Lee and Park *Journal of Inequalities and Applications* (2020) 2020:91
https://doi.org/10.1186/s13660-020-02359-z

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Asymptotic behavior of reciprocal sum of two products of Fibonacci numbers

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Abstract

Let $\{f_k\}_{k=1}^{\infty}$ be a Fibonacci sequence with $f_1 = f_2 = 1$. In this paper, we find a simple form g_n such that

$$\lim_{n\to\infty}\left\{\left(\sum_{k=n}^{\infty} a_k\right)^{-1} - g_n\right\} = 0,$$

where $a_k = \frac{1}{f_k^2}$, $\frac{1}{f_k f_{k+m}}$, or $\frac{1}{f_{3k}^2}$. For example, we show that

$$\lim_{n\to\infty}\left\{\left(\sum_{k=n}^{\infty}\frac{1}{f_{3k}^2}\right)^{-1} - \left(f_{3n}^2 - f_{3n-3}^2 + \frac{4}{9}(-1)^n\right)\right\} = 0.$$

MSC: 11B39; 11Y55; 40A05

Keywords: Fibonacci number; Reciprocal sum; Catalan's identity; Convergent series

1 Introduction

Last decade many mathematicians were interested in finding the formula for the integer part of the reciprocal tails of the convergent series. Precisely, one can see the explicit value of $\lfloor(\sum_{k=n}^{\infty} a_k)^{-1}\rfloor$ when $\sum_{k=1}^{\infty} a_k$ converges. This problem starts from the reciprocal sum of Fibonacci numbers. Let $f_0 = 0, f_1 = f_2 = 1$, and $f_{n+2} = f_n + f_{n+1}$ for any $n \in \mathbb{N}$. In [6], Ohtsuka and Nakamura proved

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{f_k}\right)^{-1}\right\rfloor = \begin{cases} f_{n-2}, & n \geq 2 \text{ is even};\\ f_{n-2} - 1, & n \geq 1 \text{ is odd},\end{cases} \tag{1.1}$$

and

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{f_k^2}\right)^{-1}\right\rfloor = \begin{cases} f_{n-1}f_n - 1, & n \geq 2 \text{ is even};\\ f_{n-1}f_n, & n \geq 1 \text{ is odd},\end{cases} \tag{1.2}$$

where $\lfloor x \rfloor$ is the floor function. See more results for subsequences of Fibonacci numbers in [11, 12], Pell numbers in [1, 13], and Mathieu series in [4]. Also, recently many interesting results on special numbers have been obtained in [7–9]. The following natural question on the asymptotic behavior can be raised.

Question Let $\sum_{k=1}^{\infty} a_k$ be a convergent series. Can we find a suitable function g_n such that

$$\left(\sum_{k=n}^{\infty} a_k\right)^{-1} \sim g_n?$$

Here the notation $A_n \sim B_n$ means that $\lim_{n\to\infty}(A_n - B_n) = 0$.

In [3], we proved that

$$\left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} \sim f_{mn-\ell} - f_{m(n-1)-\ell}$$

for any $m \in \mathbb{N}$ and $0 \leq \ell \leq m-1$. In fact, we proved

$$f_{mn-\ell} - f_{m(n-1)-\ell} - \frac{1}{f_n} < \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} < f_{mn-\ell} - f_{m(n-1)-\ell} + \frac{1}{f_n}.$$

In the special case when $m = 1$ and $\ell = 0$, the above equation is reduced to

$$\lim_{n\to\infty} \left\{ \left(\sum_{k=n}^{\infty} \frac{1}{f_k}\right)^{-1} - f_{n-2} \right\} = 0.$$

In [3], we also proved the generalization of (1.1) as the following formula:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_{mk-\ell}}\right)^{-1} \right\rfloor = \begin{cases} f_{mn-\ell} - f_{m(n-1)-\ell} - 1, & m(n+1)+\ell \text{ is even;} \\ f_{mn-\ell} - f_{m(n-1)-\ell}, & m(n+1)+\ell \text{ is odd,} \end{cases}$$

for any $m \in \mathbb{N}$ and $0 \leq \ell \leq m-1$. One can see the results for the product of two Fibonacci numbers in [5].

In this paper, we study the asymptotic behavior of the reciprocal sum of

$$f_k^2, f_k f_{k+m}, f_{3k}^2$$

for $k, m \in \mathbb{N}$. Precisely, we prove that

$$\left(\sum_{k=n}^{\infty} \frac{1}{f_k f_{k+2\ell}}\right)^{-1} \sim g_{n,\ell}, \quad \ell = 0, 1, 2, \ldots,$$

$$\left(\sum_{k=n}^{\infty} \frac{1}{f_k f_{k+2\ell-1}}\right)^{-1} \sim h_{n,\ell}, \quad \ell = 1, 2, 3, \ldots,$$

where

$$g_{n,\ell} := f_{n+\ell-1}f_{n+\ell} - \bigl(f_\ell^2 + (-1)^\ell\bigr)\frac{(-1)^n}{3},$$
$$h_{n,\ell} := f_{n+\ell-1}^2 - \bigl(f_{\ell-1}f_\ell + (-1)^\ell\bigr)\frac{(-1)^n}{3}.$$

For the proof of our main theorem, we prove the following inequalities:

(i) $g_{n,0} < (\sum_{k=n}^{\infty}\frac{1}{f_k^2})^{-1} < g_{n,0} + c_n$ for all $n \geq 1$,

(ii) $g_{n,\ell} - c_n < (\sum_{k=n}^{\infty}\frac{1}{f_k f_{k+2\ell}})^{-1} \leq g_{n,\ell}$ for all $\ell \geq 1$ and $n \geq 2\ell - 1$,

(iii) $h_{n,\ell} - c_n < (\sum_{k=n}^{\infty}\frac{1}{f_k f_{k+2\ell-1}})^{-1} < h_{n,\ell}$ for all $\ell \geq 1$ and $n \geq 2\ell - 2$.

