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Hermite–Hadamard and Hermite–Hadamard–Fejer type inequalities for p-convex functions via conformable fractional integrals

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Abstract
In this paper, we obtain the Hermite–Hadamard and Hermite–Hadamard–Fejer type inequalities for p-convex functions via conformable fractional integrals. We also discuss some special cases.

Keywords: Hermite–Hadamard inequality; Hermite–Hadamard–Fejer inequality; p-convex functions; Conformable fractional integrals

1 Introduction and preliminaries

A function $\Upsilon : \mathcal{W} \to \mathbb{R}$ on an interval of real line, for all $w_1, w_2 \in \mathcal{W}$ and $\kappa \in [0,1]$, is called convex if the following inequality holds:

$$\Upsilon\big(\kappa w_1 + (1-\kappa)w_2\big) \le \kappa\Upsilon(w_1) + (1-\kappa)\Upsilon(w_2). \tag{1}$$

Due to the importance of convex functions, many authors have given results not only for convex functions but also for their generalizations. The Hermite–Hadamard inequality [9] on a real interval was defined by

$$\Upsilon\left(\frac{w_1+w_2}{2}\right) \le \frac{1}{w_2-w_1}\int_{w_1}^{w_2}\Upsilon(u)\,du \le \frac{\Upsilon(w_1)+\Upsilon(w_2)}{2} \tag{2}$$

for all $w_1, w_2 \in \mathcal{W}$ with $w_1 < w_2$. Then Fejér [8] proved the following inequality:

$$\begin{aligned}\Upsilon\left(\frac{w_1+w_2}{2}\right)\int_{w_1}^{w_2}\curlyvee(u)\,du &\le \frac{1}{w_2-w_1}\int_{w_1}^{w_2}\Upsilon(u)\curlyvee(u)\,du\\ &\le \frac{\Upsilon(w_1)+\Upsilon(w_2)}{2}\int_{w_1}^{w_2}\curlyvee(u)\,du,\end{aligned} \tag{3}$$

where $\curlyvee : [w_1, w_2] \to \mathbb{R}$ is nonnegative, integrable, and symmetric to $(w_1 + w_2)/2$, called Hermite–Hadamard–Fejér inequality. Inequalities (2) and (3) have been further generalized in different ways not only for classical integral but also for other generalized integrals such as Riemann–Liouville fractional integral, Katugampola, ψ-Riemann–Liouville, and conformable fractional integrals etc. For more results and details see [1, 4–7, 17–23, 26–30].

Definition 1.1 ([11, 12]) Suppose an interval $\mathcal{W} \subset (0,\infty) = \mathbb{R}_+$ and $p \in \mathbb{R} \setminus \{0\}$. Then a function $\Upsilon : \mathcal{W} \to \mathbb{R}$ is called p-convex if

$$\Upsilon\bigl(\bigl[\kappa w_1^p + (1-\kappa)w_2^p\bigr]^{\frac{1}{p}}\bigr) \leq \kappa\Upsilon(w_1) + (1-\kappa)\Upsilon(w_2) \tag{4}$$

holds for all $w_1, w_2 \in \mathcal{W}$ and $\kappa \in [0,1]$. If inequality (4) is in opposite order, then Υ is called p-concave function.

Definition 1.2 ([14]) Let $\Upsilon \in L[w_1,w_2]$. The left- and right-sided Riemann–Liouville fractional integrals $J_{w_1+}^{\alpha}\Upsilon$ and $J_{w_2-}^{\alpha}\Upsilon$ of order $\alpha \in \mathbb{C}$ with $\mathbb{R}(\alpha) > 0$ and $w_2 > w_1 \geq 0$ are given by

$$J_{w_1+}^{\alpha}\Upsilon(u) = \frac{1}{\Gamma(\alpha)}\int_{w_1}^{u}(u-v)^{\alpha-1}\Upsilon(v)\,dv, \quad u > w_1,$$

and

$$J_{w_2-}^{\alpha}\Upsilon(u) = \frac{1}{\Gamma(\alpha)}\int_{u}^{w_2}(v-u)^{\alpha-1}\Upsilon(v)\,dv, \quad u < w_2,$$

respectively, where $\Gamma(\cdot)$ is the gamma function.

Abdeljawad [2] defined the conformable fractional integral as follows.

Definition 1.3 ([2]) Let $\alpha \in (n, n+1]$ and $\gamma = \alpha - n$. Then the left- and right-sided conformable fractional integrals of order $\alpha > 0$ are given by

$$J_{\alpha}^{w_1}\Upsilon(u) = \frac{1}{n!}\int_{w_1}^{u}(u-v)^{n}(v-w_1)^{\gamma-1}\Upsilon(v)\,dv,$$

and

$${}^{w_2}J_{\alpha}\Upsilon(u) = \frac{1}{n!}\int_{u}^{w_2}(v-u)^{n}(w_2-v)^{\gamma-1}\Upsilon(v)\,dv,$$

respectively.

Note that for $\alpha = n+1$ then $\gamma = 1$, where $n = 0,1,2,\ldots$, and in this case conformable fractional integrals become Riemann–Liouville fractional integrals.

The classical beta function and hypergeometric function are defined, respectively, by

$$\beta(w_1,w_2) = \int_0^1 u^{w_1-1}(1-u)^{w_2-1}\,du$$

and

$$ {}_2F_1(w_1, w_2; u; v) = \frac{1}{\beta(w_2, u - w_2)} \int_0^1 u^{w_2-1}(1-u)^{u-w_2-1}(1-vu)^{-w_1}\,du, $$

with $u > w_2 > 0$, $|v| < 1$.

The incomplete beta function is defined as follows:

$$ \beta_u(w_1, w_2) = \int_0^u v^{w_1-1}(1-v)^{w_2-1}\,dv, \quad u \in [0,1]. $$

The relationship between the classical beta function and the incomplete beta function is given as follows:

$$ \beta(w_1, w_2) = \beta_u(w_1, w_2) + \beta_{1-u}(w_1, w_2). $$

2 Hermite–Hadamard type inequalities

In this section we prove some Hermite–Hadamard type inequalities for p-convex functions via conformable fractional integral.

Theorem 2.1 *Let $\Upsilon : [w_1, w_2] \subset (0,\infty) \to \mathbb{R}$ be a p-convex function such that $\Upsilon \in L[w_1, w_2]$ and $\alpha > 0$. Then*

(i) *for $p > 0$, we have*

$$ \begin{aligned} &\Upsilon\left(\left[\frac{w_1^p + w_2^p}{2}\right]^{1/p}\right) \\ &\quad \leq \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha-n)(w_2^p - w_1^p)^\alpha}\left[J_\alpha^{w_1^p}(\Upsilon\circ\phi)\left(w_2^p\right) + {}^{w_2^p}J_\alpha(\Upsilon\circ\phi)\left(w_1^p\right)\right] \\ &\quad \leq \frac{\Upsilon(w_1^p) + \Upsilon(w_2^p)}{2}, \end{aligned} \tag{5} $$

here $\phi(u) = u^{\frac{1}{p}}$ for all $u \in [w_1^p, w_2^p]$;

(ii) *for $p < 0$, we have*

$$ \begin{aligned} &\Upsilon\left(\left[\frac{w_1^p + w_2^p}{2}\right]^{1/p}\right) \\ &\quad \leq \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha-n)(w_1^p - w_2^p)^\alpha}\left[{}^{w_1^p}J_\alpha(\Upsilon\circ\phi)\left(w_2^p\right) + J_\alpha^{w_2^p}(\Upsilon\circ\phi)\left(w_1^p\right)\right] \\ &\quad \leq \frac{\Upsilon(w_1^p) + \Upsilon(w_2^p)}{2}, \end{aligned} \tag{6} $$

here $\phi(u) = u^{\frac{1}{p}}$ for all $u \in [w_2^p, w_1^p]$.

