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Refining and reversing the weighted arithmetic–geometric mean inequality involving convex functionals and application for the functional entropy

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Abstract
In this paper, we present some refinements and reverses for some inequalities involving the weighted arithmetic mean and the weighted geometric mean of two convex functionals. Inequalities involving the Heinz functional mean are also obtained. As applications, we give some refinements and reverses for the relative entropy and the Tsallis relative entropy involving operator or functional arguments.

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1 Introduction

Let $a, b > 0$ and $\lambda \in [0,1]$ be real numbers. The following expressions

$$a\nabla_\lambda b = (1-\lambda)a + \lambda b, \qquad a!_\lambda b = \left(a^{-1}\nabla_\lambda b^{-1}\right)^{-1}, \qquad a\sharp_\lambda b = a^{1-\lambda}b^\lambda \tag{1.1}$$

are known in the literature as the λ-weighted arithmetic mean, the λ-weighted harmonic mean, and the λ-weighted geometric mean, respectively. For $\lambda = 1/2$, they are simply denoted by $a\nabla b$, $a!b$, and $a\sharp b$, respectively. In the recent few years, these means have received extensive attention which led to several developments and interesting applications, see [1–3, 5, 7, 10, 13, 16–19].

The previous weighted means satisfy the following inequalities:

$$a!_\lambda b \le a\sharp_\lambda b \le a\nabla_\lambda b \tag{1.2}$$

known as the weighted arithmetic–geometric–harmonic mean inequality. Some refinements and reverses of (1.2) have been discussed in the literature. In particular, the following result has been proved in [10, 13]:

$$2r_\lambda(a\nabla b - a\sharp b) \le a\nabla_\lambda b - a\sharp_\lambda b \le 2(1-r_\lambda)(a\nabla b - a\sharp b), \tag{1.3}$$

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where we set

$$r_\lambda := \min(\lambda, 1-\lambda). \tag{1.4}$$

Another weighted mean, known as the Heinz mean, is defined by

$$\mathrm{HZ}_\lambda(a,b) = \frac{a^{1-\lambda}b^\lambda + a^\lambda b^{1-\lambda}}{2}. \tag{1.5}$$

As it is well known, $\mathrm{HZ}_\lambda(a,b)$ interpolates the arithmetic mean and the geometric mean in the sense that the inequality

$$a\sharp b \le \mathrm{HZ}_\lambda(a,b) \le a\nabla b \tag{1.6}$$

holds for any $a, b > 0$.

A refinement and reverse of the right inequality in (1.6) have been recently obtained in the literature, see [12, 13], for instance,

$$a\nabla b - \frac{1}{2}\lambda(1-\lambda)(b-a)\log(b/a) \le \mathrm{HZ}_\lambda(a,b) \le a\nabla b - 2\lambda(1-\lambda)(\sqrt{a}-\sqrt{b})^2. \tag{1.7}$$

Other reversed and refined versions of the same inequality can be found in [11, 12].

The extension of the previous means, from the case where the variables are positive real numbers to the case where the arguments are positive operators, has been investigated in the literature. Let H be a complex Hilbert space and let $\mathcal{B}(H)$ be the $\mathbb{C}^*$-algebra of bounded linear operators acting on H. We denote by $\mathcal{B}^{+*}(H)$ the open cone of all (self-adjoint) invertible positive operators in $\mathcal{B}(H)$. For $A, B \in \mathcal{B}^{+*}(H)$, the following expressions:

$$A\nabla_\lambda B := (1-\lambda)A + \lambda B = B\nabla_{1-\lambda}A,$$
$$A!_\lambda B := \big((1-\lambda)A^{-1} + \lambda B^{-1}\big)^{-1} = B!_{1-\lambda}A,$$
$$A\sharp_\lambda B := A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^\lambda A^{1/2} = B\sharp_{1-\lambda}A$$

are known in the literature as the λ-weighted arithmetic operator mean, the λ-weighted harmonic operator mean, and the λ-weighted geometric operator mean of A and B, respectively. For $\lambda = 1/2$, they are simply denoted by $A\nabla B$, $A!B$, and $A\sharp B$, respectively. These operator means satisfy the following inequality:

$$A!_\lambda B \le A\sharp_\lambda B \le A\nabla_\lambda B, \tag{1.8}$$

which is the operator version of (1.2). Here, the notation $\le$ stands for the Löwner partial order defined as follows: $T \le S$ if and only if T and S are self-adjoint and $S - T$ is positive.

An operator version of (1.3) has been established in [10] and reads as follows:

$$2r_\lambda(A\nabla B - A\sharp B) \le A\nabla_\lambda B - A\sharp_\lambda B \le 2(1-r_\lambda)(A\nabla B - A\sharp B), \tag{1.9}$$

where r_λ is defined by (1.4).

