{i^l j}(t_k^{**})}{a_{i^l i^l}(t_k^{**}) - \sum_{j=1, j\ne i^l}^{n} a_{i^l j}(t_k^{**})}\right) \\ &\ge \liminf_{t\to+\infty} \ln\left(\frac{\sum_{j=1}^{m} \beta_{i^l j}(t)}{a_{i^l i^l}(t) - \sum_{j=1, j\ne i^l}^{n} a_{i^l j}(t)}\right) \\ &> \frac{\kappa}{\gamma^-}. \end{aligned} \tag{2.23}$$

If $Le^{-L} = \min\{le^{-l}, Le^{-L}\} < le^{-l}$, (2.19) indicates that

$$1 < L \le A, \qquad Le^{-L} \ge Ae^{-A},$$

together with (2.12) and (2.22), we obtain

$$
\begin{aligned}
l &\geq \frac{Ae^{-A}}{\lim_{k\to+\infty}\frac{a_{i^l i^l}(t_k^{**})-\sum_{j=1,j\neq i^l}^{n}a_{i^l j}(t_k^{**})}{\sum_{j=1}^{m}\beta_{i^l j}(t_k^{**})}} \\
&\geq \frac{\liminf_{t\to+\infty}(\frac{\sum_{j=1}^{m}\beta_{i^l j}(t)}{a_{i^l i^l}(t)-\sum_{j=1,j\neq i^l}^{n}a_{i^l j}(t)})}{\max_{1\leq i\leq n}\limsup_{t\to+\infty}[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}}{a_{ii}(t)-\sum_{j=1,j\neq i}^{n}a_{ij}(t)}]} \\
&> \frac{\kappa}{\gamma^-}.
\end{aligned} \tag{2.24}
$$

This finishes the proof of Lemma 2.3. □

Lemma 2.4 *Assume that all the assumptions adopted in Lemma 2.3 are satisfied, and let* $x^h(t) = x^h(t; t_0, \varphi)$ *be a solution of the initial value problem* $(1.2)^h$ *and* (2.4). *Then* $x^h(t)$ *is positive and bounded on* $[t_0, +\infty)$, $\frac{\kappa}{\gamma^-} < \liminf_{t\to+\infty} x_i^h(t) \leq \limsup_{t\to+\infty} x_i^h(t) < A$, *and there is* $t_\varphi^* \in [t_0, +\infty)$ *such that*

$$
\frac{\kappa}{\gamma^-} < x_i^h(t) < A \quad \text{for all } t \in [t_\varphi^*, +\infty), i \in Q. \tag{2.25}
$$

Proof From (2.1), (2.2), (2.10), (2.11), (2.12) and the definition of asymptotically almost periodic function, one can easily find that

$$
\begin{gathered}
\liminf_{t\to+\infty}\left[a_{ii}^h(t) - \sum_{j=1,j\neq i}^{n} a_{ij}^h(t)\right] > 0, \quad i \in Q, \\
\liminf_{t\to+\infty}\left[\sum_{j=1,j\neq i}^{n}\frac{a_{ij}^h(t)}{a_{ii}^h(t)} + \sum_{j=1}^{m}\frac{\beta_{ij}^h(t)}{a_{ii}^h(t)}\right] > 1, \quad i \in Q, \\
e < \liminf_{t\to+\infty}\left[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t)}}{a_{ii}^h(t)-\sum_{j=1,j\neq i}^{n}a_{ij}^h(t)}\right] \leq \limsup_{t\to+\infty}\left[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t)}}{a_{ii}^h(t)-\sum_{j=1,j\neq i}^{n}a_{ij}^h(t)}\right] < e^2, \quad i \in Q, \\
\liminf_{t\to+\infty}\ln\left(\frac{\sum_{j=1}^{m}\beta_{ij}^h(t)}{a_{ii}^h(t)-\sum_{j=1,j\neq i}^{n}a_{ij}^h(t)}\right) > \frac{\kappa}{\gamma^-}, \\
\frac{\liminf_{t\to+\infty}(\frac{\sum_{j=1}^{m}\beta_{ij}^h(t)}{a_{ii}^h(t)-\sum_{j=1,j\neq i}^{n}a_{ij}^h(t)})}{\max_{1\leq i\leq n}\limsup_{t\to+\infty}[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t)}}{a_{ii}^h(t)-\sum_{j=1,j\neq i}^{n}a_{ij}^h(t)}]} > \frac{\kappa}{\gamma^-}, \quad i \in Q,
\end{gathered}
$$

and

$$
\delta = \frac{1}{\max_{1\leq i\leq n}\limsup_{t\to+\infty}[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}}{a_{ii}(t)-\sum_{j=1,j\neq i}^{n}a_{ij}(t)}]} = \frac{1}{\max_{1\leq i\leq n}\limsup_{t\to+\infty}[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t)}}{a_{ii}^h(t)-\sum_{j=1,j\neq i}^{n}a_{ij}^h(t)}]}.
$$

Then, by applying a similar argument as Lemma 2.3, we can obtain

$$\frac{\kappa}{\gamma^-} < \liminf_{t\to+\infty} x_i^h(t) \le \limsup_{t\to+\infty} x_i^h(t) < A, \quad i \in Q,$$

which proves Lemma 2.4. □

Lemma 2.5 *Let assumptions adopted in Lemma 2.3 hold, and $x^h(t) = x^h(t;t_0,\varphi)$ be a solution of equation $(1.2)^h$ and (2.4). Then, for any $\epsilon > 0$, we can choose a relatively dense subset P_ϵ of $\mathbb{R}$ with the property that, for each $\delta \in P_\epsilon$, there exists $T = T(\delta) > 0$ satisfying*

$$\left\|x^h(t+\delta) - x^h(t)\right\| < \frac{\epsilon}{2} \quad \textit{for all } t > T.$$

Proof According to the fact

$$\limsup_{t\to+\infty}\left[\frac{\sum_{j=1}^m \frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t)}}{a_{ii}^h(t) - \sum_{j=1,j\neq i}^n a_{ij}^h(t)}\right] < e^2,$$

we have

$$\limsup_{t\to+\infty}\left[-a_{ii}^h(t) + \sum_{j=1,j\neq i}^n a_{ij}^h(t) + \sum_{j=1}^m \frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t)e^2}\right] < 0,$$

which implies that there exists a constant $0 < \varpi < \frac{\gamma^-}{2}$ such that

$$\limsup_{t\to+\infty}\left[-a_{ii}^h(t) + \sum_{j=1,j\neq i}^n a_{ij}^h(t) + \sum_{j=1}^m \frac{\beta_{ij}^h(t)}{(\gamma_{ij}^h(t) - \varpi)e^2}\right] < 0.$$

From (2.1), (2.2), and Lemma 2.4, we can choose positive constants $T_1 > \max\{0, t_\varphi^*\}$ and ζ to satisfy that

$$\gamma_{ij}^h(t)x_i^h\big(t - \tau_{ij}^h(t)\big) > \kappa, \qquad \gamma_{ij}^h(t) < 1 + \varpi, \quad \forall t \ge T_1, i \in Q,$$

and

$$\begin{aligned}
&-a_{ii}^h(t) + \sum_{j=1,j\neq i}^n a_{ij}^h(t) + \frac{1}{e^2}\sum_{j=1}^m \beta_{ij}^h(t) \\
&\quad\le -a_{ii}^h(t) + \sum_{j=1,j\neq i}^n a_{ij}^h(t) + \frac{1}{e^2}\sum_{j=1}^m \frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t) - \varpi} \\
&\quad< -\zeta.
\end{aligned}$$