Proof (i) Since Υ is a p-convex function on $[w_1, w_2]$, we have

$$ \Upsilon\left(\left[\frac{x^p + y^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\Upsilon(x) + \Upsilon(y)}{2}. $$

Taking $x^p = \kappa w_1^p + (1-\kappa)w_2^p$ and $y^p = (1-\kappa)w_1^p + \kappa w_2^p$ with $\kappa \in [0,1]$, we get

$$\Upsilon\left(\left[\frac{w_1^p + w_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\Upsilon([\kappa w_1^p + (1-\kappa)w_2^p]^{\frac{1}{p}}) + \Upsilon([(1-\kappa)w_1^p + \kappa w_2^p]^{\frac{1}{p}})}{2}. \tag{7}$$

Multiplying (7) by $\frac{1}{n!}\kappa^n(1-\kappa)^{\alpha-n-1}$, with $\kappa \in (0,1)$, $\alpha > 0$, on both sides and then integrating about κ over $[0,1]$, we find

$$\begin{aligned} &\frac{2}{n!}\Upsilon\left(\left[\frac{w_1^p + w_2^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1}\,d\kappa \\ &\quad\leq \frac{1}{n!}\int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1}\Upsilon\left(\left[\kappa w_1^p + (1-\kappa)w_2^p\right]^{\frac{1}{p}}\right)d\kappa \\ &\qquad + \frac{1}{n!}\int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1}\Upsilon\left(\left[(1-\kappa)w_1^p + \kappa w_2^p\right]^{\frac{1}{p}}\right)d\kappa \\ &\quad= I_1 + I_2. \end{aligned} \tag{8}$$

By setting $u = \kappa w_1^p + (1-\kappa)w_2^p$, we have

$$\begin{aligned} I_1 &= \frac{1}{n!}\int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1}\Upsilon\left(\left[\kappa w_1^p + (1-\kappa)w_2^p\right]^{\frac{1}{p}}\right)d\kappa \\ &= \frac{1}{n!}\int_{w_2^p}^{w_1^p}\left(\frac{u - w_2^p}{w_1^p - w_2^p}\right)^n\left(1 - \frac{u - w_2^p}{w_1^p - w_2^p}\right)^{\alpha-n-1}(\Upsilon\circ\phi)(u)\frac{du}{w_1^p - w_2^p} \\ &= \frac{1}{n!(w_2^p - w_1^p)^\alpha}\int_{w_1^p}^{w_2^p}\left(w_2^p - u\right)^n\left(u - w_1^p\right)^{\alpha-n-1}(\Upsilon\circ\phi)(u)\,du \\ &= \frac{1}{(w_2^p - w_1^p)^\alpha}J_\alpha^{w_1^p}(\Upsilon\circ\phi)\left(w_2^p\right). \end{aligned} \tag{9}$$

Similarly, by setting $u = \kappa w_2^p + (1-\kappa)w_1^p$, we have

$$\begin{aligned} I_2 &= \frac{1}{n!}\int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1}\Upsilon\left(\left[\kappa w_2^p + (1-\kappa)w_1^p\right]^{\frac{1}{p}}\right)d\kappa \\ &= \frac{1}{n!}\int_{w_1^p}^{w_2^p}\left(\frac{u - w_1^p}{w_2^p - w_1^p}\right)^n\left(1 - \frac{u - w_1^p}{w_2^p - w_1^p}\right)^{\alpha-n-1}(\Upsilon\circ\phi)(u)\frac{du}{w_2^p - w_1^p} \\ &= \frac{1}{n!(w_2^p - w_1^p)^\alpha}\int_{w_1^p}^{w_2^p}\left(u - w_1^p\right)^n\left(w_2^p - u\right)^{\alpha-n-1}(\Upsilon\circ\phi)(u)\,du \\ &= \frac{1}{(w_2^p - w_1^p)^\alpha}\,{}^{w_2^p}J_\alpha(\Upsilon\circ\phi)\left(w_1^p\right). \end{aligned} \tag{10}$$

Thus, by putting values of I_1 and I_2 in (8), the first inequality of (5) is achieved. For another inequality, we note that

$$\Upsilon\left(\left[\kappa w_1^p + (1-\kappa)w_2^p\right]^{\frac{1}{p}}\right) + \Upsilon\left(\left[\kappa w_2^p + (1-\kappa)w_1^p\right]^{\frac{1}{p}}\right) \leq \left[\Upsilon(w_1) + \Upsilon(w_2)\right]. \tag{11}$$

Multiplying (11) by $\frac{1}{n!}\kappa^n(1-\kappa)^{\alpha-n-1}$, with $\kappa \in (0,1)$, $\alpha > 0$, on both sides and then integrating about κ over $[0,1]$, we achieve the second inequality of (5). This completes the proof.

(ii) Proof is identical to that of (i). □

Remark 2.1 In Theorem 2.1:

1. If we let $p = 1$ in (i), we get Theorem 2.1 in [25].
2. If we let $p = -1$ in (ii), we get Theorem 2.1 in [3].
3. If we let $p = 1$ and $\alpha = n + 1$ in (i), we get Theorem 2 in [24].
4. If we let $p = -1$ and $\alpha = n + 1$ in (ii), we get Theorem 4 in [13].

Lemma 2.1 *Let $\Upsilon : [w_1, w_2] \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on (w_1, w_2) with $w_1 < w_2$ such that $\Upsilon' \in L[w_1, w_2]$ and $\alpha > 0$. Then*

(i) *for $p > 0$, we have*

$$\begin{aligned} &{}_1\Delta_\Upsilon(w_1, w_2; \alpha; \beta; J) \\ &\quad = \frac{w_2^p - w_1^p}{2p} \int_0^1 \bigl(\beta_{1-\kappa}(n+1, \alpha-n) - \beta_\kappa(n+1, \alpha-n)\bigr) \\ &\qquad \times A_\kappa^{\frac{1}{p}-1} \Upsilon'\bigl(\bigl[\kappa w_1^p + (1-\kappa) w_2^p\bigr]^{\frac{1}{p}}\bigr)\, d\kappa, \end{aligned} \tag{12}$$

here $A_\kappa = [\kappa w_1^p + (1-\kappa) w_2^p]$ and

$$\begin{aligned} &{}_1\Delta_\Upsilon(w_1, w_2; \alpha; \beta; J) \\ &\quad = \beta(n+1, \alpha-n)\left(\frac{\Upsilon(w_1) + \Upsilon(w_2)}{2}\right) \\ &\qquad - \frac{n!}{2(w_2^p - w_1^p)^\alpha} \bigl[J_\alpha^{w_1^p}(\Upsilon \circ \phi)\bigl(w_2^p\bigr) + {}^{w_2^p}J_\alpha(\Upsilon \circ \phi)\bigl(\tfrac{1}{p}\bigr)\bigr]; \end{aligned}$$

(ii) *for $p < 0$, we have*

$$\begin{aligned} &{}_2\Delta_\Upsilon(w_1, w_2; \alpha; \beta; J) \\ &\quad = \frac{w_1^p - w_2^p}{2p} \int_0^1 \bigl(\beta_\kappa(n+1, \alpha-n) - \beta_{1-\kappa}(n+1, \alpha-n)\bigr) \\ &\qquad \times B_\kappa^{\frac{1}{p}-1} \Upsilon'\bigl(\bigl[\kappa w_2^p + (1-\kappa) w_1^p\bigr]^{\frac{1}{p}}\bigr)\, d\kappa, \end{aligned} \tag{13}$$

here $B_\kappa = [\kappa w_2^p + (1-\kappa) w_1^p]$ and

$$\begin{aligned} &{}_2\Delta_\Upsilon(w_1, w_2; \alpha; \beta; J) \\ &\quad = \beta(n+1, \alpha-n)\left(\frac{\Upsilon(w_1) + \Upsilon(w_2)}{2}\right) \\ &\qquad - \frac{n!}{2(w_1^p - w_2^p)^\alpha} \bigl[{}^{w_1^p}J_\alpha(\Upsilon \circ \phi)\bigl(w_2^p\bigr) + J_\alpha^{w_2^p}(\Upsilon \circ \phi)\bigl(w_1^p\bigr)\bigr]. \end{aligned}$$

Proof (i) Consider

$$\begin{aligned}
&\int_0^1 \bigl(\beta_{1-\kappa}(n+1,\alpha-n)-\beta_\kappa(n+1,\alpha-n)\bigr)A_\kappa^{\frac{1}{p}-1}\Upsilon'\bigl(\bigl[\kappa w_1^p+(1-\kappa)w_2^p\bigr]^{\frac{1}{p}}\bigr)\,d\kappa\\
&\quad=\int_0^1 \beta_{1-\kappa}(n+1,\alpha-n)A_\kappa^{\frac{1}{p}-1}\Upsilon'\bigl(\bigl[\kappa w_1^p+(1-\kappa)w_2^p\bigr]^{\frac{1}{p}}\bigr)\,d\kappa\\
&\qquad-\int_0^1 \beta_{\kappa}(n+1,\alpha-n)A_\kappa^{\frac{1}{p}-1}\Upsilon'\bigl(\bigl[\kappa w_1^p+(1-\kappa)w_2^p\bigr]^{\frac{1}{p}}\bigr)\,d\kappa\\
&\quad=I_1-I_2.
\end{aligned}\tag{14}$$