By analogy with the scalar case, the Heinz operator mean is defined by

$$\mathrm{HZ}_\lambda(A,B) = \frac{A\sharp_\lambda B + A\sharp_{1-\lambda}B}{2}. \tag{1.10}$$

We also have the following operator inequalities extending (1.6):

$$A\sharp B \le \mathrm{HZ}_\lambda(A,B) \le A\nabla B. \tag{1.11}$$

In fact, according to the Kubo–Ando theory [14], (1.9) and (1.11) can be immediately deduced from (1.3) and (1.6), respectively. For more inequalities related to the Heinz mean involving matrix/operator arguments, we refer the reader to [4, 8, 9, 11, 15, 24] and the related references cited therein.

2 Functional means

The previous operator means have been extended from the case where the variables are invertible positive operators to the case that the variables are convex functionals, see [21, 22]. To explain this, we need to recall some basic notions from convex analysis. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a given functional. The notation $\operatorname{dom} f$ refers to the effective domain of f defined by

$$\operatorname{dom} f = \{x \in H, f(x) < +\infty\},$$

while f^* stands for the Fenchel conjugate of f defined through

$$\forall x^* \in H, \quad f^*(x^*) = \sup_{x\in H}\{\Re e\langle x^*, x\rangle - f(x)\}. \tag{2.1}$$

The subdifferential $\partial f(x)$ of f at $x \in \operatorname{dom} f$ is the (possibly empty) subset of H defined by

$$x^* \in \partial f(x) \quad \Longleftrightarrow \quad \forall y \in H, \quad f(y) \ge f(x) + \Re e\langle x^*, y - x\rangle,$$

and it is well known that

$$x^* \in \partial f(x) \quad \Longleftrightarrow \quad \Re e\langle x^*, x\rangle = f^*(x^*) + f(x). \tag{2.2}$$

Henceforth, and by virtue of the definition of the subdifferential, whenever we write $\partial f(x)$ it will be assumed that $x \in \operatorname{dom} f$.

Let us denote by $\Gamma_0(H)$ the cone of all functionals $f : H \to \mathbb{R} \cup \{+\infty\}$ which are convex lower semicontinuous and not identically equal to $+\infty$. It is well known that $f \in \Gamma_0(H)$ if and only if $f^{**} := (f^*)^* = f$ and, if $f \in \Gamma_0(H)$ then $x^* \in \partial f(x)$ if and only if $x \in \partial f^*(x^*)$. If we denote by $\operatorname{int}(\operatorname{dom} f)$ the topological interior of $\operatorname{dom} f$, we recall that if $f \in \Gamma_0(H)$ and $\operatorname{int}(\operatorname{dom} f)$ is nonempty then for all $x \in \operatorname{int}(\operatorname{dom} f)$, f is continuous at x and $\partial f(x)$ is nonempty.

The following example, which will be needed in the sequel, explains the previous notions in more detail.

Example 2.1 Let $A \in \mathcal{B}(H)$ be a self-adjoint operator. Usually, we denote by f_A the quadratic form generated by the operator A, i.e.,

$$\forall x \in H, \quad f_A(x) = \frac{1}{2}\langle Ax, x\rangle.$$

Such functional f_A enjoys the following properties:

(i) Let $A, B \in \mathcal{B}(H)$ be self-adjoint. For any $\alpha, \beta \in \mathbb{R}$ we have $\alpha f_A + \beta f_B = f_{\alpha A+\beta B}$. Further, it is clear that $f_A \le f_B$ if and only if $A \le B$. If, moreover, A is positive then $f_A \in \Gamma_0(H)$ and $\partial f_A(x) = \{Ax\}$ for any $x \in H$.

(ii) If $A \in \mathcal{B}^{+*}(H)$ then we have

$$\forall x \in H, \quad f_A^*(x) = \frac{1}{2}\langle A^{-1}x, x\rangle,$$

which can be written as $f_A^* = f_{A^{-1}}$.

Throughout this paper, we use the following notation

$$\mathcal{D}(H) = \{(f,g) \in \Gamma_0(H) \times \Gamma_0(H) : \operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset \text{ and } \operatorname{dom} f^* \cap \operatorname{dom} g^* \neq \emptyset\}.$$

Now, let $(f,g) \in \mathcal{D}(H)$ and $\lambda \in (0,1)$. The following expressions:

$$\mathcal{A}_\lambda(f,g) := (1-\lambda)f + \lambda g, \tag{2.3}$$

$$\mathcal{H}_\lambda(f,g) := \big((1-\lambda)f^* + \lambda g^*\big)^*, \tag{2.4}$$

$$\mathcal{G}_\lambda(f,g) := \frac{\sin(\lambda\pi)}{\pi}\int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda}\mathcal{H}_t(f,g)\,dt \tag{2.5}$$

are called (by analogy) the λ-weighted functional arithmetic mean, the λ-weighted functional harmonic mean, and the λ-weighted functional geometric mean of f and g, respectively. For $\lambda = 1/2$, they are simply denoted by $\mathcal{A}(f,g)$, $\mathcal{H}(f,g)$, and $\mathcal{G}(f,g)$, respectively. We also have the following double inequality:

$$\mathcal{H}_\lambda(f,g) \le \mathcal{G}_\lambda(f,g) \le \mathcal{A}_\lambda(f,g), \tag{2.6}$$

known, by analogy, as the weighted arithmetic–geometric–harmonic functional mean inequality. Here, the symbol $\le$ refers to the pointwise order defined as follows: $f \le g$ if and only if $f(x) \le g(x)$ for all $x \in H$.