This means there exist two constants $\eta > 0$ and $\lambda \in (0,1]$ such that, for $i \in Q$,

$$\sup_{t\in[T_1,+\infty)}\left\{-\left[a_{ii}^h(t) - \lambda\right] + \sum_{j=1,j\neq i}^n a_{ij}^h(t) + \sum_{j=1}^m \frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t) - \varpi}\frac{1}{e^2}e^{\lambda\sigma_i}\right\} < -\eta. \tag{2.26}$$

Define

$$x_i^h(t) \equiv x_i^h(t_0 - \sigma_i), \quad \text{for all } t \in (-\infty, t_0 - \sigma_i], i \in Q, \tag{2.27}$$

and

$$\begin{aligned}
&A_i(\delta, t) \\
&\quad = -\big[a_{ii}^h(t+\delta) - a_{ii}^h(t)\big]x_i^h(t+\delta) + \sum_{j=1, j\neq i}^{n} \big[a_{ij}^h(t+\delta) - a_{ij}^h(t)\big]x_j^h(t+\delta) \\
&\qquad + \sum_{j=1}^{m} \big[\beta_{ij}^h(t+\delta) - \beta_{ij}^h(t)\big]x_i^h\big(t+\delta - \tau_{ij}^h(t+\delta)\big)e^{-\gamma_{ij}^h(t+\delta)x_i^h(t+\delta-\tau_{ij}^h(t+\delta))} \\
&\qquad + \sum_{j=1}^{m} \beta_{ij}^h(t)\big[x_i^h\big(t+\delta - \tau_{ij}^h(t+\delta)\big)e^{-\gamma_{ij}^h(t+\delta)x_i^h(t+\delta-\tau_{ij}^h(t+\delta))} \\
&\qquad - x_i^h\big(t - \tau_{ij}^h(t) + \delta\big)e^{-\gamma_{ij}^h(t+\delta)x_i^h(t-\tau_{ij}^h(t)+\delta)}\big] \\
&\qquad + \sum_{j=1}^{m} \beta_{ij}^h(t)\big[x_i^h\big(t - \tau_{ij}^h(t) + \delta\big)e^{-\gamma_{ij}^h(t+\delta)x_i^h(t-\tau_{ij}^h(t)+\delta)} \\
&\qquad - x_i^h\big(t - \tau_{ij}^h(t) + \delta\big)e^{-\gamma_{ij}^h(t)x_i^h(t-\tau_{ij}^h(t)+\delta)}\big] \quad \text{for all } t \in \mathbb{R}, i \in Q.
\end{aligned} \tag{2.28}$$

In view of Lemma 2.4, one can see that $x^h(t)$ and the right-hand side of $(1.2)^h$ are bounded. It follows from (2.27) that $x^h(t)$ is uniformly continuous on $\mathbb{R}$. Therefore, for any $\epsilon > 0$, we can choose a sufficiently small constant $\epsilon^* > 0$ such that

$$\left.\begin{array}{ll}
|a_{ij}^h(t) - a_{ij}^h(t+\delta)| < \epsilon^*, & |\beta_{ij}^h(t) - \beta_{ij}^h(t+\delta)| < \epsilon^*, \\
|\gamma_{ij}^h(t) - \gamma_{ij}^h(t+\delta)| < \epsilon^*, & |\tau_{ij}^h(t) - \tau_{ij}^h(t+\delta)| < \epsilon^*,
\end{array}\right\}$$

follows that

$$\big|A_i(\delta, t)\big| < \frac{1}{2}\eta\epsilon, \tag{2.29}$$

where $t \in \mathbb{R}, i \in Q, j \in I$.

Furthermore, for $\epsilon^* > 0$, from the uniformly almost periodic family theory in [2, p. 19, Corollary 2.3], one can choose a relatively dense subset P_{ϵ^*} of $\mathbb{R}$ such that

$$\begin{aligned}
&\left.\begin{array}{ll}
|a_{ij}^h(t) - a_{ij}^h(t+\delta)| < \epsilon^*, & |\beta_{ij}^h(t) - \beta_{ij}^h(t+\delta)| < \epsilon^*, \\
|\gamma_{ij}^h(t) - \gamma_{ij}^h(t+\delta)| < \epsilon^*, & |\tau_{ij}^h(t) - \tau_{ij}^h(t+\delta)| < \epsilon^*,
\end{array}\right\} \\
&\delta \in P_{\epsilon^*}, t \in \mathbb{R}, i \in Q, j \in I.
\end{aligned} \tag{2.30}$$

Denote $P_\epsilon = P_{\epsilon^*}$ for any $\delta \in P_\epsilon$, from (2.29) and (2.30), we have

$$\big|A_i(\delta, t)\big| < \frac{1}{2}\eta\epsilon \quad \text{for all } t \in \mathbb{R}, i \in Q. \tag{2.31}$$

Let $\Lambda_0 \geq \max\{|t_0| + T_1 + \max_{i\in Q}\sigma_i, |t_0| + T_1 + \max_{i\in Q}\sigma_i - \delta\}$. For $t \in \mathbb{R}$, denote

$$u(t) = \big(u_1(t), u_2(t), \ldots, u_n(t)\big), \quad u_i(t) = x_i^h(t+\delta) - x_i^h(t),$$

and

$$U(t) = \big(U_1(t), U_2(t), \ldots, U_n(t)\big), \quad U_i(t) = e^{\lambda t} u_i(t),$$

where $i \in Q$. Let i_t be such an index that

$$\big|U_{i_t}(t)\big| = \big\|U(t)\big\|. \tag{2.32}$$

Then, for all $t \geq \Lambda_0$, we have

$$\begin{aligned} u_i'(t) = {} & -a_{ii}^h(t)\big[x_i^h(t+\delta) - x_i^h(t)\big] + \sum_{j=1, j\neq i}^{n} a_{ij}^h(t)\big[x_j^h(t+\delta) - x_j^h(t)\big] \\ & + \sum_{j=1}^{m} \beta_{ij}^h(t)\big[x_i^h\big(t - \tau_{ij}^h(t) + \delta\big)e^{-\gamma_{ij}^h(t)x_i^h(t-\tau_{ij}^h(t)+\delta)} \\ & - x_i^h\big(t - \tau_{ij}^h(t)\big)e^{-\gamma_{ij}^h(t)x_i^h(t-\tau_{ij}^h(t))}\big] + A_i(\delta, t). \end{aligned} \tag{2.33}$$