Here $c_n = 1/f_n$. We believe that the conditions $n \geq 2\ell - 1$ and $n \geq 2\ell - 2$ can be removed. However, it is enough to prove the inequalities *for sufficiently large n* for the study of asymptotic behavior as $n \to \infty$. As an application of the above results, we can obtain

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{f_k f_{k+m}}\right)^{-1} \right\rfloor$$

for all $m \in \mathbb{N}$. For example, our formulas imply that

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{f_k f_{k+8}}\right)^{-1} \right\rfloor = \begin{cases} f_{n+3}f_{n+4} - 4, & n \text{ is even;} \\ f_{n+3}f_{n+4} + 3, & n \text{ is odd,} \end{cases}$$

and

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{f_k f_{k+7}}\right)^{-1} \right\rfloor = \begin{cases} f_{n+3}^2 - 3, & n \text{ is even;} \\ f_{n+3}^2 + 2, & n \text{ is odd.} \end{cases}$$

In the final section, we discuss the reciprocal sum of f_{mk}^2 for $m \geq 2$. If $m = 3$, then we prove that

$$\left(\sum_{k=n}^{\infty}\frac{1}{f_{3k}^2}\right)^{-1} \sim \widetilde{g}_n,$$

where $\widetilde{g}_n = f_{3n}^2 - f_{3n-3}^2 + \frac{4}{9}(-1)^n$.

As in [3, 6], the following lemma plays an important role in proving the essential inequalities.

Lemma 1.1 ([6]) *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences with* $\lim_{n\to\infty} a_n = 0$. *If* $a_n < b_n + a_{n+1}$ *holds for any* $n \in \mathbb{N}$, *then*

$$a_n < \sum_{k=n}^{\infty} b_k$$

holds for any $n \in \mathbb{N}$.

We use the following relation when we calculate Fibonacci numbers.

Lemma 1.2 ([2], Catalan's identity) *For any $n, k \in \mathbb{N}$, we have*

$$f_n^2 = f_{n+k}f_{n-k} + (-1)^{n+k}f_k^2.$$

The following lemma is useful when we get a lower bound of formulas containing Fibonacci numbers. It comes from the identity

$$f_{m+n} = f_{m-1}f_n + f_m f_{n+1}.$$

Lemma 1.3 *For any $m, n \in \mathbb{N}$, we have $f_{m+n} > f_{m+1}f_n$.*

Remark 1.4 The Fibonacci numbers f_n can be written as the closed form by Binet's formula [10]

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha > \beta$ are two solutions of $x^2 - x - 1 = 0$. The infinite sum $\sum_{k=1}^{\infty} \frac{1}{f_k}$ is known to be irrational, but it is unknown whether $\sum_{k=1}^{\infty} \frac{1}{f_k^2}$ is irrational or not.

2 Reciprocal sum of f_k^2

In [6], Ohtsuka and Nakamura proved the following inequalities:

(i) $f_{n-1}f_n - 1 < (\sum_{k=n}^{\infty} \frac{1}{f_k^2})^{-1} < f_{n-1}f_n$, when n is even,

(ii) $f_{n-1}f_n < (\sum_{k=n}^{\infty} \frac{1}{f_k^2})^{-1} < f_{n-1}f_n + 1$, when n is odd.

Now we will prove the inequalities which are sharper than (i) and (ii). In this section, let

$$g_n = g_{n,0} = f_{n-1}f_n - \frac{1}{3}(-1)^n.$$

In fact, g_n can be written as

$$\begin{aligned} g_n &= f_{n-1}f_n - \frac{1}{3}(-1)^n \\ &= \{f_{n-2}f_{n+1} + (-1)^n\} - \frac{1}{3}(-1)^n \\ &= f_n^2 - f_{n-1}^2 + \frac{2}{3}(-1)^n. \end{aligned} \tag{2.1}$$

Theorem 2.1 *For any $n \in \mathbb{N}$, we have*

$$g_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_k^2}\right)^{-1} < g_n + c_n,$$

where $c_n = 1/f_n$.

To prove Theorem 2.1, we need the following formula.

Proposition 2.2 *For any $n \in \mathbb{N}$, we have*

$$(g_{n+1} - g_n)f_n^2 - g_n g_{n+1} = \frac{1}{9}.$$

Proof Note that

$$\begin{aligned}(g_{n+1}-g_n)f_n^2 &= \left\{f_nf_{n+1}-f_{n-1}f_n+\frac{2}{3}(-1)^n\right\}f_n^2\\ &= f_n^4+\frac{2}{3}(-1)^nf_n^2.\end{aligned}$$

By Catalan's identity, we have

$$\begin{aligned}g_ng_{n+1} &= \left\{f_{n-1}f_n-\frac{1}{3}(-1)^n\right\}\left\{f_nf_{n+1}+\frac{1}{3}(-1)^n\right\}\\ &= f_n^2(f_{n+1}f_{n-1})-\frac{1}{3}(-1)^nf_n(f_{n+1}-f_{n-1})-\frac{1}{9}\\ &= f_n^2\left(f_n^2+(-1)^n\right)-\frac{1}{3}(-1)^nf_n^2-\frac{1}{9}\\ &= f_n^4+\frac{2}{3}(-1)^nf_n^2-\frac{1}{9}.\end{aligned}$$

The desired result comes from the above two identities. □

Now we prove Theorem 2.1. By Proposition 2.2, we have

$$\frac{1}{g_n}-\frac{1}{g_{n+1}}-\frac{1}{f_n^2}=\frac{(g_{n+1}-g_n)f_n^2-g_ng_{n+1}}{g_ng_{n+1}f_n^2}=\frac{1}{9g_ng_{n+1}f_n^2}>0.$$