We can extend the previous functional means on the whole interval $[0,1]$ by setting

$$\mathcal{A}_0(f,g) = \mathcal{G}_0(f,g) = \mathcal{H}_0(f,g) = f, \qquad \mathcal{A}_1(f,g) = \mathcal{G}_1(f,g) = \mathcal{H}_1(f,g) = g. \tag{2.7}$$

For all $(f,g) \in \mathcal{D}(H)$ and $\lambda \in [0,1]$ we introduce the following expression [23]:

$$\mathcal{HZ}_\lambda(f,g) = \frac{1}{2}\big(\mathcal{G}_\lambda(f,g) + \mathcal{G}_{1-\lambda}(f,g)\big) \tag{2.8}$$

which is the Heinz functional mean of f and g. It is clear that $\mathcal{HZ}_\lambda(f,g)$ is symmetric in f and g and $\mathcal{HZ}_\lambda(f,g) = \mathcal{HZ}_{1-\lambda}(f,g)$. Further, $\mathcal{HZ}_0(f,g) = \mathcal{HZ}_1(f,g) = \mathcal{A}(f,g)$ and

$\mathcal{HZ}_{1/2}(f,g) = \mathcal{G}(f,g)$. Otherwise, from (2.6) and (2.8) we can check that the inequality

$$\mathcal{HZ}_\lambda(f,g)(x) \le \mathcal{A}(f,g)(x) \tag{2.9}$$

holds for any $\lambda \in [0,1]$ and $x \in H$.

The following remark may be of interest for the reader.

Remark 2.1 We notice that the mentioned functionals can take the value $+\infty$. For this, we observe the following:

(i) The equalities (2.7) cannot be deduced from (2.3), (2.4) and (2.5), respectively. This is because the involved functionals can take the value $+\infty$, with the convention $0 \cdot (+\infty) = (+\infty) - (+\infty) = (+\infty) + (-\infty) := +\infty$, as is usual in convex analysis.

(ii) The functional inequality (2.9) can be easily proved, but with some precautions. These precautions should be taken in consideration in the proof of any functional equality or inequality. This latter point is explained in the following items.

(iii) For instance, the equality $f - f = 0$ is not always true for any $f : H \to \mathbb{R} \cup \{+\infty\}$ by virtue of the same reason as in (i). Precisely, we have $f - f = \Psi_{\mathrm{dom} f}$ for any $f : H \to \mathbb{R} \cup \{+\infty\}$, where the notation Ψ_C refers to the indicator function of the subset C of H defined as $\Psi_C(z) = 0$ if $z \in C$ and $\Psi_C(z) = +\infty$ if $z \notin C$.

(iv) Similarly, the equality $f - g = -(g - f)$ is not always true, unless f or g is everywhere finite. For the same reason, the functional inequality $f \le g$ is equivalent to $g - f \ge 0$ but it is not equivalent to $f - g \le 0$.

Clearly, $\mathcal{A}_\lambda(f,f) = f$ for any $f : H \to \mathbb{R} \cup \{+\infty\}$. Indeed, when $f \in \Gamma_0(H)$ then $f^{**} = f$ and so the previous functional means satisfy the following relations:

$$\mathcal{H}_\lambda(f,f) = \mathcal{G}_\lambda(f,f) = \mathcal{HZ}_\lambda(f,f) = f.$$

Further, all the previous functional means are respectively extensions of their related operator means in the following sense:

$$\begin{aligned} &\mathcal{A}_\lambda(f_A,f_B) = f_{A\nabla_\lambda B}, \qquad \mathcal{H}_\lambda(f_A,f_B) = f_{A!_\lambda B}, \\ &\mathcal{G}_\lambda(f_A,f_B) = f_{A\sharp_\lambda B}, \qquad \mathcal{HZ}_\lambda(f_A,f_B) = f_{\mathrm{HZ}_\lambda(A,B)}, \end{aligned} \tag{2.10}$$

where the notation f_A (resp. f_B) was defined in Example 2.1. With this, (1.8) is an immediate consequence of (2.6); see [21, 22], for instance.

The main goal of this paper is to give some refined and reversed inequalities between the weighted arithmetic mean and the weighted geometric mean involving convex functionals. We also obtain some refinements and reverses of (1.11) when the operator variables A and B are replaced by convex functionals. As applications, we obtain some refinements and reverses for some inequalities involving the relative entropy and the Tsallis entropy with operator or functional arguments.

3 Some needed results

Before stating our main results, we need to recall the following lemmas.