From (2.26), (2.33), and the inequality

$$\big|\alpha e^{-\alpha} - \beta e^{-\beta}\big| \leq \frac{1}{e^2}|\alpha - \beta|, \quad \text{where } \alpha, \beta \in [\kappa, +\infty), \tag{2.34}$$

we obtain

$$\begin{aligned} & D^-\big(\big|U_{i_s}(s)\big|\big)\big|_{s=t} \\ & \quad \leq \lambda e^{\lambda t}\big|u_{i_t}(t)\big| + e^{\lambda t}\Bigg\{-a_{i_t i_t}^h(t)\big[x_{i_t}^h(t+\delta) - x_{i_t}^h(t)\big]\operatorname{sgn}\big(x_{i_t}^h(t+\delta) - x_{i_t}^h(t)\big) \\ & \qquad + \sum_{j=1, j\neq i_t}^{n} a_{i_t j}^h(t)\big|x_j^h(t+\delta) - x_j^h(t)\big| + \sum_{j=1}^{m}\beta_{i_t j}^h(t) \\ & \qquad \times \big|x_{i_t}^h\big(t - \tau_{i_t j}^h(t) + \delta\big)e^{-\gamma_{i_t j}^h(t)x_{i_t}^h(t-\tau_{i_t j}^h(t)+\delta)} - x_{i_t}^h\big(t - \tau_{i_t j}^h(t)\big)e^{-\gamma_{i_t j}^h(t)x_{i_t}^h(t-\tau_{i_t j}^h(t))}\big| \\ & \qquad + \big|A_{i_t}(\delta, t)\big|\Bigg\} \\ & \quad = \lambda e^{\lambda t}\big|u_{i_t}(t)\big| + e^{\lambda t}\Bigg\{-a_{i_t i_t}^h(t)\big[x_{i_t}^h(t+\delta) - x_{i_t}^h(t)\big]\operatorname{sgn}\big(x_{i_t}^h(t+\delta) - x_{i_t}^h(t)\big) \\ & \qquad + \sum_{j=1, j\neq i_t}^{n} a_{i_t j}^h(t)\big|x_j^h(t+\delta) - x_j^h(t)\big| + \sum_{j=1}^{m}\frac{\beta_{i_t j}^h(t)}{\gamma_{i_t j}^h(t)} \\ & \qquad \times \big|\gamma_{i_t j}^h(t)x_{i_t}^h\big(t - \tau_{i_t j}^h(t) + \delta\big)e^{-\gamma_{i_t j}^h(t)x_{i_t}^h(t-\tau_{i_t j}^h(t)+\delta)} \end{aligned}$$

$$
\begin{aligned}
&- \gamma_{i_t j}^h(t) x_{i_t}^h\bigl(t-\tau_{i_t j}^h(t)\bigr) e^{-\gamma_{i_t j}^h(t) x_{i_t}^h(t-\tau_{i_t j}^h(t))}\bigr| + \bigl|A_{i_t}(\delta,t)\bigr|\Biggr\} \\
&\leq \lambda e^{\lambda t}\bigl|u_{i_t}(t)\bigr| + e^{\lambda t}\Biggl\{-a_{i_t i_t}^h(t)\bigl|u_{i_t}(t)\bigr| + \sum_{j=1,j\neq i_t}^{n} a_{i_t j}^h(t)\bigl|u_j(t)\bigr| \\
&\quad + \sum_{j=1}^{m} \beta_{i_t j}^h(t)\frac{1}{e^2}\bigl|u_{i_t}\bigl(t-\tau_{i_t j}^h(t)\bigr)\bigr| + \bigl|A_{i_t}(\delta,t)\bigr|\Biggr\} \\
&\leq \lambda e^{\lambda t}\bigl|u_{i_t}(t)\bigr| + e^{\lambda t}\Biggl\{-a_{i_t i_t}^h(t)\bigl|u_{i_t}(t)\bigr| + \sum_{j=1,j\neq i_t}^{n} a_{i_t j}^h(t)\bigl|u_j(t)\bigr| \\
&\quad + \sum_{j=1}^{m} \frac{\beta_{i_t j}^h(t)}{\gamma_{i_t j}^h(t)-\varpi}\frac{1}{e^2}\bigl|u_{i_t}\bigl(t-\tau_{i_t j}^h(t)\bigr)\bigr| + \bigl|A_{i_t}(\delta,t)\bigr|\Biggr\} \\
&= -\bigl[a_{i_t i_t}^h(t)-\lambda\bigr]\bigl|U_{i_t}(t)\bigr| + \sum_{j=1,j\neq i_t}^{n} a_{i_t j}^h(t)\bigl|U_j(t)\bigr| \\
&\quad + \sum_{j=1}^{m} \frac{\beta_{i_t j}^h(t)}{\gamma_{i_t j}^h(t)-\varpi}\frac{1}{e^2} e^{\lambda\tau_{i_t j}^h(t)}\bigl|U_{i_t}\bigl(t-\tau_{i_t j}^h(t)\bigr)\bigr| + e^{\lambda t}\bigl|A_{i_t}(\delta,t)\bigr| \quad \text{for all } t \geq \varLambda_0.
\end{aligned}
\tag{2.35}
$$

Let

$$
E(t) = \sup_{-\infty< s\leq t}\bigl\{e^{\lambda s}\bigl\|u(s)\bigr\|\bigr\}.
$$

It is obvious that $e^{\lambda t}\|u(t)\| \leq E(t)$, and $E(t)$ is nondecreasing.

Now, the remaining proof will be divided into two steps.

Step one. If $E(t) > e^{\lambda t}\|u(t)\|$ for all $t \geq \varLambda_0$, we assert that

$$
E(t) \equiv \bigl\|U(\varLambda_0)\bigr\| \quad \text{for all } t \geq \varLambda_0. \tag{2.36}
$$

In the contrary case, one can pick $\varLambda_1 > \varLambda_0$ such that $E(\varLambda_1) > E(\varLambda_0)$. From the fact that

$$
e^{\lambda t}\bigl\|u(t)\bigr\| \leq E(\varLambda_0) \quad \text{for all } t \leq \varLambda_0,
$$

we can find that there exists $\beta^* \in (\varLambda_0, \varLambda_1)$ such that

$$
e^{\lambda\beta^*}\bigl\|u\bigl(\beta^*\bigr)\bigr\| = E(\varLambda_1) \geq E\bigl(\beta^*\bigr),
$$

which contradicts the fact that $E(\beta^*) > e^{\lambda\beta^*}\|u(\beta^*)\|$ and proves (2.36). Then we can select $\varLambda_2 > \varLambda_0$ satisfying

$$
\bigl\|u(t)\bigr\| \leq e^{-\lambda t}E(t) = e^{-\lambda t}E(\varLambda_0) < \frac{\varepsilon}{2} \quad \text{for all } t \geq \varLambda_2. \tag{2.37}
$$

Step two. If there exists $\varsigma \geq \Lambda_0$ such that $E(\varsigma) = e^{\lambda\varsigma}\|u(\varsigma)\|$, from (2.35) and the definition of $E(t)$, we have

$$\begin{aligned}
0 &\leq D^-\big(\big|U_{i_s}(s)\big|\big)\big|_{s=\varsigma} \\
&\leq -\big[a^h_{i_\varsigma i_\varsigma}(t) - \lambda\big]\big|U_{i_\varsigma}(\varsigma)\big| + \sum_{j=1, j\neq i_\varsigma}^{n} a^h_{i_\varsigma j}(t)\big|U_j(\varsigma)\big| \\
&\quad + \sum_{j=1}^{m} \frac{\beta^h_{i_\varsigma j}(\varsigma)}{\gamma^h_{i_\varsigma j}(\varsigma) - \varpi}\frac{1}{e^2} e^{\lambda\tau^h_{i_\varsigma j}(\varsigma)}\big|U_{i_\varsigma}\big(\varsigma - \tau^h_{i_\varsigma j}(\varsigma)\big)\big| + e^{\lambda\varsigma}\big|A_{i_\varsigma}(\delta,\varsigma)\big| \\
&\leq \left\{-\big[a^h_{i_\varsigma i_\varsigma}(t) - \lambda\big] + \sum_{j=1, j\neq i_\varsigma}^{n} a^h_{i_\varsigma j}(t) + \sum_{j=1}^{m} \frac{\beta^h_{i_\varsigma j}(\varsigma)}{\gamma^h_{i_\varsigma j}(\varsigma) - \varpi}\frac{1}{e^2} e^{\lambda\tau^h_{i_\varsigma j}(\varsigma)}\right\}E(\varsigma) \\
&\quad + \frac{1}{2}\eta\varepsilon e^{\lambda\varsigma} \\
&< -\eta E(\varsigma) + \frac{1}{2}\eta\varepsilon e^{\lambda\varsigma},
\end{aligned} \tag{2.38}$$