It follows that

$$\frac{1}{g_n}>\frac{1}{f_n^2}+\frac{1}{g_{n+1}}$$

for all $n\in\mathbb{N}$. By Lemma 1.1, we obtain

$$\frac{1}{g_n}>\sum_{k=n}^{\infty}\frac{1}{f_k^2}. \tag{2.2}$$

For the proof of the converse inequality, we compute

$$\begin{aligned}\frac{1}{g_n+c_n}-\frac{1}{g_{n+1}+c_{n+1}}-\frac{1}{f_n^2} &= \frac{(g_{n+1}-g_n)-(c_n-c_{n+1})}{(g_n+c_n)(g_{n+1}+c_{n+1})}-\frac{1}{f_n^2}\\ &< \frac{g_{n+1}-g_n}{(g_n+c_n)(g_{n+1}+c_{n+1})}-\frac{1}{f_n^2}\\ &= \frac{A}{(g_n+c_n)(g_{n+1}+c_{n+1})f_n^2},\end{aligned}$$

where

$$A=(g_{n+1}-g_n)f_n^2-(g_n+c_n)(g_{n+1}+c_{n+1}).$$

By Proposition 2.2, we have

$$A=\frac{1}{9}-c_ng_{n+1}-c_{n+1}g_n-c_nc_{n+1}<\frac{1}{9}-c_ng_{n+1}<0,$$

Table 1 Some values of $(\sum_{k=n}^{\infty} \frac{1}{f_k^2})^{-1}$

n	$(\sum_{k=n}^{\infty} \frac{1}{f_k^2})^{-1}$	$f_n^2 - f_{n-1}^2$	n	$(\sum_{k=n}^{\infty} \frac{1}{f_k^2})^{-1}$	$f_n^2 - f_{n-1}^2$
3	2.3456...	3	4	5.6714...	5
5	15.3351...	16	6	39.6673...	39
7	104.3335...	105	8	272.6667...	272
9	714.3333...	715	10	1869.6666...	1869

since $c_n g_{n+1} = f_{n+1} - \frac{\frac{1}{3}(-1)^{n+1}}{f_n} \geq f_{n+1} - \frac{1}{3} \geq \frac{2}{3}$ for all $n \in \mathbb{N}$. Thus we obtain

$$\frac{1}{g_n + c_n} < \frac{1}{f_n^2} + \frac{1}{g_{n+1} + c_{n+1}}$$

for all $n \in \mathbb{N}$. By Lemma 1.1 again, we obtain

$$\frac{1}{g_n + c_n} < \sum_{k=n}^{\infty} \frac{1}{f_k^2}. \tag{2.3}$$

By (2.2) and (2.3), we complete the proof of Theorem 2.1.

Remark 2.3 The inequalities of Theorem 2.1 imply

$$\lim_{n\to\infty} \left\{ \left(\sum_{k=n}^{\infty} \frac{1}{f_k^2} \right)^{-1} - g_n \right\} = 0.$$

See Table 1.

3 Reciprocal sum of $f_k f_{k+m}$ when m is even

In Sects. 3 and 4, we deal with the value of

$$\left(\sum_{k=n}^{\infty} \frac{1}{f_k f_{k+m}} \right)^{-1}.$$

In fact, we can compute the explicit value when $m = 2$. Note that

$$\begin{aligned} \sum_{k=n}^{\infty} \frac{1}{f_k f_{k+2}} &= \sum_{k=n}^{\infty} \left(\frac{1}{f_k} - \frac{1}{f_{k+2}} \right) \frac{1}{f_{k+2} - f_k} \\ &= \sum_{k=n}^{\infty} \left(\frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1} f_{k+2}} \right) \\ &= \frac{1}{f_n f_{n+1}}. \end{aligned}$$

Hence it holds that, for all $n \in \mathbb{N}$,

$$\left(\sum_{k=n}^{\infty} \frac{1}{f_k f_{k+2}} \right)^{-1} = f_n f_{n+1}.$$

However, it is difficult to find the explicit value of

$$\left(\sum_{k=n}^{\infty}\frac{1}{f_k f_{k+m}}\right)^{-1}$$

except for $m = 2$.

Throughout this section, we assume that m is even, so that $m = 2\ell$ for some $\ell \in \mathbb{N}$. In this case, we define

$$g_n = g_{n,\ell} = f_{n+\ell-1}f_{n+\ell} - \left(f_\ell^2 + (-1)^\ell\right)\frac{(-1)^n}{3}.$$

For simplicity, we write $I_1 := f_\ell^2 + (-1)^\ell$.

Proposition 3.1 *For any $n \in \mathbb{N}$, we have*

$$(g_{n+1} - g_n)f_n f_{n+2\ell} - g_n g_{n+1} = -\frac{5}{9}I_1^2 + \frac{2}{3}(-1)^\ell I_1.$$

Proof Write $g_n = f_{n+\ell-1}f_{n+\ell} - \frac{(-1)^n}{3}I_1$. Note that

$$\begin{aligned}
&(g_{n+1} - g_n)f_n f_{n+2\ell} \\
&\quad = \left\{\left(f_{n+\ell}f_{n+\ell+1} + \frac{(-1)^n}{3}I_1\right) - \left(f_{n+\ell-1}f_{n+\ell} - \frac{(-1)^n}{3}I_1\right)\right\}f_n f_{n+2\ell} \\
&\quad = \left\{f_{n+\ell}^2 + \frac{2}{3}(-1)^n I_1\right\}\left\{f_{n+\ell}^2 - (-1)^n f_\ell^2\right\}.
\end{aligned}$$

Since $f_\ell^2 = I_1 - (-1)^\ell$, we have

$$\begin{aligned}
&(g_{n+1} - g_n)f_n f_{n+2\ell} \\
&\quad = f_{n+\ell}^4 + (-1)^{n+\ell}f_{n+\ell}^2 - \frac{(-1)^n}{3}I_1 f_{n+\ell}^2 - \frac{2}{3}I_1^2 + \frac{2}{3}(-1)^\ell I_1.
\end{aligned} \tag{3.1}$$

Note that

$$\begin{aligned}
g_n g_{n+1} &= \left(f_{n+\ell-1}f_{n+\ell} - \frac{(-1)^n}{3}I_1\right)\left(f_{n+\ell}f_{n+\ell+1} + \frac{(-1)^n}{3}I_1\right) \\
&= f_{n+\ell}^2 f_{n+\ell-1}f_{n+\ell+1} - \frac{(-1)^n}{3}I_1 f_{n+\ell}^2 - \frac{1}{9}I_1^2.
\end{aligned}$$

By Lemma 1.2, $f_{n+\ell-1}f_{n+\ell+1} = f_{n+\ell}^2 + (-1)^{n+\ell}$. Thus we have

$$g_n g_{n+1} = f_{n+\ell}^4 + (-1)^{n+\ell}f_{n+\ell}^2 - \frac{(-1)^n}{3}I_1 f_{n+\ell}^2 - \frac{1}{9}I_1^2. \tag{3.2}$$

By (3.1) and (3.2), we complete the proof. □

See Table 2. If $m = 4$, then $g_{n,2} = f_{n+1}f_{n+2} - \frac{2}{3}(-1)^n$.