Lemma 3.1 ([21]) *Let $(f,g) \in \mathcal{D}(H)$. For each $t,s \in (0,1)$, the following inequalities:*

$$r_{t,s}\big(\mathcal{A}_s(f,g)(x) - \mathcal{H}_s(f,g)(x)\big) \le \mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x) \\ \le R_{t,s}\big(\mathcal{A}_s(f,g)(x) - \mathcal{H}_s(f,g)(x)\big) \tag{3.1}$$

hold for any $x \in H$, where we set

$$r_{t,s} := \min\left(\frac{t}{s}, \frac{1-t}{1-s}\right) \quad and \quad R_{t,s} := \max\left(\frac{t}{s}, \frac{1-t}{1-s}\right).$$

Lemma 3.2 ([23]) *Let $(f,g) \in \mathcal{D}(H)$ and $\lambda \in (0,1)$. For all $x \in H$, one has*

$$\mathcal{A}_\lambda(f,g)(x) - \mathcal{G}_\lambda(f,g)(x) = \frac{\sin(\lambda\pi)}{\pi}\int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda}\big(\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)\big)\,dt. \tag{3.2}$$

In particular, for all $x \in H$ we have

$$\mathcal{A}(f,g)(x) - \mathcal{G}(f,g)(x) = \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{t(1-t)}}\big(\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)\big)\,dt. \tag{3.3}$$

Lemma 3.3 ([23]) *Let $(f,g) \in \mathcal{D}(H)$. Then the map*

$$t \mapsto \frac{\mathcal{A}_t(f,g) - \mathcal{H}_t(f,g)}{t(1-t)}$$

is pointwise integrable on $(0,1)$. That is, for any $x \in H$, the integral

$$\mathcal{J}(f,g)(x) := \int_0^1 \frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t(1-t)}\,dt \tag{3.4}$$

exists in $\mathbb{R} \cup \{+\infty\}$.

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ and $x, x^* \in H$. We set

$$\mathcal{F}_f\big(x,x^*\big) := f(x) + f^*\big(x^*\big) - \Re e\big\langle x^*, x\big\rangle. \tag{3.5}$$

Following the so-called Fenchel inequality, we have

$$\forall x, x^* \in H, \quad \mathcal{F}_f\big(x,x^*\big) \ge 0. \tag{3.6}$$

We also define

$$(f \diamond g)(x) := \sup_{x^* \in \partial f(x)} \big\{\Re e\big\langle x^*, x\big\rangle - g^*\big(x^*\big)\big\}, \tag{3.7}$$

with the usual convention $\sup_\emptyset(\cdot) = -\infty$. With this, it is not hard to check that the following formula:

$$(f \diamond g)(x) = \big(g^* + \Psi_{\partial f(x)}\big)^*(x) \tag{3.8}$$

holds for any $x \in H$. Using (3.7) or (3.8), we can see that $(f \diamond g)(x) \leq g(x)$ holds for any $x \in H$.

Below, we present an example which explains the previous notions in more detail. Such an example will be needed throughout this paper.

Example 3.1 Take $f(x) = (1/2)\langle Ax, x\rangle$ and $g(x) = (1/2)\langle Bx, x\rangle$, where $A, B \in \mathcal{B}^{+*}(H)$. We write $f = f_A$ and $g = f_B$, for the sake of simplicity. Since $\partial f(x) = \{Ax\}$ and $g^* = f_{B^{-1}}$, (3.7) yields

$$(f \diamond g)(x) = \Re e\langle Ax, x\rangle - f_{B^{-1}}(Ax) = \langle Ax, x\rangle - \frac{1}{2}\langle B^{-1}Ax, Ax\rangle = \frac{1}{2}\langle (A \diamond B)x, x\rangle,$$

where $A \diamond B$ is given by

$$A \diamond B = 2A - AB^{-1}A.$$

We notice that $f \diamond g$ is not always a convex functional, and so $A \diamond B$ is not always a positive operator. However, from (3.7) we can immediately deduce that $f \diamond g \leq g$, which implies that $B - A \diamond B$ is always positive.

Finally, let $a > 0$, $b \geq 0$ and $\lambda, s \in (0,1)$. For the sake of simplicity, we set

$$B_s(a,b) := \int_0^s t^{a-1}(1-t)^{b-1}\,dt \quad \text{and} \quad \Lambda_s^\lambda := \frac{1}{s}B_s(1+\lambda, 1-\lambda). \tag{3.9}$$

By simple computation of integrals, it is easy to see that

$$B_s(1,1) = s \quad \text{and} \quad B_s(2,0) = -s - \ln(1-s).$$

We can then check that

$$\Lambda_s^0 := \lim_{\lambda\downarrow 0} \Lambda_s^\lambda = 1 \quad \text{and} \quad \Lambda_s^1 := \lim_{\lambda\uparrow 1} \Lambda_s^\lambda = -1 - \frac{\ln(1-s)}{s}. \tag{3.10}$$

We have the following result.

Proposition 3.4 *For all $s \in (0,1)$, we have*

$$\Lambda_s^{1/2} = \frac{1}{s}\left(-\sqrt{s(1-s)} + \arctan\left(\sqrt{\frac{s}{1-s}}\right)\right). \tag{3.11}$$

Proof It is a simple exercise of real analysis. We first use the change of variables by setting $u = \sqrt{t/(1-t)}$ and then we proceed by an appropriate integration by parts. The details are straightforward and therefore omitted. □

4 The main results

We preserve the same notations as in the previous sections. We start this section by stating the following result.