Then, by integration by parts, we have

$$\begin{aligned}
I_1&=\int_0^1 \beta_{1-\kappa}(n+1,\alpha-n)A_\kappa^{\frac{1}{p}-1}\Upsilon'\bigl(\bigl[\kappa w_1^p+(1-\kappa)w_2^p\bigr]^{\frac{1}{p}}\bigr)\,d\kappa\\
&=\int_0^1\biggl(\int_0^{1-\kappa}u^n(1-u)^{\alpha-n-1}\,du\biggr)A_\kappa^{\frac{1}{p}-1}\Upsilon'\bigl(\bigl[\kappa w_1^p+(1-\kappa)w_2^p\bigr]^{\frac{1}{p}}\bigr)\,d\kappa\\
&=\frac{p}{w_2^p-w_1^p}\beta(n+1,\alpha-n)\Upsilon(w_2)\\
&\quad-\frac{p}{w_2^p-w_1^p}\int_0^1(1-\kappa)^n\kappa^{\alpha-n-1}\Upsilon\bigl(\bigl[\kappa w_1^p+(1-\kappa)w_2^p\bigr]^{\frac{1}{p}}\bigr)\,d\kappa\\
&=\frac{p}{w_2^p-w_1^p}\beta(n+1,\alpha-n)\Upsilon(w_2)\\
&\quad-\frac{p}{w_2^p-w_1^p}\int_{w_2^p}^{w_1^p}\biggl(1-\frac{x-w_2^p}{w_1^p-w_2^p}\biggr)^n\biggl(\frac{x-w_2^p}{w_1^p-w_2^p}\biggr)^{\alpha-n-1}\frac{(\Upsilon\circ\phi)(x)}{w_1^p-w_2^p}dx\\
&=\frac{p}{w_2^p-w_1^p}\beta(n+1,\alpha-n)\Upsilon(w_2)-\frac{n!}{(w_2^p-w_1^p)^{\alpha+1}}\,{}^{w_2^p}J_\alpha(\Upsilon\circ\phi)\bigl(w_1^p\bigr).
\end{aligned}\tag{15}$$

Similarly, we have

$$\begin{aligned}
I_2&=\int_0^1 \beta_{\kappa}(n+1,\alpha-n)\Upsilon'\bigl(\bigl[\kappa w_1^p+(1-\kappa)w_2^p\bigr]^{\frac{1}{p}}\bigr)\,d\kappa\\
&=\int_0^1\biggl(\int_0^{\kappa}u^n(1-u)^{\alpha-n-1}\,du\biggr)\Upsilon'\bigl(\bigl[\kappa w_1^p+(1-\kappa)w_2^p\bigr]^{\frac{1}{p}}\bigr)\,d\kappa\\
&=-\frac{p}{w_2^p-w_1^p}\beta(n+1,\alpha-n)\Upsilon(w_1)\\
&\quad+\frac{p}{w_2^p-w_1^p}\int_0^1\kappa^n(1-\kappa)^{\alpha-n-1}\Upsilon\bigl(\bigl[\kappa w_1^p+(1-\kappa)w_2^p\bigr]^{\frac{1}{p}}\bigr)\,d\kappa\\
&=-\frac{p}{w_2^p-w_1^p}\beta(n+1,\alpha-n)\Upsilon(w_1)\\
&\quad+\frac{p}{w_2^p-w_1^p}\int_{w_2^p}^{w_1^p}\biggl(\frac{x-w_2^p}{w_1^p-w_2^p}\biggr)^n\biggl(1-\frac{x-w_2^p}{w_1^p-w_2^p}\biggr)^{\alpha-n-1}\frac{(\Upsilon\circ\phi)(x)}{w_1^p-w_2^p}dx\\
&=-\frac{p}{w_2^p-w_1^p}\beta(n+1,\alpha-n)\Upsilon(w_1)+\frac{n!}{(w_2^p-w_1^p)^{\alpha+1}}\,J_\alpha^{w_1^p}(\Upsilon\circ\phi)\bigl(w_2^p\bigr).
\end{aligned}\tag{16}$$

By substituting values of I_1 and I_2 in (14) and then multiplying by $\frac{w_2^p-w_1^p}{2}$, we get (12).

(ii) Proof is similar to that of (i). □

Remark 2.2 By taking $p=-1$ in Lemma 2.1, we obtain Lemma 2.1 in [3].

Theorem 2.2 *Let $\Upsilon:[w_1,w_2]\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on (w_1,w_2) with $w_1<w_2$ such that $\Upsilon'\in L[w_1,w_2]$ and $\alpha>0$. If $|\Upsilon'|^q$, where $q\geq 1$, is a p-convex function, then*

(i) *for $p>0$, we have*

$$\left|{}_1\Delta_\Upsilon(w_1,w_2;\alpha;\beta;J)\right|\leq\frac{w_2^p-w_1^p}{2p}\lambda^{1-1/q}\left(\lambda_1\left|\Upsilon'(w_1)\right|^q+\lambda_2\left|\Upsilon'(w_2)\right|^q\right)^{1/q},\tag{17}$$

here

$$\lambda=\beta(n+1,\alpha-n+1)-\beta(n+1,\alpha-n)+\beta(n+2,\alpha-n),$$

$$\lambda_1=\frac{w_2^{1-p}}{2}\,{}_2F_1\left(1-\frac{1}{p},2;3;1-\frac{w_1^p}{w_2^p}\right)\quad\text{and}\quad\lambda_2=\frac{w_2^{1-p}}{2}\,{}_2F_1\left(1-\frac{1}{p},1;3;1-\frac{w_1^p}{w_2^p}\right);$$

(ii) *for $p<0$, we have*

$$\left|{}_2\Delta_\Upsilon(w_1,w_2;\alpha;\beta;J)\right|\leq\frac{w_1^p-w_2^p}{2p}\lambda_3^{1-1/q}\left(\lambda_4\left|\Upsilon'(w_1)\right|^q+\lambda_5\left|\Upsilon'(w_2)\right|^q\right)^{1/q},\tag{18}$$

here

$$\lambda_3=\beta(n+1,\alpha-n+1)-\beta(n+2,\alpha-n),$$

$$\lambda_4=\frac{w_2^{p-1}}{2}\,{}_2F_1\left(1-\frac{1}{p},1;3;1-\frac{w_2^p}{w_1^p}\right)\quad\text{and}\quad\lambda_5=\frac{w_2^{p-1}}{2}\,{}_2F_1\left(1-\frac{1}{p},2;3;1-\frac{w_2^p}{w_1^p}\right).$$

Proof (i) Let $A_\kappa=[\kappa w_1^p+(1-\kappa)w_2^p]$. Applying Lemma 2.1, power mean inequality, and p-convexity of $|\Upsilon'|^q$, we find

$$\begin{aligned}
&\left|{}_1\Delta_\Upsilon(w_1,w_2;\alpha;\beta;J)\right|\\
&=\left|\frac{w_2^p-w_1^p}{2p}\int_0^1\left\{\beta_{1-\kappa}(n+1,\alpha-n)-\beta_\kappa(n+1,\alpha-n)\right\}\right.\\
&\quad\left.\times A_\kappa^{\frac{1}{p}-1}\Upsilon'\left(\left[\kappa w_1^p+(1-\kappa)w_2^p\right]^{\frac{1}{p}}\right)d\kappa\right|\\
&\leq\frac{w_2^p-w_1^p}{2p}\left(\int_0^1\left\{\beta_{1-\kappa}(n+1,\alpha-n)-\beta_\kappa(n+1,\alpha-n)\right\}d\kappa\right)^{1-1/q}\\
&\quad\times\left(\int_0^1 A_\kappa^{\frac{1}{p}-1}\left|\Upsilon'\left(\left[\kappa w_1^p+(1-\kappa)w_2^p\right]^{\frac{1}{p}}\right)\right|^q d\kappa\right)^{1/q}\\
&\leq\frac{w_2^p-w_1^p}{2p}\lambda^{1-1/q}\left(\int_0^1 A_\kappa^{\frac{1}{p}-1}\left[\kappa\left|\Upsilon'(w_1)\right|^q+(1-\kappa)\left|\Upsilon'(w_2)\right|^q\right]d\kappa\right)^{1/q}\\
&=\frac{w_2^p-w_1^p}{2p}\lambda^{1-1/q}\left(\lambda_1\left|\Upsilon'(w_1)\right|^q+\lambda_2\left|\Upsilon'(w_2)\right|^q\right)^{1/q},
\end{aligned}\tag{19}$$