Table 2 Some values of $(\sum_{k=n}^{\infty}\frac{1}{f_k f_{k+4}})^{-1}$

n	$(\sum_{k=n}^{\infty}\frac{1}{f_k f_{k+4}})^{-1}$	$g_{n,2}$	n	$(\sum_{k=n}^{\infty}\frac{1}{f_k f_{k+4}})^{-1}$	$g_{n,2}$
3	15.6521 . . .	15.6666	4	39.3277 . . .	39.3333
5	104.6645 . . .	104.6666	6	272.3325 . . .	272.3333
7	714.6663 . . .	714.6666	8	1869.3332 . . .	1869.3333
9	4895.6666 . . .	4895.6666	10	12,815.3333 . . .	12,815.3333

Theorem 3.2 *If $m = 2\ell$ for some $\ell \in \mathbb{N}$, then for any $n \geq 2\ell - 1$, we have*

$$g_n - c_n < \left(\sum_{k=n}^{\infty}\frac{1}{f_k f_{k+m}}\right)^{-1} \leq g_n,$$

where $c_n = 1/f_n$.

For example, if $\ell = 4$, then $g_n = f_{n+3}f_{n+4} - \frac{10(-1)^n}{3}$. Thus we have

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{f_k f_{k+8}}\right)^{-1} \right\rfloor = \begin{cases} f_{n+3}f_{n+4} - 4, & n \text{ is even;} \\ f_{n+3}f_{n+4} + 3, & n \text{ is odd.} \end{cases}$$

Proof (i) By Lemma 1.1, it is enough to show that

$$\frac{1}{g_n} \leq \frac{1}{f_n f_{n+2\ell}} + \frac{1}{g_{n+1}} \tag{3.3}$$

for all $n \in \mathbb{N}$. By Proposition 3.1, we have

$$\begin{aligned}(g_{n+1} - g_n)f_n f_{n+2\ell} - g_n g_{n+1} &= -\frac{5}{9}I_1^2 + \frac{2}{3}(-1)^{\ell} I_1 \\ &= -\frac{I_1}{9}\bigl(5I_1 - 6(-1)^{\ell}\bigr) \\ &= -\frac{I_1}{9}\bigl(5f_{\ell}^2 - (-1)^{\ell}\bigr) \leq 0.\end{aligned}$$

It follows that

$$\frac{1}{g_n} - \frac{1}{g_{n+1}} - \frac{1}{f_n f_{n+2\ell}} = \frac{(g_{n+1} - g_n)f_n f_{n+2\ell} - g_n g_{n+1}}{g_n g_{n+2} f_n f_{n+2\ell}} \leq 0,$$

which implies (3.3). In fact, the equality holds only when $\ell = 1$. This is the case when $m = 2$.

(ii) By Lemma 1.1, it is enough to show that

$$\frac{1}{g_n - c_n} > \frac{1}{f_n f_{n+2\ell}} + \frac{1}{g_{n+1} - c_{n+1}}$$

for all $n \in \mathbb{N}$. Note that

$$\begin{aligned}&\frac{1}{g_n - c_n} - \frac{1}{g_{n+1} - c_{n+1}} - \frac{1}{f_n f_{n+2\ell}} \\ &\quad = \frac{(g_{n+1} - g_n) + (c_n - c_{n+1})}{(g_n - c_n)(g_{n+1} - c_{n+1})} - \frac{1}{f_n f_{n+2\ell}}\end{aligned}$$

$$\begin{aligned}
&> \frac{g_{n+1}-g_n}{(g_n-c_n)(g_{n+1}-c_{n+1})} - \frac{1}{f_n f_{n+2\ell}} \\
&= \frac{(g_{n+1}-g_n)f_n f_{n+2\ell} - (g_n-c_n)(g_{n+1}-c_{n+1})}{(g_n-c_n)(g_{n+1}-c_{n+1})f_n f_{n+2\ell}}.
\end{aligned}$$

Now we will show that, for $n \geq 2\ell - 1$, we have

$$(g_{n+1}-g_n)f_n f_{n+2\ell} - (g_n-c_n)(g_{n+1}-c_{n+1}) > 0.$$

Note that

$$\begin{aligned}
&(g_{n+1}-g_n)f_n f_{n+2\ell} - (g_n-c_n)(g_{n+1}-c_{n+1}) \\
&\quad = \left\{(g_{n+1}-g_n)f_n f_{n+2\ell} - g_n g_{n+1}\right\} + \{c_n g_{n+1} + c_{n+1} g_n - c_n c_{n+1}\}.
\end{aligned}$$

By Proposition 3.1, we have

$$\begin{aligned}
(g_{n+1}-g_n)f_n f_{n+2\ell} - g_n g_{n+1} &= -\frac{5}{9}I_1^2 + \frac{2}{3}(-1)^\ell I_1 \\
&= -\frac{5}{9}f_\ell^4 - \frac{4}{9}(-1)^\ell f_\ell^2 + \frac{1}{9} \\
&> -\frac{5}{9}f_\ell^4 - \frac{4}{9}f_\ell^2 + \frac{1}{9}. \qquad (3.4)
\end{aligned}$$