Theorem 4.1 *Let $(f,g) \in \mathcal{D}(H)$ and $t \in (0,1)$. Let $x \in H$ be such that $\partial f(x) \neq \emptyset$ and $\partial g(x) \neq \emptyset$. Then the following inequality*:

$$\frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t(1-t)} \leq \mathcal{F}_g\big(x,x^*\big) + \mathcal{F}_f\big(x,z^*\big) \tag{4.1}$$

holds for any $x^* \in \partial f(x)$ and $z^* \in \partial g(x)$.

Proof By definition, we have

$$\mathcal{H}_t(f,g)(x) := \big((1-t)f^* + tg^*\big)^*(x) := \sup_{x^* \in H}\big\{\Re e\langle x^*,x\rangle - (1-t)f^*\big(x^*\big) - tg^*\big(x^*\big)\big\},$$

from which we deduce that the inequality

$$\mathcal{H}_t(f,g)(x) \geq \Re e\langle x^*,x\rangle - (1-t)f^*\big(x^*\big) - tg^*\big(x^*\big)$$

holds for any $x, x^* \in H$. It follows that

$$\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x) \leq (1-t)f(x) + tg(x) - \Re e\langle x^*,x\rangle + (1-t)f^*\big(x^*\big) + tg^*\big(x^*\big),$$

or equivalently,

$$\begin{aligned}\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x) \leq{}& (1-t)\big(f(x) + f^*\big(x^*\big) - \Re e\langle x^*,x\rangle\big)\\ &+ t\big(g(x) + g^*\big(x^*\big) - \Re e\langle x^*,x\rangle\big).\end{aligned}$$

If $x^* \in \partial f(x)$ then $f(x) + f^*(x^*) = \Re e\langle x^*,x\rangle$ and so

$$\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x) \leq t\mathcal{F}_g\big(x,x^*\big).$$

Similarly, if $z^* \in \partial g(x)$, we have

$$\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x) \leq (1-t)\mathcal{F}_f\big(x,z^*\big).$$

It follows that if $x^* \in \partial f(x)$ and $z^* \in \partial g(x)$ then

$$\frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t} \leq \mathcal{F}_g\big(x,x^*\big) \tag{4.2}$$

and

$$\frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{1-t} \leq \mathcal{F}_f\big(x,z^*\big). \tag{4.3}$$

By adding (4.2) and (4.3), with a simple manipulation, we obtain (4.1). □

Corollary 4.2 *For any $x \in H$ such that $\partial f(x) \cap \operatorname{dom} g^* \neq \emptyset$ and $\partial g(x) \cap \operatorname{dom} f^* \neq \emptyset$, the integral (3.4) is finite, i.e., $\mathcal{J}(f,g)(x) < +\infty$. Further, the inequality*

$$\mathcal{J}(f,g)(x) \leq \mathcal{F}_g\big(x,x^*\big) + \mathcal{F}_f\big(x,z^*\big)$$

holds for any $x^* \in \partial f(x) \cap \operatorname{dom} g^*$ and $z^* \in \partial g(x) \cap \operatorname{dom} f^*$.

Proof First, by Lemma 3.3, the integral in (3.4) exists in $\mathbb{R} \cup \{+\infty\}$. This, with (4.1) and (3.5), yields the desired result. The details are simple and therefore omitted here. □

The following result will be needed in the sequel.

Proposition 4.3 *For all $x \in H$, we have*

$$\begin{aligned} &\mathcal{A}(f,g)(x) - \mathcal{HZ}_\lambda(f,g)(x) \\ &\quad = \frac{\sin(\lambda\pi)}{\pi} \int_0^1 \mathrm{HZ}_\lambda(1-t,t) \frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t(1-t)}\,dt. \end{aligned} \tag{4.4}$$

Proof By (3.2), we can write

$$\mathcal{A}_\lambda(f,g)(x) - \mathcal{G}_\lambda(f,g)(x) = \frac{\sin(\lambda\pi)}{\pi} \int_0^1 t^\lambda (1-t)^{1-\lambda} \frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t(1-t)}\,dt. \tag{4.5}$$

If we replace λ by $1-\lambda$ in this latter formula, we obtain

$$\begin{aligned} &\mathcal{A}_{1-\lambda}(f,g)(x) - \mathcal{G}_{1-\lambda}(f,g)(x) \\ &\quad = \frac{\sin(\lambda\pi)}{\pi} \int_0^1 t^{1-\lambda}(1-t)^\lambda \frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t(1-t)}\,dt. \end{aligned} \tag{4.6}$$

Adding (4.5) and (4.6), we obtain (4.4). □

We now state our second main result which reads as follows.

Theorem 4.4 *With the same hypotheses as in Theorem 4.1, the following inequalities*:

$$\mathcal{A}_\lambda(f,g)(x) - \mathcal{G}_\lambda(f,g)(x) \le \frac{\lambda(1-\lambda)}{2}\big(\mathcal{F}_g(x,x^*) + \mathcal{F}_f(x,z^*)\big), \tag{4.7}$$

$$\mathcal{A}(f,g)(x) - \mathcal{HZ}_\lambda(f,g)(x) \le \frac{\lambda(1-\lambda)}{2}\big(\mathcal{F}_g(x,x^*) + \mathcal{F}_f(x,z^*)\big) \tag{4.8}$$

hold for any $x^* \in \partial f(x)$ and $z^* \in \partial g(x)$.