where

$$\begin{aligned}
\lambda &= \int_0^1 \bigl(\beta_{1-\kappa}(n+1,\alpha-n) - \beta_\kappa(n+1,\alpha-n)\bigr)\,d\kappa \\
&= \int_0^1 \biggl(\int_0^{1-\kappa} u^n(1-u)^{\alpha-n-1}\,du\biggr)\,d\kappa + \int_0^1 \biggl(\int_0^{\kappa} u^n(1-u)^{\alpha-n-1}\,du\biggr)\,d\kappa \\
&= \kappa\biggl(\int_0^{1-\kappa} u^n(1-u)^{\alpha-n-1}\,du\biggr)\bigg|_0^1 + \int_0^1 \kappa(1-\kappa)^n\kappa^{\alpha-n-1}\,d\kappa \\
&\quad + \kappa\biggl(\int_0^{\kappa} u^n(1-u)^{\alpha-n-1}\,du\biggr)\bigg|_0^1 + \int_0^1 \kappa^{n+1}(1-\kappa)^{\alpha-n-1}\,d\kappa \\
&= \beta(n+1,\alpha-n+1) - \beta(n+1,\alpha-n) + \beta(n+2,\alpha-n), \\
\lambda_1 &= \int_0^1 \kappa A_\kappa^{\frac{1}{p}-1}\,d\kappa = \frac{w_2^{1-p}}{2}\,{}_2F_1\biggl(1-\frac{1}{p},2;3;1-\frac{w_1^p}{w_2^p}\biggr),
\end{aligned}$$

and

$$\lambda_2 = \int_0^1 (1-\kappa)A_\kappa^{\frac{1}{p}-1}\,d\kappa = \frac{w_2^{1-p}}{2}\,{}_2F_1\biggl(1-\frac{1}{p},1;3;1-\frac{w_1^p}{w_2^p}\biggr).$$

Hence the proof is completed.

(ii) Proof is similar to that of (i). □

Remark 2.3 By letting $p=-1$ in Theorem 2.2, we obtain Theorem 2.2 in [3].

Now, for the next two results, we consider the case when $p>0$ and leave the case when $p<0$ for the reader.

Theorem 2.3 *Let $\Upsilon:[w_1,w_2]\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on (w_1,w_2) with $w_1<w_2$ such that $\Upsilon'\in L[w_1,w_2]$ and $\alpha>0$. If $|\Upsilon'|^q$, where $q\geq 1$, is a p-convex function, then for $p>0$, we have*

$$\begin{aligned}
&\bigl|{}_1\Delta_\Upsilon(w_1,w_2;\alpha;\beta;J)\bigr| \\
&\quad\leq \frac{w_2^p-w_1^p}{2p}\mu^{1-1/q}\bigl((\mu_1-\mu_2)\bigl|\Upsilon'(w_1)\bigr|^q + (\mu_3-\mu_4)\bigl|\Upsilon'(w_2)\bigr|^q\bigr)^{1/q},
\end{aligned} \tag{20}$$

here

$$\begin{aligned}
\mu &= \frac{w_2^{1-p}}{2}\,{}_2F_1\biggl(1-\frac{1}{p},1;2;1-\frac{w_1^p}{w_2^p}\biggr), \\
\mu_1 &= \frac{1}{2}\beta(n+1,\alpha-n+2), \\
\mu_2 &= \frac{1}{2}\bigl(\beta(n+1,\alpha-n) - \beta(n+3,\alpha-n)\bigr), \\
\mu_3 &= \beta(n+2,\alpha-n+1) - \frac{1}{2}\beta(n+1,\alpha-n+2),
\end{aligned}$$

and

$$\mu_4 = \frac{1}{2}\beta(n+1,\alpha-n) + \frac{1}{2}\beta(n+3,\alpha-n) - \beta(n+2,\alpha-n).$$

Proof Applying Lemma 2.1, power mean inequality, and p-convexity of $|\Upsilon'|^q$, we have

$$\begin{aligned}
&\left|{}_1\Delta_{\Upsilon}(w_1,w_2;\alpha;\beta;J)\right| \\
&= \left|\frac{w_2^p - w_1^p}{2p}\int_0^1 \left\{\beta_{1-\kappa}(n+1,\alpha-n) - \beta_\kappa(n+1,\alpha-n)\right\}\right. \\
&\quad \left.\times A_\kappa^{\frac{1}{p}-1}\Upsilon'\left(\left[\kappa w_1^p + (1-\kappa)w_2^p\right]^{\frac{1}{p}}\right)d\kappa\right| \\
&\le \frac{w_2^p - w_1^p}{2p}\left(\int_0^1 A_\kappa^{\frac{1}{p}-1}\,d\kappa\right)^{1-1/q} \\
&\quad \times \left(\int_0^1 \left\{\beta_{1-\kappa}(n+1,\alpha-n) - \beta_\kappa(n+1,\alpha-n)\right\}\left|\Upsilon'\left(\left[\kappa w_1^p + (1-\kappa)w_2^p\right]^{\frac{1}{p}}\right)\right|^q d\kappa\right)^{1/q} \\
&\le \frac{w_2^p - w_1^p}{2p}\mu^{1-1/q}\left(\int_0^1 \left\{\beta_{1-\kappa}(n+1,\alpha-n) - \beta_\kappa(n+1,\alpha-n)\right\}\right. \\
&\quad \left.\times\left[\kappa\left|\Upsilon'(w_1)\right|^q + (1-\kappa)\left|\Upsilon'(w_2)\right|^q\right]d\kappa\right)^{1/q} \\
&= \frac{w_2^p - w_1^p}{2p}\mu^{1-1/q}\left((\mu_1-\mu_2)\left|\Upsilon'(w_1)\right|^q + (\mu_3-\mu_4)\left|\Upsilon'(w_2)\right|^q\right)^{1/q},
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\mu &= \int_0^1 A_\kappa^{\frac{1}{p}-1}\,d\kappa = \frac{w_2^{1-p}}{2}\,{}_2F_1\left(1-\frac{1}{p},1;2;1-\frac{w_1^p}{w_2^p}\right), \\
\mu_1 &= \int_0^1 \kappa\beta_{1-\kappa}(n+1,\alpha-n)\,d\kappa = \frac{1}{2}\beta(n+1,\alpha-n+2), \\
\mu_2 &= \int_0^1 \kappa\beta_\kappa(n+1,\alpha-n) = \frac{1}{2}\left(\beta(n+1,\alpha-n) - \beta(n+3,\alpha-n)\right), \\
\mu_3 &= \int_0^1 (1-\kappa)\beta_{1-\kappa}(n+1,\alpha-n)\,d\kappa = \beta(n+2,\alpha-n+1) - \frac{1}{2}\beta(n+1,\alpha-n+2),
\end{aligned}$$

and

$$\begin{aligned}
\mu_4 &= \int_0^1 (1-\kappa)\beta_\kappa(n+1,\alpha-n)\,d\kappa \\
&= \frac{1}{2}\beta(n+1,\alpha-n) + \frac{1}{2}\beta(n+3,\alpha-n) - \beta(n+2,\alpha-n).
\end{aligned}$$

Hence the proof is completed. □

Theorem 2.4 *Let $\Upsilon : [w_1,w_2] \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on (w_1,w_2) with $w_1 < w_2$ such that $\Upsilon' \in L[w_1,w_2]$ and $\alpha > 0$. If $|\Upsilon'|^q$, where $q,l > 1$ with $\frac{1}{q} + \frac{1}{l} = 1$, is a* *p-convex function, then*

$$\begin{aligned}
&\left|{}_1\Delta_{\Upsilon}(w_1,w_2;\alpha;\beta;J)\right| \\
&\quad\le \frac{w_2^p-w_1^p}{2p}\nu^{\frac{1}{l}}\left(\nu_1\left|\Upsilon'(w_1)\right|^q+\nu_2\left|\Upsilon'(w_2)\right|^q\right)^{1/q},
\end{aligned}\tag{22}$$

here

$$\begin{aligned}
\nu &= 2\int_0^{\frac{1}{2}}\left(\int_\kappa^{1-\kappa}u^n(1-u)^{\alpha-n-1}\,du\right)d\kappa, \\
\nu_1 &= \frac{w_2^{q(1-p)}}{2}\,{}_2F_1\left(q\left(1-\frac{1}{p}\right),2;3;1-\frac{w_1^p}{w_2^p}\right), \\
\nu_2 &= \frac{w_2^{q(1-p)}}{2}\,{}_2F_1\left(q\left(1-\frac{1}{p}\right),1;3;1-\frac{w_1^p}{w_2^p}\right).
\end{aligned}$$