By Lemma 1.3, we have

$$\begin{aligned}
&c_n g_{n+1} + c_{n+1} g_n - c_n c_{n+1} \\
&\quad = \frac{f_{n+\ell} f_{n+\ell+1}}{f_n} + \frac{f_{n+\ell-1} f_{n+\ell}}{f_{n+1}} + \frac{(-1)^n}{3}\left(\frac{1}{f_n} - \frac{1}{f_{n+1}}\right)I_1 - \frac{1}{f_n f_{n+1}} \\
&\quad > f_{\ell+1} f_{n+\ell+1} + f_\ell f_{n+\ell-1} - \frac{1}{3}I_1 - 1 \\
&\quad > f_\ell (f_{n+\ell+1} + f_{n+\ell-1}) - \frac{1}{3}I_1 - 1 \\
&\quad > f_\ell^4 (f_{n-2\ell+4} + f_{n-2\ell+2}) - \frac{1}{3}I_1 - 1.
\end{aligned}$$

If $n \geq 2\ell - 1$, then $f_{n-2\ell+4} + f_{n-2\ell+2} \geq f_3 + f_1 = 3$. Thus, we have

$$c_n g_{n+1} + c_{n+1} g_n - c_n c_{n+1} > 3f_\ell^4 - \frac{1}{3}I_1 - 1 > 3f_\ell^4 - \frac{1}{3}f_\ell^2 - \frac{4}{3} \qquad (3.5)$$

for $n \geq 2\ell - 1$. From (3.4) and (3.5), we obtain

$$(g_{n+1}-g_n)f_n f_{n+2\ell} - (g_n-c_n)(g_{n+1}-c_{n+1}) > \frac{22}{9}f_\ell^4 - \frac{7}{9}f_\ell^2 - \frac{11}{9} \geq \frac{4}{9} > 0,$$

since $f_\ell \geq 1$. □

Remark 3.3 If $\ell = 0$, then $I_1 = f_0^2 + 1 = 1$. Thus in the proof of Theorem 3.2 we have

$$(g_{n+1}-g_n)f_n f_{n+2\ell} - g_n g_{n+1} = -\frac{I_1}{9}\left(5f_\ell^2 - (-1)^\ell\right) = \frac{1}{9} > 0.$$

In fact, this is an identity in Proposition 2.2. Thus the inequality in Theorem 3.2 is opposite to Theorem 2.1.

4 Reciprocal sum of $f_k f_{k+m}$ when m is odd

Throughout this section, we assume that m is odd, so that $m = 2\ell - 1$ for some $\ell \in \mathbb{N}$. In this case, we define

$$h_n = h_{n,\ell} = f_{n+\ell-1}^2 - \left(f_{\ell-1}f_\ell + (-1)^\ell\right)\frac{(-1)^n}{3}.$$

For simplicity, we write $I_2 := f_{\ell-1}f_\ell + (-1)^\ell$.

Proposition 4.1 *For any $n \in \mathbb{N}$, we have*

$$(h_{n+1} - h_n)f_n f_{n+2\ell-1} - h_n h_{n+1} = -1 + \frac{4}{3}(-1)^\ell I_2 - \frac{5}{9}I_2^2.$$

Proof By Catalan's identity, we have

$$\begin{aligned} f_n f_{n+2\ell-1} &= f_n(f_{n+2\ell} - f_{n+2\ell-2}) \\ &= f_n f_{n+2\ell} - f_n f_{n+2\ell-2} \\ &= \left(f_{n+\ell}^2 - (-1)^n f_\ell^2\right) - \left(f_{n+\ell-1}^2 - (-1)^n f_{\ell-1}^2\right) \\ &= f_{n+\ell+1}f_{n+\ell-2} - (-1)^n f_{\ell+1}f_{\ell-2}. \end{aligned}$$

Note that

$$\begin{aligned} &(h_{n+1} - h_n)f_n f_{n+2\ell-1} \\ &\quad = \left\{\left(f_{n+\ell}^2 - f_{n+\ell-1}^2\right) + \frac{2}{3}(-1)^n I_2\right\} f_n f_{n+2\ell-1} \\ &\quad = \left\{f_{n+\ell+1}f_{n+\ell-2} + \frac{2}{3}(-1)^n I_2\right\}\left\{f_{n+\ell+1}f_{n+\ell-2} - (-1)^n f_{\ell+1}f_{\ell-2}\right\} \\ &\quad = (f_{n+\ell+1}f_{n+\ell-2})^2 + f_{n+\ell+1}f_{n+\ell-2}(-1)^n\left\{\frac{2}{3}I_2 - f_{\ell+1}f_{\ell-2}\right\} - \frac{2}{3}f_{\ell+1}f_{\ell-2}I_2. \end{aligned}$$

Note that

$$\begin{aligned} h_n h_{n+1} &= \left\{f_{n+\ell-1}^2 - \frac{(-1)^n}{3}I_2\right\}\left\{f_{n+\ell}^2 + \frac{(-1)^n}{3}I_2\right\} \\ &= (f_{n+\ell-1}f_{n+\ell})^2 - \left(f_{n+\ell}^2 - f_{n+\ell-1}^2\right)\frac{(-1)^n}{3}I_2 - \frac{1}{9}I_2^2 \\ &= (f_{n+\ell-1}f_{n+\ell})^2 - f_{n+\ell+1}f_{n+\ell-2}\frac{(-1)^n}{3}I_2 - \frac{1}{9}I_2^2. \end{aligned}$$

It follows that

$$\begin{aligned} (h_{n+1} - h_n)f_n f_{n+2\ell-1} - h_n h_{n+1} &= (f_{n+\ell+1}f_{n+\ell-2})^2 - (f_{n+\ell-1}f_{n+\ell})^2 \\ &\quad + f_{n+\ell+1}f_{n+\ell-2}(-1)^n\{I_2 - f_{\ell+1}f_{\ell-2}\} \\ &\quad - \frac{2}{3}f_{\ell+1}f_{\ell-2}I_2 + \frac{1}{9}I_2^2. \end{aligned}$$

Table 3 Some values of $(\sum_{k=n}^{\infty} \frac{1}{f_k f_{k+5}})^{-1}$

n	$(\sum_{k=n}^{\infty} \frac{1}{f_k f_{k+5}})^{-1}$	$h_{n,3}$	n	$(\sum_{k=n}^{\infty} \frac{1}{f_k f_{k+5}})^{-1}$	$h_{n,3}$
3	25.3042...	25.3333...	4	63.6554...	63.6666...
5	169.3290...	169.3333...	6	440.6650...	440.6666...
7	1156.3327...	1156.3333...	8	3024.6664...	3024.6666...
9	7921.3332...	7921.3333...	10	20,735.6666...	20,735.6666...