Proof By (4.5) and (4.1), we can write

$$\frac{\pi}{\sin(\lambda\pi)}\big(\mathcal{A}_\lambda(f,g)(x) - \mathcal{G}_\lambda(f,g)(x)\big) \le \left(\int_0^1 t^\lambda(1-t)^{1-\lambda}\,dt\right)\big(\mathcal{F}_g(x,x^*) + \mathcal{F}_f(x,z^*)\big).$$

If B and Γ denote the standard beta and gamma special functions, respectively, then we have

$$\begin{aligned} \int_0^1 t^\lambda(1-t)^{1-\lambda}\,dt = B(1+\lambda, 2-\lambda) &= \frac{\Gamma(1+\lambda)\Gamma(2-\lambda)}{\Gamma(3)} = \frac{\lambda\Gamma(\lambda)(1-\lambda)\Gamma(1-\lambda)}{2!} \\ &= \frac{\lambda(1-\lambda)}{2}\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\lambda(1-\lambda)}{2}\frac{\pi}{\sin(\lambda\pi)}. \end{aligned}$$

Thus inequality (4.7) follows.

If we replace λ by $1-\lambda$ in (4.7), we obtain

$$\mathcal{A}_{1-\lambda}(f,g)(x)-\mathcal{G}_{1-\lambda}(f,g)(x)\leq\frac{\lambda(1-\lambda)}{2}\big(\mathcal{F}_g\big(x,x^*\big)+\mathcal{F}_f\big(x,z^*\big)\big).\tag{4.9}$$

Adding (4.7) and (4.9) and remarking that for $x\in H$ such that $f(x)<+\infty$ we have

$$\mathcal{A}_\lambda(f,g)(x)+\mathcal{A}_{1-\lambda}(f,g)(x)=(f+g)(x),$$

we obtain (4.8). The proof is completed. □

We can write the previous theorem in another form, which is symmetric in f and g. Precisely, the following result may be stated.

Theorem 4.5 *With the same hypotheses as in Theorem 4.1, the following inequalities*:

$$\mathcal{A}_\lambda(f,g)(x)-\mathcal{G}_\lambda(f,g)(x)\leq\lambda(1-\lambda)\big(\mathcal{A}(f,g)(x)-\mathcal{A}(f\diamond g,g\diamond f)(x)\big),\tag{4.10}$$

$$\mathcal{A}(f,g)(x)-\mathcal{HZ}_\lambda(f,g)(x)\leq\lambda(1-\lambda)\big(\mathcal{A}(f,g)(x)-\mathcal{A}(f\diamond g,g\diamond f)(x)\big)\tag{4.11}$$

hold, *where* $f\diamond g$ *is defined by* (3.7).

Proof According to (4.7), the functional

$$\partial f(x)\times\partial g(x)\ni\big(x^*,z^*\big)\mapsto\mathcal{F}_g\big(x,x^*\big)+\mathcal{F}_f\big(x,z^*\big)$$

is lower-bounded, with

$$\inf_{(x^*,z^*)\in\partial f(x)\times\partial g(x)}\big(\mathcal{F}_g\big(x,x^*\big)+\mathcal{F}_f\big(x,z^*\big)\big)=\inf_{x^*\in\partial f(x)}\mathcal{F}_g\big(x,x^*\big)+\inf_{z^*\in\partial g(x)}\mathcal{F}_f\big(x,z^*\big).$$

Following (3.5) and (3.7), we have

$$\inf_{x^*\in\partial f(x)}\mathcal{F}_g\big(x,x^*\big)=g(x)+\inf_{x^*\in\partial f(x)}\big(g^*\big(x^*\big)-\Re e\langle x^*,x\rangle\big)=g(x)-(f\diamond g)(x).$$

Similarly, one has

$$\inf_{z^*\in\partial g(x)}\mathcal{F}_f\big(x,z^*\big)=f(x)-(g\diamond f)(x).$$

Summarizing, we have proved that

$$\begin{aligned}\inf_{(x^*,z^*)\in\partial f(x)\times\partial g(x)}\big(\mathcal{F}_g\big(x,x^*\big)+\mathcal{F}_f\big(x,z^*\big)\big)&=f(x)+g(x)-(f\diamond g)(x)-(g\diamond f)(x)\\&=2\big(\mathcal{A}(f,g)(x)-\mathcal{A}(f\diamond g,g\diamond f)(x)\big).\end{aligned}\tag{4.12}$$

Substituting this into (4.7), we obtain (4.10).

Similarly, (4.11) can be deduced from (4.8). The proof is complete. □

The operator versions of Theorems 4.1 and 4.4 (or Theorem 4.5) are given in the following.

