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Some inequalities related to 2×2 block sector partial transpose matrices

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Abstract

In this article, two inequalities related to 2×2 block sector partial transpose matrices are proved, and we also present a unitarily invariant norm inequality for the Hua matrix which is sharper than an existing result.

MSC: 15A45; 15A60

Keywords: Sector partial transpose matrices; Unitarily invariant norms; Linear maps

1 Introduction

We denote by $\mathbb{M}_n$ the set of $n \times n$ complex matrices. $\mathbb{M}_n(\mathbb{M}_k)$ is the set of $n \times n$ block matrices with each block in $\mathbb{M}_k$. The $n \times n$ identity matrix is denoted by I_n. We use $\|\cdot\|$ for an arbitrary unitarily invariant norm. A positive semidefinite matrix A will be expressed as $A \geq 0$. Likewise, we write $A > 0$ to refer that A is a positive definite matrix. The singular values of A, denoted by $s_1(A), s_2(A), \ldots, s_n(A)$, are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$, arranged in decreasing order and repeated according to multiplicity as $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. When A is Hermitian, we enumerate eigenvalues of A in nonincreasing order $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A)$. Recall that $C \in \mathbb{M}_{m\times n}$ is (strictly) contractive if $(I_n > C^*C)$ $I_n \geq C^*C$. The geometric mean of two positive definite matrices $A, B \in \mathbb{M}_n$, denoted by $A\sharp B$, is the positive definite solution of the Riccati equation $XB^{-1}X = A$ and has the explicit expression $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$. More details on the matrix geometric mean can be found in [2, Chap. 4].

The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax \mid x \in \mathbb{C}^n, x^*x = 1\}.$$

For basic properties of numerical range, see [5]. Also, we define a sector on the complex plane

$$S_\alpha = \{z \in \mathbb{C} \mid \Re z \geq 0, |\Im z| \leq (\Re z)\tan(\alpha)\}, \quad \alpha \in \left[0, \frac{\pi}{2}\right).$$

Actually, the class of matrices T with $W(T) \subseteq S_\alpha$ and the class of T with positive definite real part (i.e. accretive matrices) are both called sector matrices. Sector matrices have been the subject of a number of recent papers [3, 8, 14].

A matrix $H = (H_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$ is said to be positive partial transpose (i.e. PPT) if H is positive semidefinite and its partial transpose $H^\tau = (H_{ji})_{j,i=1}^n$ is also positive semidefinite. Inspired by PPT, Kuai [6] defined a new conception called sectorial partial transpose (i.e. SPT). That is, if $W(A) \subseteq S_\alpha$ for $A = (A_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$, then $W(A^\tau) \subseteq S_\alpha$. Thus, it is natural to extend the results for PPT matrices to SPT matrices.

Hiroshima [4, Theorem 1] proved the following result.

Theorem 1.1 *Let* $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ *be PPT. Then*

$$\|H\| \leq \|A + B\|. \tag{1}$$

As the application of Theorem 1.1, Lin and Hiroshima [10, Theorem 3.3] presented a relation between the norm of diagonal blocks of the Hua matrix, e.g., [12] and the norm of its off diagonal blocks.

Theorem 1.2 *If the Hua matrix is given by*

$$H := \begin{pmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{pmatrix},$$

where $A, B \in \mathbb{M}_{m\times n}$ *are strictly contractive, then*

$$2\big\|(I - A^*B)^{-1}\big\| \leq \big\|(I - A^*A)^{-1} + (I - B^*B)^{-1}\big\| \tag{2}$$

for any unitarily invariant norm.

Actually, it was only recently observed that H is PPT; see [1].

Lin [7] obtained a singular value inequality for PPT matrices related to a linear map.

Theorem 1.3 *Let* $A, B, X \in \mathbb{M}_n$*. If*

$$M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

is PPT, then for the linear map $\Phi : C \to C + \mathrm{Tr}(C)I$,

$$s_j\big(\Phi(X)\big) \leq s_j\big(\Phi(A \sharp B)\big), \quad j = 1, \ldots, n.$$

In this paper, we extend Theorem 1.1 and Theorem 1.3 to SPT matrices and show a stronger inequality than (2).

2 Main results

We start with some lemmas. The first three lemmas are quite standard in matrix analysis.