which leads to

$$e^{\lambda\varsigma}\big\|u(\varsigma)\big\| = E(\varsigma) < \frac{\varepsilon}{2}e^{\lambda\varsigma} \quad \text{and} \quad \big\|u(\varsigma)\big\| < \frac{\varepsilon}{2}. \tag{2.39}$$

For any $t > \varsigma$ satisfying $E(t) = e^{\lambda t}\|u(t)\|$, by using the similar method to the proof of (2.39), we can get

$$e^{\lambda t}\big\|u(t)\big\| < \frac{\varepsilon}{2}e^{\lambda t} \quad \text{and} \quad \big\|u(t)\big\| < \frac{\varepsilon}{2}. \tag{2.40}$$

Furthermore, if $E(t) > e^{\lambda t}\|u(t)\|$ and $t > \varsigma$, one can pick $\Lambda_3 \in [\varsigma, t)$ such that

$$E(\Lambda_3) = e^{\lambda\Lambda_3}\big\|u(\Lambda_3)\big\| \quad \text{and} \quad E(s) > e^{\lambda s}\big\|u(s)\big\| \text{ for all } s \in (\Lambda_3, t],$$

together with (2.39) and (2.40), we have

$$\big\|u(\Lambda_3)\big\| < \frac{\varepsilon}{2}. \tag{2.41}$$

With a similar proof in step one, we can entail that

$$E(s) \equiv E(\Lambda_3) \quad \text{is a constant for all } s \in (\Lambda_3, t],$$

which together with (2.41) leads to

$$\big\|u(t)\big\| < e^{-\lambda t}E(t) = e^{-\lambda t}E(\Lambda_3) = \big\|u(\Lambda_3)\big\|e^{-\lambda(t-\Lambda_3)} < \frac{\varepsilon}{2}.$$

Finally, the above discussion infers that there exists $\hat{\Lambda} > \max\{\varsigma, \Lambda_0, \Lambda_2\}$ obeying that

$$\big\|u(t)\big\| \leq \frac{\varepsilon}{2} < \varepsilon \quad \text{for all } t > \hat{\Lambda},$$

which finishes the proof of Lemma 2.5. □

3 Main result

Theorem 3.1 *Assume that the assumptions in Lemma 2.3 hold. Then system* $(1.2)^h$ *has exactly one positive almost periodic solution* $x^*(t)$, *and every solution of* (1.2) *with initial condition* (2.4) *is asymptotically almost periodic on* $\mathbb{R}^+$ *and converges to* $x^*(t)$ *as* $t \to +\infty$.

Proof Let $v(t)$ be a solution of system $(1.2)^h$ with initial condition (2.4), and

$$v_i(t) \equiv v_i(t_0 - \sigma_i) \quad \text{for all } t \in (-\infty, t_0 - \sigma_i], i \in Q.$$

We also define

$$\begin{aligned} B_i(q,t) = {} & -\left[a_{ii}^h(t+t_q) - a_{ii}^h(t)\right]v_i(t+t_q) + \sum_{j=1, j\neq i}^{n}\left[a_{ij}^h(t+t_q) - a_{ij}^h(t)\right]v_j(t+t_q) \\ & + \sum_{j=1}^{m}\left[\beta_{ij}^h(t+t_q) - \beta_{ij}^h(t)\right]v_i\left(t+t_q-\tau_{ij}^h(t+t_q)\right)e^{-\gamma_{ij}^h(t+t_q)v_i(t+t_q-\tau_{ij}^h(t+t_q))} \\ & + \sum_{j=1}^{m}\beta_{ij}^h(t)\left[v_i\left(t+t_q-\tau_{ij}^h(t+t_q)\right)e^{-\gamma_{ij}^h(t+t_q)v_i(t+t_q-\tau_{ij}^h(t+t_q))}\right. \\ & \left. - v_i\left(t-\tau_{ij}^h(t)+t_q\right)e^{-\gamma_{ij}^h(t+t_q)v_i(t-\tau_{ij}^h(t)+t_q)}\right] \\ & + \sum_{j=1}^{m}\beta_{ij}^h(t)\left[v_i\left(t-\tau_{ij}^h(t)+t_q\right)e^{-\gamma_{ij}^h(t+t_q)v_i(t-\tau_{ij}^h(t)+t_q)}\right. \\ & \left. - v_i\left(t-\tau_{ij}^h(t)+t_q\right)e^{-\gamma_{ij}^h(t)v_i(t-\tau_{ij}^h(t)+t_q)}\right] \quad \text{for all } t \in \mathbb{R}, i \in Q, \end{aligned} \tag{3.1}$$

where $\{t_q\}_{q\geq 1} \subseteq \mathbb{R}$ is a sequence. Then

$$\begin{aligned} v_i'(t+t_q) = {} & -a_{ii}^h(t)v_i(t+t_q) + \sum_{j=1, j\neq i}^{n} a_{ij}^h(t)v_j(t+t_q) \\ & + \sum_{j=1}^{m}\beta_{ij}^h(t)v_i\left(t-\tau_{ij}^h(t)+t_q\right)e^{-\gamma_{ij}^h(t)v_i(t-\tau_{ij}^h(t)+t_q)} + B_i(q,t) \end{aligned} \tag{3.2}$$

for all $t + t_q \geq t_0$, $i \in Q$. By using the proof similar to Lemma 2.5, we can choose $\{t_q\}_{q\geq 1}$ such that

$$\left|B_i(q,t)\right| < \frac{1}{q}. \tag{3.3}$$

From Arzela–Ascoli lemma and the fact that the function sequence $\{v(t+t_q)\}_{q\geq 1}$ is uniformly bounded and equi-uniformly continuous, we can choose a subsequence $\{t_{q_j}\}_{j\geq 1}$ of $\{t_q\}_{q\geq 1}$ such that $\{v(t+t_{q_j})\}_{j\geq 1}$ (for convenience, we still denote it by $\{v(t+t_q)\}_{q\geq 1}$) uniformly converges to a continuous function $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$ on any compact set of $\mathbb{R}$. Then, from Lemma 2.4, we have

$$\frac{\kappa}{\gamma^-} < \min_{i\in Q}\liminf_{t\to+\infty} v_i(t) \leq x_i^*(t) \leq \max_{i\in Q}\limsup_{t\to+\infty} v_i(t) < A \quad \forall t \in \mathbb{R}, i \in Q, \tag{3.4}$$