Corollary 4.6 *Let $A, B \in \mathcal{B}^{+*}(H)$ and $\lambda \in [0,1]$. Then we have the following operator inequalities*:

$$\begin{aligned} & A\nabla_\lambda B - A!_\lambda B \leq \lambda(1-\lambda)\bigl(AB^{-1}A + BA^{-1}B - A - B\bigr), \\ & A\nabla_\lambda B - A\sharp_\lambda B \leq \frac{\lambda(1-\lambda)}{2}\bigl(AB^{-1}A + BA^{-1}B - A - B\bigr), \\ & A\nabla B - \mathrm{HZ}_\lambda(A,B) \leq \frac{\lambda(1-\lambda)}{2}\bigl(AB^{-1}A + BA^{-1}B - A - B\bigr). \end{aligned} \tag{4.13}$$

Now, let us observe the following remark which explains the interest of our functional approach and its application for scalar/operator means.

Remark 4.1 The inequalities of the preceding corollary are important due to the fact that they refine some existing operator inequalities. By virtue of the Kubo–Ando theory for monotone operator means, it is enough to discuss the scalar case when justifying our claim. We restrict ourselves to inequality (4.13), since the others can be investigated in a similar manner.

Indeed, the scalar version of (4.13) is reduced to

$$a\nabla_\lambda b - a\sharp_\lambda b \leq \frac{\lambda(1-\lambda)}{2}\frac{(a-b)^2(a+b)}{ab}. \tag{4.14}$$

We would like to compare (4.14) with the right inequality in (1.3). In fact, by the techniques of real analysis, it is not hard to show that there exist λ_0 and $\lambda_1 \in (0,1)$ such that if $0 \leq \lambda_0 \leq \lambda$ or $\lambda_1 \leq \lambda \leq 1$ then (4.14) is better than the right inequality of (1.3).

Now we will present some results giving simultaneously refinements and reverses of the inequalities $\mathcal{G}_\lambda(f,g) \leq \mathcal{A}_\lambda(f,g)$ and $\mathcal{HZ}_\lambda(f,g) \leq \mathcal{A}(f,g)$.

Theorem 4.7 *Let $(f,g) \in \mathcal{D}(H)$. For each $\lambda, s \in (0,1)$, the following inequalities*:

$$\begin{aligned} & \frac{\sin(\lambda\pi)}{\pi}\bigl(\Lambda_s^\lambda + \Lambda_{1-s}^{1-\lambda}\bigr)\bigl(\mathcal{A}_s(f,g)(x) - \mathcal{H}_s(f,g)(x)\bigr) \\ & \quad \leq \mathcal{A}_\lambda(f,g)(x) - \mathcal{G}_\lambda(f,g)(x) \leq \frac{\sin(\lambda\pi)}{\pi}\bigl((1-\lambda)\sharp_\lambda\lambda\bigr)\mathcal{J}(f,g)(x) \end{aligned} \tag{4.15}$$

hold for any $x \in H$.

Proof First, we show the right inequality of (4.15). By virtue of (2.6), we can write, for all $x \in H$,

$$\begin{aligned} \frac{t^{\lambda-1}}{(1-t)^\lambda}\bigl(\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)\bigr) &= \bigl((1-t)^{1-\lambda}t^\lambda\bigr)\frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t(1-t)} \\ &\leq \Bigl(\max_{0\leq t\leq 1}(1-t)^{1-\lambda}t^\lambda\Bigr)\frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t(1-t)}. \end{aligned}$$

It is not hard to check that

$$\max_{0\leq t\leq 1}(1-t)^{1-\lambda}t^\lambda = (1-\lambda)^{1-\lambda}\lambda^\lambda := (1-\lambda)\sharp_\lambda\lambda.$$

The right inequality of (4.15) follows by Lemma 3.3 and (3.2).

We now show the left inequality of (4.15). For the right hand-side of (3.2), we use the left inequality in (3.1). For any fixed $s \in (0,1)$, it is easy to see that

$$\int_0^1 \frac{t^{\lambda-1}}{(1-t)^{\lambda}} r_{t,s}\,dt = \int_0^s \frac{t^{\lambda-1}}{(1-t)^{\lambda}} \frac{t}{s}\,dt + \int_s^1 \frac{t^{\lambda-1}}{(1-t)^{\lambda}} \frac{1-t}{1-s}\,dt.$$

If in this latter integral, we use a simple change of variables, we deduce the desired inequality with the help of (4.7) and a simple manipulation. □

The operator version of Theorem 4.7 reads as follows.

Corollary 4.8 *Let $A, B \in \mathcal{B}^{+*}(H)$ and $\lambda, s \in (0,1)$. Then we have*

$$\frac{\sin(\lambda\pi)}{\pi}\big(\Lambda_s^{\lambda} + \Lambda_{1-s}^{1-\lambda}\big)(A\nabla_s B - A!_s B) \leq A\nabla_\lambda B - A\sharp_\lambda B \leq \frac{\sin(\lambda\pi)}{\pi}\big((1-\lambda)\sharp_\lambda \lambda\big) J(A,B), \tag{4.16}$$

where we set

$$J(A,B) := \int_0^1 \frac{A\nabla_t B - A!_t B}{t(1-t)}\,dt = (B-A)A^{-1}S(A|B), \tag{4.17}$$

with

$$S(A|B) := A^{1/2}\log\big(A^{-1/2}BA^{-1/2}\big)A^{1/2}.$$

Proof If in (4.15) we take $f = f_A$ and $g = f_B$, we obtain the left inequality of (4.16) by (2.10). To obtain the right inequality of (4.16), we can check that, see [23],

$$\mathcal{J}(f_A, f_B) = f_{J(A,B)},$$

where $J(A,B)$ is given by (4.17). The details are simple and therefore omitted here. □