Lemma 2.1 ([13, p. 63]) *If $H \in \mathbb{M}_n$, then*

$$\sigma_j(\operatorname{Re} H) \le s_j(H), \quad j = 1, \dots, n. \tag{3}$$

Lemma 2.2 ([11, Theorem 1]) *Let $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_{m+n}$ be positive semidefinite with $A \in \mathbb{M}_m, B \in \mathbb{M}_n$. Then*

$$2s_j(X) \le s_j(H), \quad j = 1, \dots, \min\{m, n\}. \tag{4}$$

Lemma 2.3 ([2, p.106]) *Let $A, B \in \mathbb{M}_n$ be positive definite matrices. Then, for all $X \in M_n$,*

$$X^*(A\sharp B)X \le \bigl(X^*AX\bigr)\sharp\bigl(X^*BX\bigr). \tag{5}$$

The next lemma is due to Zhang [14, Lemma 3.1].

Lemma 2.4 *Let $A \in \mathbb{M}_n$ have $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then*

$$\|A\| \le \sec(\alpha)\|\operatorname{Re} A\| \tag{6}$$

for any unitarily invariant norm.

The following result about geometric mean has been proved by Lin and Sun [9].

Lemma 2.5 *Let $A, B \in \mathbb{M}_n$ be matrices with positive semidefinite real part. Then*

$$(\operatorname{Re} A)\sharp(\operatorname{Re} B) \le \operatorname{Re}(A\sharp B). \tag{7}$$

Now we are ready to present our results. The first theorem is an extension of Theorem 1.1.

Theorem 2.6 *Let $H_{11}, H_{12}, H_{21}, H_{22} \in \mathbb{M}_n$. If $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ is SPT, then*

$$\|H\| \le \sec(\alpha)\|H_{11} + H_{22}\|$$

for any unitarily invariant norm.

Proof Since H is a sector partial transpose matrix, then we know that

$$\operatorname{Re} H = \begin{pmatrix} \frac{H_{11}+H_{11}^*}{2} & \frac{H_{12}+H_{21}^*}{2} \\ \frac{H_{21}+H_{12}^*}{2} & \frac{H_{22}+H_{22}^*}{2} \end{pmatrix}$$

is PPT.

So by (6) we have

$$\begin{aligned} \|H\| &\le \sec(\alpha)\|\operatorname{Re} H\| \\ &\le \sec(\alpha)\|\operatorname{Re} H_{11} + \operatorname{Re} H_{22}\| \quad \bigl(\text{by } (1)\bigr) \\ &\le \sec(\alpha)\|H_{11} + H_{22}\|. \end{aligned}$$ □

Remark 2.7 When $H_{12} = H_{21}^*$ and $\alpha = 0$, then H is PPT in Theorem 2.6. Thus, our result is Hiroshima's inequality (1).

Next will give a stronger inequality than Theorem 1.2.

Theorem 2.8 *Let the Hua matrix be given by*

$$H := \begin{pmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{pmatrix},$$

where $A, B \in \mathbb{M}_{m\times n}$ *are strictly contractive. Then*

$$\| (I - A^*B)^{-1} \| \le \| (I - A^*A)^{-1} \sharp (I - B^*B)^{-1} \|$$

for any unitarily invariant norm.

Proof Since H is PPT, then

$$\begin{pmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{pmatrix}, \qquad \begin{pmatrix} (I - A^*A)^{-1} & (I - A^*B)^{-1} \\ (I - B^*A)^{-1} & (I - B^*B)^{-1} \end{pmatrix}$$

are both positive semidefinite matrices.

Hence,

$$(I - B^*B)^{-1} \ge (I - A^*B)^{-1}(I - A^*A)(I - B^*A)^{-1}$$

and

$$(I - B^*B)^{-1} \ge (I - B^*A)^{-1}(I - A^*A)(I - A^*B)^{-1}. \tag{8}$$

Clearly, by unitary similarity transformation,

$$\begin{pmatrix} (I - B^*B)^{-1} & (I - A^*B)^{-1} \\ (I - B^*A)^{-1} & (I - A^*A)^{-1} \end{pmatrix}$$

is also positive semidefinite.