and

$$\left.\begin{array}{ll} -a_{ii}^h(t)v_i(t+t_q) \rightrightarrows -a_{ii}^h(t)x_i^*(t), & i\in Q,\\ \sum_{j=1,j\neq i}^n a_{ij}^h(t)v_j(t+t_q) \rightrightarrows \sum_{j=1,j\neq i}^n a_{ij}^h(t)x_j^*(t), & i\in Q,\\ \sum_{j=1}^m \beta_{ij}^h(t)v_i(t-\tau_{ij}^h(t)+t_q)e^{-\gamma_{ij}^h(t)v_i(t-\tau_{ij}^h(t)+t_q)} & \\ \quad\rightrightarrows \sum_{j=1}^m \beta_{ij}^h(t)x_i^*(t-\tau_{ij}^h(t))e^{-\gamma_{ij}^h(t)x^*(t-\tau_{ij}^h(t))}, & i\in Q, \end{array}\right\}\quad \text{as } q\to+\infty, \tag{3.5}$$

on any compact set of $\mathbb{R}$, where "$\rightrightarrows$" denotes "uniformly converge". Thus, (3.2), (3.3), and (3.5) produce that $\{v_i'(t+t_q)\}_{q\geq1}$ uniformly converges to

$$-a_{ii}^h(t)x_i^*(t)+\sum_{j=1,j\neq i}^n a_{ij}^h(t)x_j^*(t)+\sum_{j=1}^m \beta_{ij}^h(t)x_i^*\bigl(t-\tau_{ij}^h(t)\bigr)e^{-\gamma_{ij}^h(t)x^*(t-\tau_{ij}^h(t))},\quad i\in Q,$$

on any compact set of $\mathbb{R}$. According to the properties of the uniform convergence function sequence, we obtain that $x^*(t)$ is a solution of $(1.2)^h$ and

$$\begin{aligned} \bigl(x_i^*(t)\bigr)' &= -a_{ii}^h(t)x_i^*(t)+\sum_{j=1,j\neq i}^n a_{ij}^h(t)x_j^*(t)\\ &\quad+\sum_{j=1}^m \beta_{ij}^h(t)x_i^*\bigl(t-\tau_{ij}^h(t)\bigr)e^{-\gamma_{ij}^h(t)x^*(t-\tau_{ij}^h(t))}\quad \text{for all } t\in\mathbb{R}, i\in Q. \end{aligned}\tag{3.6}$$

From Lemma 2.5, for any $\epsilon>0$, we can choose a relatively dense subset P_ϵ of $\mathbb{R}$ with the property that, for each $\delta\in P_\epsilon$, there exists $T=T(\delta)>0$ satisfying

$$\bigl\|v(s+t_q+\delta)-v(s+t_q)\bigr\|<\frac{\epsilon}{2}\quad \text{for all } s+t_q>T$$

and

$$\lim_{q\to+\infty}\bigl\|v(s+t_q+\delta)-v(s+t_q)\bigr\| = \bigl\|x^*(s+\delta)-x^*(s)\bigr\|\leq\frac{\epsilon}{2}<\epsilon\quad \text{for all } s\in\mathbb{R},$$

which implies that $x^*(t)$ is a positive almost periodic solution of $(1.2)^h$.

Next, we show that all the solutions of (1.2) converge to $x^*(t)$ as $t\to+\infty$. Let $x(t)$ be an arbitrary solution of system (1.2) with initial value (2.4). Define $y(t)=x(t)-x^*(t)$ and $x_i(t)\equiv x_i(t_0-\sigma_i)$ for all $t\in(-\infty,t_0-\sigma_i]$, let

$$\begin{aligned} &F_i(t)\\ &\quad= -\bigl[\bigl(a_{ii}^h(t)+a_{ii}^g(t)\bigr)x_i(t)-a_{ii}^h(t)x_i(t)\bigr]\\ &\qquad+\sum_{j=1,j\neq i}^n\bigl[\bigl(a_{ij}^h(t)+a_{ij}^g(t)\bigr)x_j(t)-a_{ij}^h(t)x_j(t)\bigr]\\ &\qquad+\sum_{j=1}^m\bigl[\bigl(\beta_{ij}^h(t)+\beta_{ij}^g(t)\bigr)x_i\bigl(t-\bigl(\tau_{ij}^h(t)+\tau_{ij}^g(t)\bigr)\bigr)e^{-(\gamma_{ij}^h(t)+\gamma_{ij}^g(t))x_i(t-(\tau_{ij}^h(t)+\tau_{ij}^g(t)))}\\ &\qquad-\beta_{ij}^h(t)x_i\bigl(t-\tau_{ij}^h(t)\bigr)e^{-\gamma_{ij}^h(t)x_i(t-\tau_{ij}^h(t))}\bigr]. \end{aligned}$$

Then

$$y_i'(t) = -a_{ii}^h(t)y_i(t) + \sum_{j=1,j\neq i}^{n} a_{ij}^h(t)y_j(t) + \sum_{j=1}^{m} \beta_{ij}^h(t)\bigl[x_i\bigl(t-\tau_{ij}^h(t)\bigr)e^{-\gamma_{ij}^h(t)x_i(t-\tau_{ij}^h(t))} - x_i^*\bigl(t-\tau_{ij}^h(t)\bigr)e^{-\gamma_{ij}^h(t)x_i^*(t-\tau_{ij}^h(t))}\bigr] + F_i(t) \quad \text{for all } t \geq t_0, i \in Q. \tag{3.7}$$

For any $\epsilon > 0$, in view of the global existence and uniform continuity of x and the fact that $a_{ij}^g, \beta_{ij}^g, \gamma_{ij}^g, \tau_{ij}^g \in W_0(\mathbb{R}^+, \mathbb{R}^+)$, we can choose a constant $T_\varphi^{**} > \max\{T_1, t_\varphi^*\}$ such that

$$\bigl|F_i(t)\bigr| < \eta\frac{\epsilon}{2} \quad \text{for all } t > T_\varphi^{**}. \tag{3.8}$$

Set

$$G(t) = \sup_{-\infty<s\leq t}\bigl\{e^{\lambda s}\bigl\|y(s)\bigr\|\bigr\} \quad \text{for all } t \in \mathbb{R},$$

and let i_t be such an index that

$$e^{\lambda t}\bigl|y_{i_t}(t)\bigr| = \bigl\|e^{\lambda t}y(t)\bigr\|.$$

According to (3.4) and Lemma 2.3, one can find $T_{\varphi,x^*} > T_\varphi^{**}$ such that

$$\frac{\kappa}{\gamma^-} < x_i(t), x_i^*(t), x_i^h\bigl(t-\tau_{ij}^h(t)\bigr) \quad \text{for all } t > T_{\varphi,x^*} - \sigma_i, i \in Q. \tag{3.9}$$

Combined with (2.34) and (3.7), we gain

$$\begin{aligned} &D^-\bigl(e^{\lambda s}\bigl|y_{i_s}(s)\bigr|\bigr)\big|_{s=t} \\ &\quad\leq -\bigl[a_{i_ti_t}^h(t)-\lambda\bigr]e^{\lambda t}\bigl|y_{i_t}(t)\bigr| + \sum_{j=1,j\neq i_t}^{n} a_{i_tj}^h(t)e^{\lambda t}\bigl|y_j(t)\bigr| + \sum_{j=1}^{m}\beta_{i_tj}^h(t)\frac{1}{e^2}e^{\lambda\tau_{i_tj}^h(t)} \\ &\qquad\times e^{\lambda(t-\tau_{i_tj}^h(t))}\bigl|y_{i_t}\bigl(t-\tau_{i_tj}^h(t)\bigr)\bigr| + e^{\lambda t}\bigl|F_{i_t}(t)\bigr| \quad \text{for all } t \geq T_{\varphi,x^*}, i \in Q. \end{aligned} \tag{3.10}$$