Proof Let $A_\kappa=[\kappa w_1^p+(1-\kappa)w_2^p]$. Applying Lemma 2.1, Hölder's inequality, and p-convexity of $|\Upsilon'|^q$, we have

$$\begin{aligned}
&\left|{}_1\Delta_{\Upsilon}(w_1,w_2;\alpha;\beta;J)\right| \\
&\quad= \left|\frac{w_2^p-w_1^p}{2p}\int_0^1\left\{\beta_{1-\kappa}(n+1,\alpha-n)-\beta_\kappa(n+1,\alpha-n)\right\}\right. \\
&\qquad\left.\times A_\kappa^{\frac{1}{p}-1}\Upsilon'\left(\left[\kappa w_1^p+(1-\kappa)w_2^p\right]^{\frac{1}{p}}\right)d\kappa\right| \\
&\quad\le \frac{w_2^p-w_1^p}{2p}\left(\int_0^1\left|\beta_{1-\kappa}(n+1,\alpha-n)-\beta_\kappa(n+1,\alpha-n)\right|^l d\kappa\right)^{\frac{1}{l}} \\
&\qquad\times\left(\int_0^1 A_\kappa^{q(\frac{1}{p}-1)}\left|\Upsilon'\left(\left[\kappa w_1^p+(1-\kappa)w_2^p\right]^{\frac{1}{p}}\right)\right|^q d\kappa\right)^{1/q} \\
&\quad\le \frac{w_2^p-w_1^p}{2p}\nu^{\frac{1}{l}}\left(\int_0^1 A_\kappa^{q(\frac{1}{p}-1)}\left[\kappa\left|\Upsilon'(w_1)\right|^q+(1-\kappa)\left|\Upsilon'(w_2)\right|^q\right]d\kappa\right)^{1/q} \\
&\quad= \frac{w_2^p-w_1^p}{2p}\nu^{\frac{1}{p}}\left(\nu_1\left|\Upsilon'(w_1)\right|^q+\nu\left|\Upsilon'(w_2)\right|^q\right)^{1/q},
\end{aligned}\tag{23}$$

where

$$\begin{aligned}
\nu &= \int_0^1\left|\beta_{1-\kappa}(n+1,\alpha-n)-\beta_\kappa(n+1,\alpha-n)\right|^l d\kappa \\
&= \int_0^{\frac{1}{2}}\left(\beta_{1-\kappa}(n+1,\alpha-n)-\beta_\kappa(n+1,\alpha-n)\right)^l d\kappa \\
&\quad+\int_{\frac{1}{2}}^1\left(\beta_\kappa(n+1,\alpha-n)-\beta_{1-\kappa}(n+1,\alpha-n)\right)^l d\kappa \\
&= \int_0^{\frac{1}{2}}\left(\int_\kappa^{1-\kappa}u^n(1-u)^{\alpha-n-1}\,du\right)^l d\kappa+\int_{\frac{1}{2}}^1\left(\int_{1-\kappa}^{\kappa}u^n(1-u)^{\alpha-n-1}\,du\right)^l d\kappa \\
&= 2\int_0^{\frac{1}{2}}\left(\int_\kappa^{1-\kappa}u^n(1-u)^{\alpha-n-1}\,du\right)^l d\kappa,
\end{aligned}$$

$$\nu_1 = \int_0^1 \kappa A_\kappa^{q(\frac{1}{p}-1)}\,d\kappa = \frac{w_2^{q(1-p)}}{2}\,{}_2F_1\left(q\left(1-\frac{1}{p}\right), 2; 3; 1-\frac{w_1^p}{w_2^p}\right),$$

and

$$\nu_2 = \int_0^1 (1-\kappa) A_\kappa^{q(\frac{1}{p}-1)}\,d\kappa = \frac{w_2^{q(1-p)}}{2}\,{}_2F_1\left(q\left(1-\frac{1}{p}\right), 1; 3; 1-\frac{w_1^p}{w_2^p}\right).$$

Hence the proof is completed. □

3 Hermite–Hadamard–Fejér type inequalities

In this section we prove some Hermite–Hadamard–Fejér type inequalities for p-convex functions via conformable fractional integral. First we give the following useful definition.

Definition 3.1 ([15]) Let $p \in \mathbb{R} \setminus \{0\}$. A function $\Upsilon : [w_1, w_2] \subseteq (0,\infty) \to \mathbb{R}$ is called p-symmetric around $[\frac{w_1^p + w_2^p}{2}]^{1/p}$ if

$$\Upsilon(x) = \Upsilon\left(\left[w_1^p + w_2^p - x^p\right]^{\frac{1}{p}}\right)$$

holds for all $x \in [w_1, w_2]$.

Now we prove the following identity.

Lemma 3.1 *Let $p \in \mathbb{R} \setminus \{0\}$. If $\Upsilon : [w_1, w_2] \subseteq (0,\infty) \to \mathbb{R}$ is integrable and p-symmetric around $[\frac{w_1^p + w_2^p}{2}]^{1/p}$, then*

(i) *for $p > 0$, we have*

$$\begin{aligned} J_\alpha^{w_1^p}(\Upsilon \circ \phi)\left(w_2^p\right) &= {}^{w_2^p}J_\alpha(\Upsilon \circ \phi)\left(w_1^p\right) \\ &= \frac{1}{2}\left[J_\alpha^{w_1^p}(\Upsilon \circ \phi)\left(w_2^p\right) + {}^{w_2^p}J_\alpha(\Upsilon \circ \phi)\left(w_1^p\right)\right], \end{aligned} \tag{24}$$

with $\alpha > 0$ and $\phi(u) = u^{\frac{1}{p}}$, for all $u \in [w_1^p, w_2^p]$;

(ii) *for $p < 0$, we have*

$$\begin{aligned} J_\alpha^{w_2^p}(\Upsilon \circ \phi)\left(w_1^p\right) &= {}^{w_1^p}J_\alpha(\Upsilon \circ \phi)\left(w_2^p\right) \\ &= \frac{1}{2}\left[J_\alpha^{w_2^p}(\Upsilon \circ \phi)\left(w_1^p\right) + {}^{w_1^p}J_\alpha(\Upsilon \circ \phi)\left(w_2^p\right)\right], \end{aligned} \tag{25}$$

with $\alpha > 0$ and $\phi(u) = u^{\frac{1}{p}}$, for all $u \in [w_2^p, w_1^p]$.

Proof (i) Since Υ is p-symmetric around $[\frac{w_1^p + w_2^p}{2}]^{1/p}$, then by definition we have $\Upsilon(x^{\frac{1}{p}}) = \Upsilon([w_1^p + w_2^p - x]^{\frac{1}{p}})$ for all $x \in [w_1^p, w_2^p]$. In the following integral, setting $u = w_1^p + w_2^p - x$

gives

$$\begin{aligned} J_\alpha^{w_1^p}(\curlyvee \circ \phi)(w_2^p) &= \frac{1}{n!}\int_{w_1^p}^{w_2^p}(w_2^p-u)^n(u-w_1^p)^{\alpha-n-1}\curlyvee(u^{\frac{1}{p}})\,du \\ &= \frac{1}{n!}\int_{w_1^p}^{w_2^p}(x-w_1^p)^n(w_2^p-x)^{\alpha-n-1}\curlyvee([w_1^p+w_2^p-x]^{\frac{1}{p}})dx \\ &= \frac{1}{n!}\int_{w_1^p}^{w_2^p}(x-w_1^p)^n(w_2^p-x)^{\alpha-n-1}\curlyvee(x^{\frac{1}{p}})dx \\ &= {}^{w_2^p}J_\alpha(\curlyvee \circ \phi)(w_1^p). \end{aligned} \tag{26}$$

This completes the proof.

(ii) Proof is similar to that of (i). □

Remark 3.1 In Lemma 3.1:

1. By taking $\alpha = n+1$, we obtain Lemma 1 in [16].
2. By taking $\alpha = n+1$ and $p = 1$, we find Lemma 3 of [10].