Since $f_{n+\ell-1}f_{n+\ell} = f_{n+\ell+1}f_{n+\ell-2} + (-1)^{n+\ell}$, we have

$$\begin{aligned}
&(f_{n+\ell+1}f_{n+\ell-2})^2 - (f_{n+\ell-1}f_{n+\ell})^2 \\
&\quad = (f_{n+\ell+1}f_{n+\ell-2} + f_{n+\ell-1}f_{n+\ell})(f_{n+\ell+1}f_{n+\ell-2} - f_{n+\ell-1}f_{n+\ell}) \\
&\quad = -1 - 2(-1)^{n+\ell} f_{n+\ell+1}f_{n+\ell-2}.
\end{aligned}$$

Since $I_2 - f_{\ell+1}f_{\ell-2} = f_{\ell-1}f_\ell - f_{\ell+1}f_{\ell-2} + (-1)^\ell = 2(-1)^\ell$,

$$\begin{aligned}
&(h_{n+1} - h_n)f_n f_{n+2\ell-1} - h_n h_{n+1} \\
&\quad = -1 - 2(-1)^{n+\ell} f_{n+\ell+1}f_{n+\ell-2} \\
&\qquad + f_{n+\ell+1}f_{n+\ell-2}(-1)^n\{2(-1)^\ell\} - \frac{2}{3}f_{\ell+1}f_{\ell-2}I_2 + \frac{1}{9}I_2^2 \\
&\quad = -1 - \frac{2}{3}\big(I_2 - 2(-1)^\ell\big)I_2 + \frac{1}{9}I_2^2 \\
&\quad = -1 + \frac{4}{3}(-1)^\ell I_2 - \frac{5}{9}I_2^2.
\end{aligned}$$

□

See Table 3. If $m = 5$, then $h_{n,3} = f_{n+2}^2 - \frac{1}{3}(-1)^n$.

Theorem 4.2 *If $m = 2\ell - 1$, then for any $n \geq 2\ell - 2$, we have*

$$h_n - c_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_k f_{k+m}}\right)^{-1} < h_n,$$

where $c_n = 1/f_n$.

For example, if $\ell = 4$, then $h_n = f_{n+3}^2 - \frac{7(-1)^n}{3}$. Thus we have

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_k f_{k+7}}\right)^{-1} \right\rfloor = \begin{cases} f_{n+3}^2 - 3, & n \text{ is even;} \\ f_{n+3}^2 + 2, & n \text{ is odd.} \end{cases}$$

Proof (i) We will show that

$$\frac{1}{h_n} < \frac{1}{f_n f_{n+2\ell-1}} + \frac{1}{h_{n+1}}.$$

Note that

$$\frac{1}{h_n} - \frac{1}{h_{n+1}} - \frac{1}{f_n f_{n+2\ell-1}} = \frac{(h_{n+1} - h_n)f_n f_{n+2\ell-1} - h_n h_{n+1}}{h_n h_{n+1} f_n f_{n+2\ell-1}}.$$

By Proposition 4.1, we have

$$\begin{aligned}(h_{n+1}-h_n)f_nf_{n+2\ell-1}-h_nh_{n+1} &= -1+\frac{4}{3}(-1)^{\ell}I_2-\frac{5}{9}I_2^2\\ &= -\frac{5}{9}(f_{\ell-1}f_{\ell})^2+\frac{2}{9}(-1)^{\ell}(f_{\ell-1}f_{\ell})-\frac{2}{9}\\ &\le -\frac{5}{9}(f_{\ell-1}f_{\ell})^2+\frac{2}{9}(f_{\ell-1}f_{\ell})-\frac{2}{9}\\ &< -\frac{5}{9}<0.\end{aligned}$$

(ii) We will show that

$$\frac{1}{h_n-c_n}>\frac{1}{f_nf_{n+2\ell-1}}+\frac{1}{h_{n+1}-c_{n+1}}.$$

Note that

$$\begin{aligned}&\frac{1}{h_n-c_n}-\frac{1}{h_{n+1}-c_{n+1}}-\frac{1}{f_nf_{n+2\ell-1}}\\ &\quad=\frac{h_{n+1}-h_n+(c_n-c_{n+1})}{(h_n-c_n)(h_{n+1}-c_{n+1})}-\frac{1}{f_nf_{n+2\ell-1}}\\ &\quad>\frac{h_{n+1}-h_n}{(h_n-c_n)(h_{n+1}-c_{n+1})}-\frac{1}{f_nf_{n+2\ell-1}}\\ &\quad=\frac{(h_{n+1}-h_n)f_nf_{n+2\ell-1}-(h_n-c_n)(h_{n+1}-c_{n+1})}{(h_n-c_n)(h_{n+1}-c_{n+1})f_nf_{n+2\ell-1}}.\end{aligned}$$

It is enough to show that

$$(h_{n+1}-h_n)f_nf_{n+2\ell-1}-(h_n-c_n)(h_{n+1}-c_{n+1})>0.$$

Note that

$$\begin{aligned}&(h_{n+1}-h_n)f_nf_{n+2\ell-1}-(h_n-c_n)(h_{n+1}-c_{n+1})\\ &\quad=\bigl\{(h_{n+1}-h_n)f_nf_{n+2\ell-1}-h_nh_{n+1}\bigr\}+\{c_nh_{n+1}+c_{n+1}h_n-c_nc_{n+1}\}.\end{aligned}$$

By Proposition 4.1, we have

$$\begin{aligned}(h_{n+1}-h_n)f_nf_{n+2\ell-1}-h_nh_{n+1} &= -1+\frac{4}{3}(-1)^{\ell}I_2-\frac{5}{9}I_2^2\\ &= -\frac{5}{9}(f_{\ell-1}f_{\ell})^2+\frac{2}{9}(-1)^{\ell}(f_{\ell-1}f_{\ell})-\frac{2}{9}\\ &\ge -\frac{5}{9}(f_{\ell-1}f_{\ell})^2-\frac{2}{9}(f_{\ell-1}f_{\ell})-\frac{2}{9}.\end{aligned}$$