Then, from (2.26) and (3.8), by employing the argument of Lemma 2.5, we know that there is a constant $\widetilde{T} \geq T_{\varphi,x^*}$ such that

$$\bigl\|y(t)\bigr\| < \frac{\epsilon}{2} \quad \text{for all } t \geq \widetilde{T},$$

which yields

$$\lim_{t\to+\infty} x(t) = x^*(t) \quad \text{and} \quad x(t) \in \mathrm{AAP}\bigl(\mathbb{R}, \mathbb{R}^n\bigr).$$

It follows from the uniqueness of the limit function that $(1.2)^h$ has exactly one positive almost periodic solution $x^*(t)$. The proof is complete. □

Then, we will establish the existence and global exponential stability of the almost periodic solution of $(1.2)^h$. To do this end, we first show the following proposition.

Proposition 3.1 *Suppose that $f(t)$ is an almost periodic function, then*

$$\limsup_{t\to+\infty} f(t) = \sup_{t\in\mathbb{R}} f(t) \quad \textit{and} \quad \liminf_{t\to+\infty} f(t) = \inf_{t\in\mathbb{R}} f(t).$$

Proof We only need to validate the case that $\limsup_{t\to+\infty} f(t) = \sup_{t\in\mathbb{R}} f(t)$, since the other case that $\liminf_{t\to+\infty} f(t) = \inf_{t\in\mathbb{R}} f(t)$ can be proved similarly.

Define

$$A = \sup_{t\in R} f(t), \qquad B = \limsup_{t\to+\infty} f(t).$$

It is easy to see that $B \le A$. We claim

$$B = A.$$

Otherwise, $B < A$, let $\varepsilon_0 = \frac{A-B}{8}$, from the definition of upper limit, there exists $T = T(\varepsilon_0) > 0$ such that

$$f(t) < B + \varepsilon_0 < A - 2\varepsilon_0 \quad \text{for all } t \ge T.$$

According to the definition of the upper bound, one can take $t_0 \in R$ to satisfy that

$$f(t_0) > A - \varepsilon_0 > B + 2\varepsilon_0.$$

Furthermore, there exists a constant $l = l(\varepsilon_0) > 0$ such that, $\forall[\alpha, \alpha + l] \subset \mathbb{R}$ with $\alpha \in \mathbb{R}$, one can pick $\tau \in [\alpha, \alpha + l]$ satisfying that

$$\left|f(t+\tau) - f(t)\right| < \varepsilon_0 \quad \text{for all } t \in \mathbb{R}.$$

Letting $\alpha = T - t_0$ and $\tau \in [T - t_0, T - t_0 + l]$ leads to

$$f(t_0 + \tau) > f(t_0) - \varepsilon_0 > A - 2\varepsilon_0 > B + \varepsilon_0 \quad \text{and} \quad t_0 + \tau \ge t_0 + T - t_0 = T,$$

which is contrary to the fact that $f(t) < B + \varepsilon_0 < A - 2\varepsilon_0$ for all $t \ge T$. This finishes the proof of Proposition 3.1. □

Theorem 3.2 *Suppose that, for $i \in Q$, $j \in I$,*

$$\gamma^- = \min_{i\in Q} \inf_{t\in\mathbb{R}} \gamma_{ij}(t) > 0, \qquad \sup_{t\in\mathbb{R}} \gamma_{ij}(t) \le 1, \qquad \inf_{t\in\mathbb{R}} \left[a_{ii}^h(t) - \sum_{j=1, j\ne i}^{n} a_{ij}^h(t) \right] > 0, \tag{3.11}$$

$$\inf_{t\in\mathbb{R}} \left[\sum_{j=1, j\ne i}^{n} \frac{a_{ij}^h(t)}{a_{ii}^h(t)} + \sum_{j=1}^{m} \frac{\beta_{ij}^h(t)}{a_{ii}^h(t)} \right] > 1, \tag{3.12}$$

$$e < \inf_{t\in\mathbb{R}} \left[\frac{\sum_{j=1}^{m} \frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t)}}{a_{ii}^h(t) - \sum_{j=1, j\ne i^L}^{n} a_{ij}^h(t)} \right] \le \sup_{t\in\mathbb{R}} \left[\frac{\sum_{j=1}^{m} \frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t)}}{a_{ii}^h(t) - \sum_{j=1, j\ne i^L}^{n} a_{ij}^h(t)} \right] < e^2, \tag{3.13}$$

$$\begin{aligned}
&\inf_{t\in\mathbb{R}} \ln\left(\frac{\sum_{j=1}^{m}\beta_{ij}^{h}(t)}{a_{ii}^{h}(t)-\sum_{j=1,j\neq i}^{n}a_{ij}^{h}(t)}\right) > \frac{\kappa}{\gamma^{-}},\\
&\frac{\inf_{t\in\mathbb{R}}(\frac{\sum_{j=1}^{m}\beta_{ij}^{h}(t)}{a_{ii}^{h}(t)-\sum_{j=1,j\neq i}^{n}a_{ij}^{h}(t)})}{\max_{1\leq i\leq n}\sup_{t\in\mathbb{R}}[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}^{h}(t)}{\gamma_{ij}^{h}(t)}}{a_{ii}^{h}(t)-\sum_{j=1,j\neq iL}^{n}a_{ij}^{h}(t)}]} > \frac{\kappa}{\gamma^{-}},
\end{aligned} \tag{3.14}$$

and

$$\delta = \frac{1}{\max_{1\leq i\leq n}\sup_{t\in\mathbb{R}}[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}}{a_{ii}(t)-\sum_{j=1,j\neq i}^{n}a_{ij}(t)}]} = \frac{1}{\max_{1\leq i\leq n}\sup_{t\in\mathbb{R}}[\frac{\sum_{j=1}^{m}\frac{\beta_{ij}^{h}(t)}{\gamma_{ij}^{h}(t)}}{a_{ii}^{h}(t)-\sum_{j=1,j\neq i}^{n}a_{ij}^{h}(t)}]} \tag{3.15}$$

are satisfied. Then system $(1.2)^h$ *has exactly one positive almost periodic solution* $x^*(t)$*, which is global exponentially stable; in other words, the solution* $N(t;t_0,\varphi)$ *of* $(1.2)^h$ *with* (2.4) *converges exponentially to* $x^*(t)$ *as* $t\to+\infty$.