Corollary 3.1 *Under the assumptions of Lemma* 3.1:

1. *If* $p = 1$ *in* (i), *then we get*

$$J_\alpha^{w_1}\curlyvee(w_2) = {}^{w_2}J_\alpha \curlyvee(w_1) = \frac{1}{2}\big[J_\alpha^{w_1}\curlyvee(w_2) + {}^{w_2}J_\alpha\curlyvee(w_1)\big]. \tag{27}$$

2. *If* $p = -1$ *in* (ii), *then we get*

$$\begin{aligned} {}^{1/w_1}J_\alpha(\curlyvee \circ \phi)(1/w_2) &= J_\alpha^{1/w_2}(\curlyvee \circ \phi)(1/w_1) \\ &= \frac{1}{2}\big[{}^{1/w_1}J_\alpha(\curlyvee \circ \phi)(1/w_2) + J_\alpha^{1/w_2}(\curlyvee \circ \phi)(1/w_1)\big]. \end{aligned} \tag{28}$$

Theorem 3.2 *Let* $p \in \mathbb{R}\setminus\{0\}$. *Let* $\Upsilon : [w_1, w_2] \subset (0,\infty) \to \mathbb{R}$ *be a p-convex function with* $w_1 < w_2$ *and* $\Upsilon \in L[w_1, w_2]$. *If* $\curlyvee : [w_1, w_2] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *is nonnegative, integrable, and p-symmetric around* $[\frac{w_1^p+w_2^p}{2}]^{1/p}$. *Then*

(i) *for* $p > 0$, *the following inequalities hold*:

$$\begin{aligned} &\Upsilon\left(\left[\frac{w_1^p+w_2^p}{2}\right]^{1/p}\right)\big[J_\alpha^{w_1^p}(\curlyvee \circ \phi)(w_2^p) + {}^{w_2^p}J_\alpha(\curlyvee \circ \phi)(w_1^p)\big] \\ &\le \big[J_\alpha^{w_1^p}(\Upsilon(\curlyvee \circ \phi))(w_2^p) + {}^{w_2^p}J_\alpha(\Upsilon(\curlyvee \circ \phi))(w_1^p)\big] \\ &\le \frac{\Upsilon(w_1)+\Upsilon(w_2)}{2}\big[J_\alpha^{w_1^p}(\curlyvee \circ \phi)(w_2^p) + {}^{w_2^p}J_\alpha(\curlyvee \circ \phi)(w_1^p)\big], \end{aligned} \tag{29}$$

with $\alpha > 0$ *and* $\phi(x) = x^{\frac{1}{p}}$, *for all* $x \in [w_1^p, w_2^p]$;

(ii) *for $p < 0$, the following inequalities hold*:

$$\begin{aligned}
&\Upsilon\left(\left[\frac{w_1^p + w_2^p}{2}\right]^{1/p}\right)\left[J_\alpha^{w_2^p}(\curlyvee \circ \phi)\left(w_1^p\right) + {}^{w_1^p}J_\alpha(\curlyvee \circ \phi)\left(w_2^p\right)\right] \\
&\quad\le \left[J_\alpha^{w_2^p}\left(\Upsilon(\curlyvee \circ \phi)\right)\left(w_1^p\right) + {}^{w_1^p}J_\alpha\left(\Upsilon(\curlyvee \circ \phi)\right)\left(w_2^p\right)\right] \\
&\quad\le \frac{\Upsilon(w_1) + \Upsilon(w_2)}{2}\left[J_\alpha^{w_2^p}(\curlyvee \circ \phi)\left(w_1^p\right) + {}^{w_1^p}J_\alpha(\curlyvee \circ \phi)\left(w_2^p\right)\right],
\end{aligned} \tag{30}$$

with $\alpha > 0$ and $\phi(x) = x^{\frac{1}{p}}$, for all $x \in [w_2^p, w_1^p]$.

Proof (i) Since Υ is a p-convex function on $[w_1, w_2]$, we have

$$\Upsilon\left(\left[\frac{x^p + y^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{\Upsilon(x) + \Upsilon(y)}{2}.$$

Taking $x^p = \kappa w_1^p + (1-\kappa)w_2^p$ and $y^p = (1-\kappa)w_1^p + \kappa w_2^p$ with $\kappa \in [0,1]$, we get

$$\Upsilon\left(\left[\frac{w_1^p + w_2^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{\Upsilon([\kappa w_1^p + (1-\kappa)w_2^p]^{\frac{1}{p}}) + \Upsilon([(1-\kappa)w_1^p + \kappa w_2^p]^{\frac{1}{p}})}{2}. \tag{31}$$

Multiplying (31) by $\frac{1}{n!}\kappa^n(1-\kappa)^{\alpha-n-1} \curlyvee ([\kappa w_1^p + (1-\kappa)w_2^p]^{\frac{1}{p}})$ on both sides, $\alpha > 0$ and then integrating about κ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{n!}\Upsilon\left(\left[\frac{w_1^p + w_2^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1} \curlyvee \left(\left[\kappa w_1^p + (1-\kappa)w_2^p\right]^{\frac{1}{p}}\right) d\kappa \\
&\quad\le \frac{1}{n!}\int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1}\Upsilon\left(\left[\kappa w_1^p + (1-\kappa)w_2^p\right]^{\frac{1}{p}}\right) \curlyvee \left(\left[\kappa w_1^p + (1-\kappa)w_2^p\right]^{\frac{1}{p}}\right) d\kappa \\
&\qquad + \frac{1}{n!}\int_0^1 \kappa^n(1-\kappa)^{\alpha-n-1}\Upsilon\left(\left[(1-\kappa)w_1^p + \kappa w_2^p\right]^{\frac{1}{p}}\right) \curlyvee \left(\left[\kappa w_1^p + (1-\kappa)w_2^p\right]^{\frac{1}{p}}\right) d\kappa.
\end{aligned} \tag{32}$$

Since $\curlyvee$ is nonnegative, integrable, and p-symmetric with respect to $[\frac{w_1^p+w_2^p}{2}]^{1/p}$, then

$$\curlyvee\left(\left[\kappa w_1^p + (1-\kappa)w_2^p\right]^{\frac{1}{p}}\right) = \curlyvee\left(\left[\kappa w_2^p + (1-\kappa)w_1^p\right]^{\frac{1}{p}}\right).$$

Also choosing $u = \kappa w_1^p + (1-\kappa)w_2^p$ leads to

$$\begin{aligned}
&\frac{2}{n!(w_2^p - w_1^p)^\alpha}\Upsilon\left(\left[\frac{w_1^p + w_2^p}{2}\right]^{\frac{1}{p}}\right)\int_{w_1^p}^{w_2^p} \left(w_2^p - u\right)^n\left(u - w_1^p\right)^{\alpha-n-1} \curlyvee \left(u^{\frac{1}{p}}\right) du \\
&\quad\le \frac{1}{n!(w_2^p - w_1^p)^\alpha}\left[\int_{w_1^p}^{w_2^p} \left(w_2^p - u\right)^n\left(u - w_1^p\right)^{\alpha-n-1}\Upsilon\left(u^{\frac{1}{p}}\right) \curlyvee \left(u^{\frac{1}{p}}\right) du\right. \\
&\qquad \left.+ \int_{w_1^p}^{w_2^p} \left(w_2^p - u\right)^n\left(u - w_1^p\right)^{\alpha-n-1}\Upsilon\left(\left[w_1^p + w_2^p - u\right]^{\frac{1}{p}}\right) \curlyvee \left(u^{\frac{1}{p}}\right) du\right]
\end{aligned}$$

$$= \frac{1}{n!(w_2^p - w_1^p)^\alpha}\left[\int_{w_1^p}^{w_2^p}(w_2^p - u)^n(u - w_1^p)^{\alpha-n-1}\Upsilon(u^{\frac{1}{p}})\curlyvee(u^{\frac{1}{p}})\,du + \int_{w_1^p}^{w_2^p}(u - w_1^p)^n(w_2^p - u)^{\alpha-n-1}\Upsilon(u^{\frac{1}{p}})\curlyvee([w_1^p + w_2^p - u]^{\frac{1}{p}})\,du\right]. \tag{33}$$

Therefore, by Lemma 3.1 we have

$$\frac{1}{(w_2^p - w_1^p)^\alpha}\Upsilon\left(\left[\frac{w_1^p + w_2^p}{2}\right]^{\frac{1}{p}}\right)\left[J_\alpha^{w_1^p}(\curlyvee\circ\phi)(b^p) + {}^{w_2^p}J_\alpha(\curlyvee\circ\phi)(w_1^p)\right] \leq \frac{1}{(w_2^p - w_1^p)^\alpha}\left[J_\alpha^{w_1^p}(\Upsilon(\curlyvee\circ\phi))(w_2^p) + {}^{w_2^p}J_\alpha(\Upsilon(\curlyvee\circ\phi))(w_1^p)\right]. \tag{34}$$

This completes the first inequality of (29). For the second inequality, we first note that if Υ is a p-convex function, then we have

$$\Upsilon([\kappa w_1^p + (1-\kappa)w_2^p]^{\frac{1}{p}}) + \Upsilon([\kappa w_2^p + (1-\kappa)w_1^p]^{\frac{1}{p}}) \leq [\Upsilon(w_1) + \Upsilon(w_2)]. \tag{35}$$