Note that

$$\begin{aligned}&c_nh_{n+1}+c_{n+1}h_n-c_nc_{n+1}\\ &\quad=\frac{f_{n+\ell}^2}{f_n}+\frac{f_{n+\ell-1}^2}{f_{n+1}}+\frac{(-1)^n}{3}I_2\left(\frac{1}{f_n}-\frac{1}{f_{n+1}}\right)-\frac{1}{f_nf_{n+1}}\end{aligned}$$

$$
\begin{aligned}
&> f_{\ell+1} f_{n+\ell} + f_{\ell-1} f_{n+\ell-1} - \frac{1}{3} I_2 - 1 \\
&> f_{\ell-1} (f_{n+\ell} + f_{n+\ell-1}) - \frac{1}{3} I_2 - 1 \\
&> (f_{\ell-1} f_\ell)^2 (f_{n-2\ell+5} + f_{n-2\ell+4}) - \frac{1}{3} I_2 - 1.
\end{aligned}
$$

If $n \geq 2\ell - 2$, then $f_{n-2\ell+5} + f_{n-2\ell+4} \geq f_3 + f_2 = 3$. It follows that

$$
c_n h_{n+1} + c_{n+1} h_n - c_n c_{n+1} > 3(f_{\ell-1} f_\ell)^2 - \frac{1}{3}(f_{\ell-1} f_\ell) - \frac{4}{3}.
$$

Thus we have

$$
\begin{aligned}
&(h_{n+1} - h_n) f_n f_{n+2\ell-1} - (h_n - c_n)(h_{n+1} - c_{n+1}) \\
&\quad > \frac{22}{9}(f_{\ell-1} f_\ell)^2 - \frac{5}{9}(f_{\ell-1} f_\ell) - \frac{14}{9} \geq \frac{1}{3} > 0.
\end{aligned}
$$

□

5 Reciprocal sum of f_{3k}^2

In this final section, we discuss the reciprocal sum of f_{mk}^2 for any $m \geq 2$. Similar to (2.1), when $m = 1$, we expected to find a suitable constant $C_m > 0$ such that

$$
\left(\sum_{k=n}^{\infty} \frac{1}{f_{mk}^2} \right)^{-1} \sim f_{mn}^2 - f_{m(n-1)}^2 + (-1)^{mn} C_m. \tag{5.1}
$$

In Sect. 2, we proved that $C_1 = \frac{2}{3}$. The integer part when $m = 3$ has been obtained as follows.

Theorem 5.1 ([5])

$$
\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_{3k}^2} \right)^{-1} \right\rfloor = \begin{cases} f_{3n}^2 - f_{3n-3}^2, & n \text{ is even}; \\ f_{3n}^2 - f_{3n-3}^2 - 1, & n \text{ is odd}. \end{cases} \tag{5.2}
$$

Now we prove that $C_3 = \frac{4}{9}$. See Table 4. Here

$$
\widetilde{g}_n := f_{3n}^2 - f_{3n-3}^2 + \frac{4}{9}(-1)^n.
$$

Our theorem contains more optimal inequality than (5.2).

Theorem 5.2 *For all $n \in \mathbb{N}$, we have*

$$
\widetilde{g}_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_{3k}^2} \right)^{-1} < \widetilde{g}_n + c_n,
$$

Table 4 Some values of $(\sum_{k=n}^{\infty} \frac{1}{f_{3k}^2})^{-1}$

n	$(\sum_{k=n}^{\infty} \frac{1}{f_{3k}^2})^{-1}$	$\widetilde{g}_n$	n	$(\sum_{k=n}^{\infty} \frac{1}{f_{3k}^2})^{-1}$	$\widetilde{g}_n$
3	1091.5561 . . .	1091.5555 . . .	4	19,580.44447 . . .	19,580.44444 . . .
5	351,363.555557 . . .	351,363.555555 . . .	6	6,304,956.4444445 . . .	6,304,956.4444444 . . .
7	113,137,859.555555 . . .	113,137,859.555555 . . .			

where $\widetilde{g}_n := f_{3n}^2 - f_{3n-3}^2 + \frac{4}{9}(-1)^n$. *Thus we have*

$$\left(\sum_{k=n}^{\infty} \frac{1}{f_{3k}^2}\right)^{-1} \sim f_{3n}^2 - f_{3n-3}^2 + \frac{4}{9}(-1)^n.$$

Lemma 5.3 *For any* $n \geq 1$, *we have*

$$f_{3n+3}^2 - 18 f_{3n}^2 + f_{3n-3}^2 = 8(-1)^n.$$

Proof Since $f_{3n+2} + f_{3n-2} = 3 f_{3n}$, we have

$$\begin{aligned} f_{3n+3} &= f_{3n+2} + f_{3n} + f_{3n-1} \\ &= f_{3n+2} + f_{3n} + f_{3n-2} + f_{3n-3} \\ &= 4 f_{3n} + f_{3n-3}. \end{aligned}$$

It follows that

$$\begin{aligned} 16 f_{3n}^2 &= (f_{3n+3} - f_{3n-3})^2 \\ &= f_{3n+3}^2 + f_{3n-3}^2 - 2 f_{3n+3} f_{3n-3} \\ &= f_{3n+3}^2 + f_{3n-3}^2 - 2\{f_{3n}^2 + (-1)^n f_3^2\}, \end{aligned}$$

which completes the proof. □

Proof (i) We will show that

$$\frac{1}{\widetilde{g}_n} > \frac{1}{f_{3n}^2} + \frac{1}{\widetilde{g}_{n+1}}.$$

Note that

$$\frac{1}{\widetilde{g}_n} - \frac{1}{\widetilde{g}_{n+1}} - \frac{1}{f_{3n}^2} = \frac{A}{\widetilde{g}_n \widetilde{g}_{n+1} f_{3n}^2},$$

where $A := (\widetilde{g}_{n+1} - \widetilde{g}_n) f_{3n}^2 - \widetilde{g}_n \widetilde{g}_{n+1}$. It is enough to show that $A > 0$.