Proof From Proposition 3.1, (3.11)–(3.14) imply that the assumptions in Lemmas 2.4 and 2.5 hold. Then, by using the similar proof in Theorem 3.1, we can obtain that system $(1.2)^h$ has exactly one positive almost periodic solution $x^*(t)$. It is sufficient to show the global exponential stability of $x^*(t)$. Set $N(t)=N(t;t_0,\varphi)$ and $y(t)=N(t)-x^*(t)$, then

$$\begin{aligned}
y_i'(t) &= -a_{ii}^{h}(t)y_i(t)+\sum_{j=1,j\neq i}^{n}a_{ij}^{h}(t)y_j(t)+\sum_{j=1}^{m}\beta_{ij}^{h}(t)\\
&\quad\times\left(N_i\left(t-\tau_{ij}^{h}(t)\right)e^{-\gamma_{ij}^{h}(t)N_i(t-\tau_{ij}^{h}(t))}-x_i^*\left(t-\tau_{ij}^{h}(t)\right)e^{-\gamma_{ij}^{h}(t)x_i^*(t-\tau_{ij}^{h}(t))}\right).
\end{aligned} \tag{3.16}$$

It follows from Lemma 2.4 that there is $M_{\varphi,x^*}>t_0$ such that

$$\frac{\kappa}{\gamma^{-}} < N_i(t),x_i^*(t) \quad \text{for all } t\in[M_{\varphi,x^*}-\sigma_i,+\infty), i\in Q. \tag{3.17}$$

Together with (3.11), we obtain

$$\begin{aligned}
&\left|\gamma_{ij}^{h}(t)N_i\left(t-\tau_{ij}^{h}(t)\right)e^{-\gamma_{ij}^{h}(t)N_i(t-\tau_{ij}^{h}(t))}-\gamma_{ij}^{h}(t)x_i^*\left(t-\tau_{ij}^{h}(t)\right)e^{-\gamma_{ij}^{h}(t)x_i^*(t-\tau_{ij}^{h}(t))}\right|\\
&\quad\leq\frac{1}{e^2}\gamma_{ij}^{h}(t)\left|N_i\left(t-\tau_{ij}^{h}(t)\right)-x_i^*\left(t-\tau_{ij}^{h}(t)\right)\right|,\quad \text{for all } t\in[M_{\varphi,x^*}-\sigma_i,+\infty),
\end{aligned} \tag{3.18}$$

where $i\in Q$, $j\in I$.

With the help of (3.13), we can choose $\lambda\in(0,1]$ such that

$$\sup_{t\in\mathbb{R}}\left\{-\left[a_{ii}^{h}(t)-\lambda\right]+\sum_{j=1,j\neq i}^{n}a_{ij}^{h}(t)+\sum_{j=1}^{m}\frac{\beta_{ij}^{h}(t)}{\gamma_{ij}^{h}(t)}\frac{1}{e^2}e^{\lambda\sigma_i}\right\}<0,\quad i\in Q. \tag{3.19}$$

Now, we define the Lyapunov functional as follows:

$$H_i(t)=\left|y_i(t)\right|e^{\lambda t},\quad i\in Q, t\in[t_0-\sigma_i,+\infty).$$

With the help of (3.16) and (3.18), we get

$$\begin{aligned}
&D^{-}\big(H_i(t)\big)\\
&\quad\le \lambda\big|y_i(t)\big|e^{\lambda t} - a_{ii}^h(t)\big|y_i(t)\big|e^{\lambda t} + \sum_{j=1,j\neq i}^{n} a_{ij}^h(t)\big|y_j(t)\big|e^{\lambda t}\\
&\qquad + \sum_{j=1}^{m} \frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t)} e^{\lambda t}\big|\gamma_{ij}^h(t)N_i\big(t-\tau_{ij}^h(t)\big)e^{-\gamma_{ij}^h(t)N_i(t-\tau_{ij}^h(t))}\\
&\qquad - \gamma_{ij}^h(t)x_i^*\big(t-\tau_{ij}^h(t)\big)e^{-\gamma_{ij}^h(t)x_i^*(t-\tau_{ij}^h(t))}\big|\\
&\quad\le \big(\lambda - a_{ii}^h(t)\big)H_i(t) + \sum_{j=1,j\neq i}^{n} a_{ij}^h(t)H_j(t) + \sum_{j=1}^{m} \frac{\beta_{ij}^h(t)}{e^2} e^{\lambda\sigma_i} H_i\big(t-\tau_{ij}^h(t)\big)\\
&\quad\le \big(\lambda - a_{ii}^h(t)\big)H_i(t)\\
&\qquad + \sum_{j=1,j\neq i}^{n} a_{ij}^h(t)H_j(t) + \sum_{j=1}^{m} \frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t)} \frac{1}{e^2} e^{\lambda\sigma_i} H_i\big(t-\tau_{ij}^h(t)\big), \quad t \ge t_0, i \in Q.
\end{aligned} \tag{3.20}$$

In the sequel, we prove that, for all $t > M_{\varphi,x^*}$,

$$H_i(t) < \sup_{t\in[t_0-\sigma_i,M_{\varphi,x^*}]} \max_{j\in J}\big(H_j(t)+1\big) := V_{\varphi,x^*}, \quad i \in Q. \tag{3.21}$$

Otherwise, there exist $K^* > M_{\varphi,x^*}$ and $\hat{i} \in Q$ such that

$$H_{\hat{i}}\big(t^*\big) = V_{\varphi,x^*}, H_j(t) < V_{\varphi,x^*} \quad \text{for all } t \in [t_0-\sigma_j, K^*), j \in Q. \tag{3.22}$$

It follows from (3.19), (3.20), and (3.22) that

$$\begin{aligned}
&0 \le D^{-}\big(H_{\hat{i}}(t)\big)\big|_{t=K^*}\\
&\quad\le \big(\lambda - a_{\hat{i}\hat{i}}^h(K^*)\big)H_{\hat{i}}(K^*)\\
&\qquad + \sum_{j=1,j\neq\hat{i}}^{n} a_{\hat{i}j}^h(K^*)H_j(K^*) + \sum_{j=1}^{m} \frac{\beta_{\hat{i}j}^h(K^*)}{\gamma_{\hat{i}j}^h(K^*)} \frac{1}{e^2} e^{\lambda\sigma_{\hat{i}}} H_{\hat{i}}\big(K^* - \tau_{\hat{i}j}^h(K^*)\big)\\
&\quad\le \left[\big(\lambda - a_{\hat{i}\hat{i}}^h(K^*)\big) + \sum_{j=1,j\neq\hat{i}}^{n} a_{\hat{i}j}^h(K^*) + \sum_{j=1}^{m} \frac{\beta_{\hat{i}j}^h(K^*)}{\gamma_{\hat{i}j}^h(K^*)} \frac{1}{e^2} e^{\lambda\sigma_{\hat{i}}}\right] V_{\varphi,x^*}\\
&\quad< 0,
\end{aligned}$$

which is a contradiction. Thus (3.21) holds, and it follows that

$$\big|N_i(t) - x_i^*(t)\big| = \big|y_i(t)\big| < V_{\varphi,x^*}e^{-\lambda t} \quad \text{for all } t > M_{\varphi,x^*}, i \in Q.$$

This completes the proof of Theorem 3.2. □

4 Some numerical simulations

In this section, we give two examples with simulations to demonstrate the feasibility and the validity of our theoretical results.