Therefore,

$$(I - A^*A)^{-1} \ge (I - B^*A)^{-1}(I - B^*B)(I - A^*B)^{-1}. \tag{9}$$

Thus,

$$\begin{aligned} &(I - B^*B)^{-1} \sharp (I - A^*A)^{-1} \\ &\quad - (I - B^*A)^{-1}\big((I - A^*A)^{-1} \sharp (I - B^*B)^{-1}\big)^{-1}(I - A^*B)^{-1} \\ &\ge (I - B^*B)^{-1} \sharp (I - A^*A)^{-1} \\ &\quad - \big((I - B^*A)^{-1}(I - A^*A)(I - A^*B)^{-1}\big) \sharp \big((I - B^*A)^{-1}(I - B^*B)(I - A^*B)^{-1}\big) \end{aligned}$$

$$\begin{aligned}&\bigl(\text{by (5) and monotonicity}\bigr)\\ &\geq \bigl(I-B^*B\bigr)^{-1}\sharp\bigl(I-A^*A\bigr)^{-1}-\bigl(I-B^*B\bigr)^{-1}\sharp\bigl(I-A^*A\bigr)^{-1}\quad\bigl(\text{by (8) and (9)}\bigr)\\ &=0.\end{aligned}$$

In a similar way, we can prove

$$\begin{aligned}&\bigl(I-B^*B\bigr)^{-1}\sharp\bigl(I-A^*A\bigr)^{-1}\\ &\quad-\bigl(I-A^*B\bigr)^{-1}\bigl(\bigl(I-A^*A\bigr)^{-1}\sharp\bigl(I-B^*B\bigr)^{-1}\bigr)^{-1}\bigl(I-B^*A\bigr)^{-1}\geq 0.\end{aligned}$$

So

$$K:=\begin{pmatrix}(I-A^*A)^{-1}\sharp(I-B^*B)^{-1} & (I-A^*B)^{-1}\\ (I-B^*A)^{-1} & (I-B^*B)^{-1}\sharp(I-A^*A)^{-1}\end{pmatrix}$$

is PPT.

Therefore,

$$\begin{aligned}2\bigl\|\bigl(I-A^*B\bigr)^{-1}\bigr\| &\leq \|K\|\quad\bigl(\text{by (4)}\bigr)\\ &\leq \bigl\|\bigl(\bigl(I-A^*A\bigr)^{-1}\sharp\bigl(I-B^*B\bigr)^{-1}\bigr)+\bigl(\bigl(I-B^*B\bigr)^{-1}\sharp\bigl(I-A^*A\bigr)^{-1}\bigr)\bigr\|\\ &\quad\bigl(\text{by (1)}\bigr)\\ &=2\bigl\|\bigl(I-B^*B\bigr)^{-1}\sharp\bigl(I-A^*A\bigr)^{-1}\bigr\|.\end{aligned}$$

□

Remark 2.9 Obviously, our result is sharper than (2).

Finally, we present an extension of Theorem 1.3.

Theorem 2.10 *Let* $A,B,X,Y\in\mathbb{M}_n$. *If* $M=\bigl(\begin{smallmatrix}A & X\\ Y^* & B\end{smallmatrix}\bigr)$ *is SPT, then*

$$s_j\biggl(\Phi\biggl(\frac{X+Y}{2}\biggr)\biggr)\leq s_j\bigl(\Phi(A\sharp B)\bigr),\tag{10}$$

where $\Phi: C\to C+\operatorname{Tr}(C)I$.

Proof Since M is SPT, then

$$\operatorname{Re}M=\begin{pmatrix}\operatorname{Re}A & (X+Y)/2\\ (X+Y)^*/2 & \operatorname{Re}B\end{pmatrix}$$

and

$$\operatorname{Re}\bigl(M^\tau\bigr)=\begin{pmatrix}\operatorname{Re}A & (X+Y)^*/2\\ (X+Y)/2 & \operatorname{Re}B\end{pmatrix}=(\operatorname{Re}M)^\tau$$

are both positive semidefinite matrices. Thus, $\operatorname{Re}M$ is PPT.

By Theorem 1.3, we have

$$s_j\left(\Phi\left(\frac{X+Y}{2}\right)\right) \le s_j\big(\Phi\big((\operatorname{Re}A)\sharp(\operatorname{Re}B)\big)\big).$$

Compute

$$\begin{aligned} s_j\left(\Phi\left(\frac{X+Y}{2}\right)\right) &\le s_j\big(\Phi(\operatorname{Re}A\sharp\operatorname{Re}B)\big) \\ &\le s_j\big(\Phi\big(\operatorname{Re}(A\sharp B)\big)\big) \quad \big(\text{by } (7)\big) \\ &= s_j\big(\operatorname{Re}\big(\Phi(A\sharp B)\big)\big) \\ &\le s_j\big(\Phi(A\sharp B)\big) \quad \big(\text{by } (3)\big). \end{aligned}$$ □

Remark 2.11 If M is PPT, then (10) becomes Lin's result in Theorem 1.3.

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Authors' contributions
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