Note that

$$(\widetilde{g}_{n+1} - \widetilde{g}_n) f_{3n}^2 = f_{3n+3}^2 f_{3n}^2 - 2 f_{3n}^4 + f_{3n-3}^2 f_{3n}^2 - \frac{8}{9}(-1)^n f_{3n}^2.$$

Note also that

$$\begin{aligned} \widetilde{g}_n \widetilde{g}_{n+1} &= \left(f_{3n}^2 - f_{3n-3}^2 + \frac{4}{9}(-1)^n\right)\left(f_{3n+3}^2 - f_{3n}^2 - \frac{4}{9}(-1)^n\right) \\ &= f_{3n}^2 f_{3n+3}^2 - f_{3n}^4 - f_{3n-3}^2 f_{3n+3}^2 + f_{3n-3}^2 f_{3n}^2 - \frac{16}{81} \\ &\quad + \frac{4}{9}(-1)^n \left(f_{3n+3}^2 - 2 f_{3n}^2 + f_{3n-3}^2\right). \end{aligned}$$

It follows that

$$\begin{aligned} A &= f_{3n-3}^2 f_{3n+3}^2 - f_{3n}^4 - \frac{4}{9}(-1)^n \{f_{3n+3}^2 + f_{3n-3}^2\} + \frac{16}{81} \\ &= \left(f_{3n}^2 + (-1)^n f_3^2\right)^2 - f_{3n}^4 - \frac{4}{9}(-1)^n \{f_{3n+3}^2 + f_{3n-3}^2\} + \frac{16}{81} \\ &= -\frac{4}{9}(-1)^n \{f_{3n+3}^2 - 18 f_{3n}^2 + f_{3n-3}^2\} + \frac{1312}{81}. \end{aligned}$$

By Lemma 5.3, we have

$$A = \frac{1024}{81}. \tag{5.3}$$

(ii) We will prove that

$$\frac{1}{\widetilde{g}_n + c_n} < \frac{1}{f_{3n}^2} + \frac{1}{\widetilde{g}_{n+1} + c_{n+1}}.$$

Note that

$$\begin{aligned} \frac{1}{\widetilde{g}_n + c_n} - \frac{1}{\widetilde{g}_{n+1} + c_{n+1}} - \frac{1}{f_{3n}^2} &= \frac{(\widetilde{g}_{n+1} - \widetilde{g}_n) - (c_n - c_{n+1})}{(\widetilde{g}_n + c_n)(\widetilde{g}_{n+1} + c_{n+1})} - \frac{1}{f_{3n}^2} \\ &< \frac{\widetilde{g}_{n+1} - \widetilde{g}_n}{(\widetilde{g}_n + c_n)(\widetilde{g}_{n+1} + c_{n+1})} - \frac{1}{f_{3n}^2} \\ &= \frac{B}{(\widetilde{g}_n + c_n)(\widetilde{g}_{n+1} + c_{n+1}) f_{3n}^2}, \end{aligned}$$

where $B = (\widetilde{g}_{n+1} - \widetilde{g}_n) f_{3n}^2 - (\widetilde{g}_n + c_n)(\widetilde{g}_{n+1} + c_{n+1})$. Now it is enough to show that $B < 0$.

By using (5.3), we have

$$B = A - c_n \widetilde{g}_{n+1} - c_{n+1} \widetilde{g}_n - c_n c_{n+1} < \frac{1024}{81} - c_n \widetilde{g}_{n+1}.$$

Note that

$$\begin{aligned} c_n \widetilde{g}_{n+1} &= \frac{f_{3n+3}^2 - f_{3n}^2 - \frac{4}{9}(-1)^n}{f_n} \\ &> \frac{(f_{3n+3} + f_{3n})(f_{3n+3} - f_{3n})}{f_n} - \frac{4}{9} \\ &= \frac{(f_{3n+3} + f_{3n})(f_{3n+2} + f_{3n-1})}{f_n} - \frac{4}{9} \\ &> 2(f_{3n+3} + f_{3n}) - 1 \geq 19. \end{aligned}$$

It follows that $B < 0$. □

Remark 5.4 It looks not easy to find the explicit values of C_m satisfying (5.1) except for $m = 1, 3$. By using a computer software program (Maple17 and wolframalpha.com), we found the following. If m is even, then

$$C_2 \sim 0.298130320\ldots,$$

$$C_4 \sim 0.383325938\ldots,$$

$$C_6 \sim 0.397523195\ldots,$$

$$C_8 \sim 0.399637681\ldots.$$

If m is odd, then

$$C_5 \sim 0.406504065\ldots,$$

$$C_7 \sim 0.400948991\ldots,$$

$$C_9 \sim 0.400138456\ldots.$$

We might expect that C_m tends to $\frac{2}{5}$ as $n \to \infty$.

6 Conclusion

We summarize all the results that have been proved in this paper.

(i) $(\sum_{k=n}^{\infty} \frac{1}{f_k^2})^{-1} \sim f_n^2 - f_{n-1}^2 + \frac{2}{3}(-1)^n$ as $n \to \infty$,

(ii) $(\sum_{k=n}^{\infty} \frac{1}{f_k f_{k+2\ell}})^{-1} \sim f_{n+\ell-1} f_{n+\ell} - (f_\ell^2 + (-1)^\ell)\frac{(-1)^n}{3}$ as $n \to \infty$,

(iii) $(\sum_{k=n}^{\infty} \frac{1}{f_k f_{k+2\ell-1}})^{-1} \sim f_{n+\ell-1}^2 - (f_{\ell-1} f_\ell + (-1)^\ell)\frac{(-1)^n}{3}$ as $n \to \infty$,

(iv) $(\sum_{k=n}^{\infty} \frac{1}{f_{3k}^2})^{-1} \sim f_{3n}^2 - f_{3n-3}^2 + \frac{4}{9}(-1)^n$ as $n \to \infty$.

Acknowledgements
Not applicable.

Funding
This work was supported by NRF-2018R1D1A1B07050044 from the National Research Foundation of Korea.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Publisher's Note

Received: 3 January 2020 Accepted: 24 March 2020 Published online: 01 April 2020

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