Multiplying (35) by $\frac{1}{n!}\kappa^n(1-\kappa)^{\alpha-n-1}\curlyvee([\kappa w_1^p + (1-\kappa)w_2^p]^{\frac{1}{p}})$ on both sides, $\alpha > 0$ and then integrating about κ over $[0,1]$, we obtain

$$\frac{1}{n!}\int_0^1\kappa^n(1-\kappa)^{\alpha-n-1}\Upsilon([\kappa_1^p + (1-\kappa)w_2^p]^{\frac{1}{p}})\curlyvee([\kappa w_1^p + (1-\kappa)w_2^p]^{\frac{1}{p}})\,d\kappa + \frac{1}{n!}\int_0^1\kappa^n(1-\kappa)^{\alpha-n-1}\Upsilon([\kappa w_2^p + (1-\kappa)w_1^p]^{\frac{1}{p}})\curlyvee([\kappa w_1^p + (1-\kappa)w_2^p]^{\frac{1}{p}})\,d\kappa \leq [\Upsilon(w_1) + \Upsilon(w_2)]\frac{1}{n!}\int_0^1\kappa^n(1-\kappa)^{\alpha-n-1}\curlyvee([\kappa w_1^p + (1-\kappa)w_2^p]^{\frac{1}{p}})\,d\kappa. \tag{36}$$

That is,

$$\frac{1}{(w_2^p - w_1^p)^\alpha}\left[J_\alpha^{w_1^p}(\Upsilon(\curlyvee\circ\phi))(w_2^p) + {}^{w_2^p}J_\alpha(\Upsilon(\curlyvee\circ\phi))(w_1^p)\right] \leq \frac{1}{(w_2^p - w_1^p)^\alpha}\left[\frac{\Upsilon(w_1) + \Upsilon(w_2)}{2}\right]\left[J_\alpha^{w_1^p}(\curlyvee\circ\phi)(w_2^p) + {}^{w_2^p}J_\alpha(\curlyvee\circ\phi)(w_1^p)\right]. \tag{37}$$

This completes the proof.

(ii) Proof is similar to that of (i). □

Remark 3.2 In Theorem 3.2:

1. If $\alpha = n+1$, we obtain Theorem 9 in [16].
2. If $\alpha = n+1$ and $p = 1$, we find Theorem 4 in [10].

Corollary 3.3 *Under the assumptions of Theorem* 3.2:

1. *If* $p = 1$, *then*

$$\begin{aligned}
&\Upsilon\left(\frac{w_1+w_2}{2}\right)\left[J_\alpha^{w_1} \curlyvee (w_2) + {}^{w_2}J_\alpha \curlyvee (w_1)\right] \\
&\quad\le \left[J_\alpha^{w_1}(\Upsilon\curlyvee)(w_2) + {}^{w_2}J_\alpha(\Upsilon\curlyvee)(w_1)\right] \\
&\quad\le \frac{\Upsilon(w_1)+\Upsilon(w_2)}{2}\left[J_\alpha^{w_1} \curlyvee (w_2) + {}^{w_2}J_\alpha \curlyvee (w_1)\right].
\end{aligned}\tag{38}$$

2. *If* $p = -1$, *then*

$$\begin{aligned}
&\Upsilon\left(\frac{2w_1w_2}{w_1+w_2}\right)\left[{}^{1/w_1}J_\alpha(\curlyvee\circ\phi)(1/w_2) + J_\alpha^{1/w_2}(\curlyvee\circ\phi)(1/w_1)\right] \\
&\quad\le \left[{}^{1/w_1}J_\alpha\big(\Upsilon(\curlyvee\circ\phi)\big)(1/w_2) + J_\alpha^{1/w_2}\big(\Upsilon(\curlyvee\circ\phi)\big)(1/w_1)\right] \\
&\quad\le \frac{\Upsilon(w_1)+\Upsilon(w_2)}{2}\left[{}^{1/w_1}J_\alpha(\curlyvee\circ\phi)(1/w_2) + J_\alpha^{1/w_2}(\curlyvee\circ\phi)(1/w_1)\right].
\end{aligned}\tag{39}$$

Remark 3.3 In Corollary 3.3(1), if we take $\alpha = n + 1$, we get inequality (3).

Lemma 3.2 *Let* $p \in \mathbb{R} \setminus \{0\}$ *and* $\alpha > 0$. *Let* $\Upsilon : [w_1, w_2] \subset (0,\infty) \to \mathbb{R}$ *be a differentiable mapping and* $\Upsilon \in L[w_1, w_2]$. *If* $\curlyvee : [w_1, w_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *is nonnegative, integrable, and p-symmetric around* $[\frac{w_1^p+w_2^p}{2}]^{1/p}$, *then*

(i) *for* $p > 0$, *the following inequality holds*:

$$\begin{aligned}
&\frac{\Upsilon(w_1)+\Upsilon(w_2)}{2}\left[J_\alpha^{w_1^p}(\curlyvee\circ\phi)\big(w_2^p\big) + {}^{w_2^p}J_\alpha(\curlyvee\circ\phi)\big(w_1^p\big)\right] \\
&\qquad - \left[J_\alpha^{w_1^p}\big(\Upsilon(\curlyvee\circ\phi)\big)\big(w_2^p\big) + {}^{w_2^p}J_\alpha\big(\Upsilon(\curlyvee\circ\phi)\big)\big(w_1^p\big)\right] \\
&\quad\le \frac{1}{n!}\int_{w_1^p}^{w_2^p}\left[\int_{w_1^p}^{t}\big(w_2^p-s\big)^n\big(s-w_1^p\big)^{\alpha-n-1}(\curlyvee\circ\phi)(s)\,ds\right. \\
&\qquad \left. - \int_t^{w_2^p}\big(s-w_1^p\big)^n\big(w_2^p-s\big)^{\alpha-n-1}(\curlyvee\circ\phi)(s)\,ds\right](\Upsilon\circ\phi)'(t)\,dt,
\end{aligned}\tag{40}$$

where $\phi(x) = x^{1/p}$ *for all* $x \in [w_1^p, w_2^p]$;

(ii) *for* $p < 0$, *the following inequality holds*:

$$\begin{aligned}
&\frac{\Upsilon(w_1)+\Upsilon(w_2)}{2}\left[J_\alpha^{w_2^p}(\curlyvee\circ\phi)\big(w_1^p\big) + {}^{w_1^p}J_\alpha(\curlyvee\circ\phi)\big(w_2^p\big)\right] \\
&\qquad - \left[J_\alpha^{w_2^p}\big(\Upsilon(\curlyvee\circ\phi)\big)\big(w_1^p\big) + {}^{w_1^p}J_\alpha\big(\Upsilon(\curlyvee\circ\phi)\big)\big(w_2^p\big)\right] \\
&\quad\le \frac{1}{n!}\int_{w_2^p}^{w_1^p}\left[\int_{w_2^p}^{t}\big(w_1^p-s\big)^n\big(s-w_2^p\big)^{\alpha-n-1}(\curlyvee\circ\phi)(s)\,ds\right. \\
&\qquad \left. - \int_t^{w_1^p}\big(s-w_2^p\big)^n\big(w_1^p-s\big)^{\alpha-n-1}(\curlyvee\circ\phi)(s)\,ds\right](\Upsilon\circ\phi)'(t)\,dt,
\end{aligned}\tag{41}$$

where $\phi(x) = x^{1/p}$ *for all* $x \in [w_2^p, w_1^p]$.