Example 4.1 Consider the following delayed Nicholson-type system involving patch structure and asymptotically almost periodic environments:

$$\begin{cases} x_1'(t) = -(10\sin^2\sqrt{2}t + 2 + \frac{100}{1+|2t|})x_1(t) \\ \qquad + (0.01\sin^2\sqrt{3}t + 0.02 + \frac{300}{1+|2t|})x_2(t) \\ \qquad + 2(10\sin^2\sqrt{2}t + 2 + \frac{100}{1+|3t|})(1.1 + 0.01\cos t) \\ \qquad \times x_1(t - 100\sin^2 t)e^{-(0.9+0.01\sin\sqrt{3}t+\frac{100}{1+|2t|})x_1(t-100\sin^2 t)} \\ \qquad + (10\sin^2\sqrt{2}t + 2 + \frac{100}{1+|5t|})(1.1 + 0.01\cos\sqrt{3}t) \\ \qquad \times x_1(t - 100\cos^2 t)e^{-(0.9+0.01\cos\sqrt{3}t+\frac{100}{1+|2t|})x_1(t-100\cos^2 t)}, \\ x_2'(t) = -(10\cos^2\sqrt{2}t + 2 + \frac{100}{1+|4t|})x_2(t) \\ \qquad + (0.01\sin^2 t + 0.02 + \frac{100}{1+|5t|})x_1(t) \\ \qquad + 2(10\cos^2 t + 2 + \frac{100}{1+|2t|})(1.1 + 0.01\cos\sqrt{7}t) \\ \qquad \times x_2(t - 150\sin^2 t)e^{-(0.9+0.01\sin t+\frac{100}{1+|8t|})x_2(t-150\sin^2 t)} \\ \qquad + (10\cos^2\sqrt{2} + 2 + \frac{100}{1+|2t|})(1.1 + 0.01\cos\sqrt{5}t) \\ \qquad \times x_2(t - 150\cos^2 t)e^{-(0.9+0.01\cos t+\frac{200}{1+|7t|})x_2(t-150\cos^2 t)}, \end{cases} \tag{4.1}$$

where $t_0 = 0$.

One can easily check that (2.1), (2.2), and (2.10)–(2.13) hold for system (4.1). From Theorem 3.1, we can obtain that every solution of (4.1) with initial value $\varphi = (\varphi_1, \varphi_2) \in C([-100, 0], \mathbb{R}^+) \times C([-150, 0], \mathbb{R}^+)$ and $\varphi_i(0) > 0$ $(i = 1, 2)$ is asymptotically almost periodic on $\mathbb{R}^+$ and converges to the same almost periodic function as $t \to +\infty$. The numeric simulations in Fig. 1 support this theoretical results.

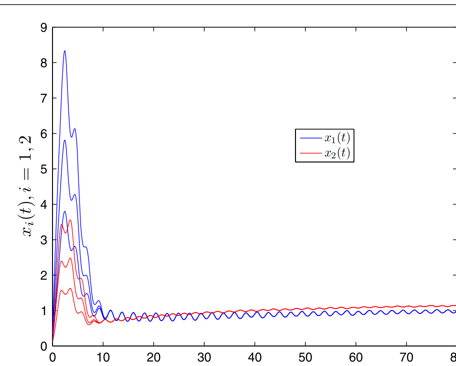
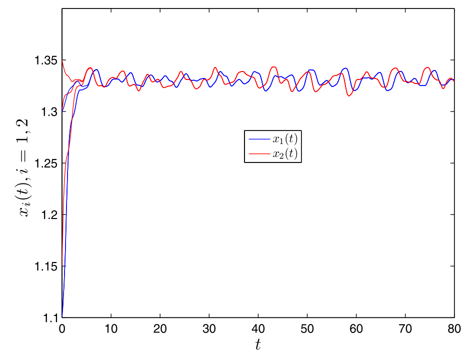


Figure 1 Numerical solutions of (4.1) for initial value (0.1, 0.3), (0.1, 0.15), (0.3, 0.35)

Figure 2 Numerical solutions of (4.2) for initial value (1.1, 1.3), (1.1, 1.15), (1.3, 1.35)



Example 4.2 Consider the following delayed Nicholson-type system involving patch structure and almost periodic environments:

$$\begin{cases} x_1'(t) = -(10\sin^2\sqrt{2}t + 2)x_1(t) + (0.01\sin^2\sqrt{3}t + 0.02)x_2(t) \\ \qquad + 2(10\sin^2\sqrt{2}t + 2)(1.1 + 0.01\cos t)x_1(t - 100\sin^2 t) \\ \qquad \times e^{-(0.9+0.01\sin\sqrt{3}t)x_1(t-100\sin^2 t)} \\ \qquad + (10\sin^2\sqrt{2}t + 2)(1.1 + 0.01\cos\sqrt{3}t)x_1(t - 100\cos^2 t) \\ \qquad \times e^{-(0.9+0.01\cos\sqrt{3}t)x_1(t-100\cos^2 t)}, \\ x_2'(t) = -(10\cos^2\sqrt{2}t + 2)x_2(t) + (0.01\sin^2 t + 0.02)x_1(t) \\ \qquad + 2(10\cos^2 t + 2)(1.1 + 0.01\cos\sqrt{7}t)x_2(t - 150\sin^2 t) \\ \qquad \times e^{-(0.9+0.01\sin t)x_2(t-150\sin^2 t)} \\ \qquad + (10\cos^2\sqrt{2}t + 2)(1.1 + 0.01\cos\sqrt{5}t)x_2(t - 150\cos^2 t) \\ \qquad \times e^{-(0.9+0.01\cos t)x_2(t-150\cos^2 t)}, \end{cases} \tag{4.2}$$

where $t_0 = 0$.

Obviously, system (4.2) satisfies all the assumptions made in (3.11)–(3.15). Therefore, by Theorem 3.2, we obtain that system (4.2) has exactly one positive almost periodic solution $x^*(t)$. In particular, the solution $N(t; t_0, \varphi)$ of (4.2) with initial value $\varphi = (\varphi_1, \varphi_2) \in C([-100, 0], \mathbb{R}^+) \times C([-150, 0], \mathbb{R}^+)$ and $\varphi_i(0) > 0$ $(i = 1, 2)$ converges exponentially to $x^*(t)$ as $t \to +\infty$. Figure 2 reveals the above consequences through numerical solutions of different initial values.

Remark 4.1 In the above examples, $\limsup_{t\in\mathbb{R}} \gamma_{ij}(t) \le 0.91 < 1$, $i, j = 1, 2$, does not satisfy assumption (1.5). Moreover, when $\frac{\kappa}{\gamma} > 1.5 > \widetilde{\kappa}$, one can find that, in Theorems 3.1 and 3.2, the existence region of almost periodic solution and the attractive region of asymptotically almost periodic solutions are outside of $\underbrace{[\kappa, \widetilde{\kappa}] \times \cdots \times [\kappa, \widetilde{\kappa}]}_{n} = \underbrace{[0.7215355, 1.342276] \times \cdots \times [0.7215355, 1.342276]}_{n}$. Therefore, it is not difficult to see that all the results in references [5–7] and [15–100] cannot be applied to show the almost periodic dynamics for system (4.1) and system (4.2).

5 Conclusions

In this paper, we combine the Lyapunov function method with the differential inequality method to establish some new criteria ensuring the existence and attractivity of positive asymptotically almost periodic solutions for a class of delayed Nicholson's blowflies systems with patch structure. The assumptions adopted in this present paper are different from some previously known literature. Numerical simulations have been given to illustrate the obtained results. The approach presented in this article can be used as a possible way to study the asymptotically almost periodic patch structure population models such as neoclassical growth model, Mackey–Glass model, epidemical system or age-structured population model, and so on. We leave this as our future work.

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