Proof (i) Note that

$$\begin{aligned} I &= \int_{w_1^p}^{w_2^p} \left(\int_{w_1^p}^{t} (w_2^p - s)^n (s - w_1^p)^{\alpha-n-1} (\curlyvee \circ \phi)(s)\,ds \right) (\Upsilon \circ \phi)'(t)\,dt \\ &\quad - \int_{w_1^p}^{w_2^p} \left(\int_{t}^{w_2^p} (s - w_1^p)^n (w_2^p - s)^{\alpha-n-1} (\curlyvee \circ \phi)(s)\,ds \right) (\Upsilon \circ \phi)'(t)\,dt \\ &= I_1 - I_2. \end{aligned} \tag{42}$$

Integrating by parts and using Lemma 3.1, we get

$$\begin{aligned} I_1 &= \left(\int_{w_1^p}^{t} (w_2^p - s)^n (s - w_1^p)^{\alpha-n-1} (\curlyvee \circ \phi)(s)\,ds \right) (\Upsilon \circ \phi)(t) \Big|_{w_1^p}^{w_2^p} \\ &\quad - \int_{w_1^p}^{w_2^p} (w_2^p - t)^n (t - w_1^p)^{\alpha-n-1} (\curlyvee \circ \phi)(t) (\Upsilon \circ \phi)(t)\,dt \\ &= n! \left[(\Upsilon \circ \phi)(w_2^p) J_\alpha^{w_1^p} (\curlyvee \circ \phi)(w_2^p) - J_\alpha^{w_1^p} \big(\Upsilon(\curlyvee \circ \phi)\big)(w_2^p) \right] \\ &= n! \left[\frac{(\Upsilon \circ \phi)(w_2^p)}{2} \left\{ {}^{w_2^p}J_\alpha (\curlyvee \circ \phi)(w_1^p) + J_\alpha^{w_1^p} (\curlyvee \circ \phi)(w_2^p) \right\} - J_\alpha^{w_1^p} \big(\Upsilon(\curlyvee \circ \phi)\big)(w_2^p) \right]. \end{aligned} \tag{43}$$

Similarly,

$$\begin{aligned} I_2 &= \left(\int_{t}^{w_2^p} (s - w_1^p)^n (w_2^p - s)^{\alpha-n-1} (\curlyvee \circ \phi)(s)\,ds \right) (\Upsilon \circ \phi)(t) \Big|_{w_1^p}^{w_2^p} \\ &\quad + \int_{w_1^p}^{w_2^p} (t - w_1^p)^n (w_2^p - t)^{\alpha-n-1} (\curlyvee \circ \phi)(t) (\Upsilon \circ \phi)(t)\,dt \\ &= n! \left[-(\Upsilon \circ \phi)(w_1^p)\, {}^{w_2^p}J_\alpha (\curlyvee \circ \phi)(w_1^p) - {}^{w_2^p}J_\alpha \big(\Upsilon(\curlyvee \circ \phi)\big)(w_1^p) \right] \\ &= n! \left[\frac{-(\Upsilon \circ \phi)(w_1^p)}{2} \left\{ {}^{w_2^p}J_\alpha (\curlyvee \circ \phi)(w_1^p) + J_\alpha^{w_1^p} (\curlyvee \circ \phi)(w_2^p) \right\} \right. \\ &\quad \left. + {}^{w_2^p}J_\alpha \big(\Upsilon(\curlyvee \circ \phi)\big)(w_1^p) \right]. \end{aligned} \tag{44}$$

Thus from (43) and (44) we get

$$\begin{aligned} I &= I_1 - I_2 \\ &= n! \left[\frac{\Upsilon(w_1) + \Upsilon(w_2)}{2} \left[J_\alpha^{w_1^p} (\curlyvee \circ \phi)(w_2^p) + {}^{w_2^p}J_\alpha (\curlyvee \circ \phi)(w_1^p) \right] \right. \\ &\quad \left. - \left[J_\alpha^{w_1^p} \big(\Upsilon(\curlyvee \circ \phi)\big)(w_2^p) + {}^{w_2^p}J_\alpha \big(\Upsilon(\curlyvee \circ \phi)\big)(w_1^p) \right] \right]. \end{aligned} \tag{45}$$

Multiplying (45) by $\frac{1}{n!}$, we obtain (40).

(ii) Proof is similar to that of (i). □

Remark 3.4 In Lemma 3.2:

1. If we take $\alpha = n + 1$, we get Lemma 2 in [16].
2. If we take $\alpha = n + 1$ and $p = 1$, we get Lemma 4 in [10].

Lemma 3.2 also holds for convex functions and harmonically convex functions just by taking $p = 1$ and $p = -1$, respectively. Also, from Lemma 3.2 we can establish more useful results.

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References

1. Abbas, G., Farid, G.: Hadamard and Fejér–Hadamard type inequalities for harmonically convex functions via generalized fractional integrals. J. Anal. **25**(1), 107–119 (2017)
2. Abdeljawad, T.: On confermable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015)
3. Awan, M.U., Noor, M.A., Mihai, M.V., Noor, K.I.: Inequalities via harmonically convex functions: conformable fractional calculus approach. J. Math. Inequal. **12**(1), 143–153 (2008)
4. Chen, F.: Extension of the Hermite–Hadamard inequality for harmonically convex functions via fractional integrals. Appl. Math. Comput. **268**, 121–128 (2015)
5. Dragomir, S.S.: Hermite–Hadamard's type inequality for operator convex functions. Appl. Math. Comput. **218**(3), 766–772 (2011)
6. Farid, G.: Hadamard and Fejér–Hadamard inequalities for generalized fractional integrals involving special functions. Konuralp J. Math. **4**(1), 108–113 (2016)
7. Farid, G.: A treatment of the Hadamard inequality due to *m*-convexity via generalized fractional integral. J. Fract. Calc. Appl. **9**(1), 8–14 (2018)
8. Fejér, L.: Über die fourierreihen, II. Math. Naturwiss, Anz. Ungar. Akad. Wiss. **24**, 369–390 (1906) (in Hungarian)
9. Hadamard, J.: Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. J. Math. Pures Appl., 171–215 (1893)
10. Iscan, I.: Hermite–Hadamard–Fejér type inequalities for convex functions via fractional integrals. Stud. Univ. Babeş–Bolyai, Math. **60**(3), 355–366 (2015)
11. Iscan, I.: Hermite–Hadamard type inequalities for *p*-convex functions. Int. J. Anal. Appl. **11**(2), 137–145 (2016)
12. Iscan, I.: Ostrowski type inequalities for *p*-convex functions. New Trends Math. Sci. **4**(3), 140–150 (2016)
13. Iscan, I., Wu, S.: Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals. Appl. Math. Comput. **237**, 237–244 (2014)
14. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
15. Kunt, M., Iscan, I.: Hermite–Hadamard–Fejér type inequalities for *p*-convex functions. Arab J. Math. Sci. **23**, 215–230 (2017)
16. Kunt, M., Iscan, I.: Hermite–Hadamard–Fejér type inequalities for *p*-convex functions via fractional integrals. Iran. J. Sci. Technol., Trans. A, Sci. **42**, 2079–2089 (2018)
17. Latif, M.A., Shoaib, M.: Hermite–Hadamard type integral inequalities for differentiable *m*-preinvex and α, *m*-preinvex functions. J. Egypt. Math. Soc. **23**(2), 236–241 (2015)
18. Mehreen, N., Anwar, M.: Integral inequalities for some convex functions via generalized fractional integrals. J. Inequal. Appl. **2018**, 208 (2018)
19. Mehreen, N., Anwar, M.: Hermite–Hadamard and Hermite–Hadamard–Fejér type inequalities for *p*-convex functions via new fractional conformable integral operators. J. Math. Comput. Sci. **19**, 230–240 (2019)
20. Mehreen, N., Anwar, M.: Hermite–Hadamard type inequalities via exponentially *p*-convex functions and exponentially *s*-convex functions in second sense with applications. J. Inequal. Appl. **2019**, 92 (2019)
21. Mehreen, N., Anwar, M.: Some inequalities via ψ-Riemann–Liouville fractional integrals. AIMS Math. **4**(5), 1403–1415 (2019)

22. Mehreen, N., Anwar, M.: On some Hermite–Hadamard type inequalities for *tgs*-convex functions via generalized fractional integrals. Adv. Differ. Equ. **2020**, 6 (2020)
23. Mehreen, N., Anwar, M.: Hermite–Hadamard type inequalities via exponentially (p,h)-convex functions. IEEE Access **8**, 37589–37595 (2020)
24. Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N.: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**(9), 2403–2407 (2013)
25. Set, E., Akdemir, A.O., Mumcu, I.: The Hermite–Hadamar's inequality and its extensions for conformable fractional integrals of any order $\alpha > 0$. Preprint (2016)
26. Set, E., Ozdemir, M.E., Dragomir, S.S.: On Hadamard-type inequalities involving several kinds of convexity. J. Inequal. Appl. **2010**, 286845 (2010)
27. Set, E., Sarikaya, M.Z., Gozpinar, A.: Some Hermite–Hadamard type inequalities for convex functions via conformable fractional integrals and related inequalities. Preprint (2016)
28. Set, E., Sarikaya, M.Z., Ozdemir, M.E., Yaldirm, H.: The Hermite–Hadamard's inequality for some convex functions via fractional integrals and related results. J. Appl. Math. Stat. Inform. **10**(2), 69–83 (2014)
29. Toader, G.H.: Some generalizations of the convexity. In: Proc. Colloq. Approx. Optim., Cluj-Napoca (Romania), pp. 329–338 (1984)
30. Ullah, S., Farid, G., Khan, K.A., Waheed, A., Mehmood, S.: Generalized fractional inequalities for quasi-convex functions. Adv. Differ. Equ. **2019**, 15 (2019)