We need to introduce the following notation:

$$\Theta_s(f,g) := \frac{\mathcal{H}_s(f,g) + \mathcal{H}_{1-s}(f,g)}{2}. \tag{4.18}$$

The following inequalities:

$$\mathcal{H}(f,g)(x) \leq \Theta_s(f,g)(x) \leq \mathcal{HZ}_s(f,g)(x) \leq \mathcal{A}(f,g)(x)$$

hold for any $x \in H$. For the left inequality, see [23]. For the other inequalities, they are immediate from (2.6), (2.8), and (2.9).

As an application of the previous theorem, we may state the following corollary which concerns a reverse and refinement of (2.9).

Corollary 4.9 *Let $(f,g)\in\mathcal{D}(H)$ and $\lambda,s\in(0,1)$. Then the following inequalities*:

$$\begin{aligned}&\frac{\sin(\lambda\pi)}{\pi}\big(\Lambda_s^{\lambda}+\Lambda_{1-s}^{1-\lambda}\big)\big(\mathcal{A}(f,g)(x)-\Theta_s(f,g)(x)\big)\\&\quad\le\mathcal{A}(f,g)(x)-\mathcal{HZ}_{\lambda}(f,g)(x)\le\frac{\sin(\lambda\pi)}{\pi}\big((1-\lambda)\sharp_{\lambda}\lambda\big)\mathcal{J}(f,g)(x)\end{aligned}\tag{4.19}$$

hold for any $x\in H$.

Proof First, if $x\in H$ is such that $f(x)=+\infty$ then the inequalities in (4.19) are reduced to an equality, since all sides of (4.19) are equal to $+\infty$ by virtue of the usual convention $+\infty+c=+\infty$ for any $c\in[-\infty,+\infty]$. Assume that below $f(x)<+\infty$. If in (4.15) we replace λ by $1-\lambda$ and s by $1-s$ and add the results, we obtain the desired inequalities with the help of (2.8), (4.18), and the fact that

$$\mathcal{A}_s(f,g)(x)+\mathcal{A}_{1-s}(f,g)(x)=(f+g)(x)$$

is valid for any $s\in(0,1)$ and $x\in\operatorname{dom}f$. The proof is completed. □

The following result is of interest.

Corollary 4.10 *Let $(f,g)\in\mathcal{D}(H)$. Then the following inequalities*:

$$\frac{\pi-2}{\pi}\big(\mathcal{A}(f,g)(x)-\mathcal{H}(f,g)(x)\big)\le\mathcal{A}(f,g)(x)-\mathcal{G}(f,g)(x)\le\frac{1}{2\pi}\mathcal{J}(f,g)(x)\tag{4.20}$$

hold for any $x\in H$.

Proof Taking $\lambda=s=1/2$ in (4.19) and using Proposition 3.4, we immediately obtain the desired inequalities after a simple computation and reduction. The details are straightforward and therefore omitted here. □

The right inequality of (4.20) gives a lower bound of $\mathcal{J}(f,g)$, and the left one yields a reverse of $\mathcal{A}(f,g)-\mathcal{G}(f,g)\le\mathcal{A}(f,g)-\mathcal{H}(f,g)$. Note that Corollary 4.2 contains an upper bound of $\mathcal{J}(f,g)$.

Also, we have the following result.

Theorem 4.11 *Let $(f,g)\in\mathcal{D}(H)$ and $\lambda\in(0,1)$. Then, for all $x\in H$, we have the following inequalities*:

$$\begin{aligned}&4(1-r_{\lambda})\pi\big(\mathcal{A}(f,g)(x)-\mathcal{G}(f,g)(x)\big)-|2\lambda-1|\mathcal{J}(f,g)(x)\\&\quad\le\frac{2\pi}{\sin(\lambda\pi)}\big(\mathcal{A}(f,g)(x)-\mathcal{HZ}_{\lambda}(f,g)(x)\big)\\&\quad\le4r_{\lambda}\pi\big(\mathcal{A}(f,g)(x)-\mathcal{G}(f,g)(x)\big)+|2\lambda-1|\mathcal{J}(f,g)(x).\end{aligned}\tag{4.21}$$

Proof From (1.3) we immediately deduce that, for all $a,b>0$,

$$r_{\lambda}(a+b-2\sqrt{ab})\le\frac{a+b}{2}-\mathrm{HZ}_{\lambda}(a,b)\le(1-r_{\lambda})(a+b-2\sqrt{ab}),$$

or equivalently,

$$\frac{a+b}{2} - (1-r_\lambda)(a+b-2\sqrt{ab}) \leq \mathrm{HZ}_\lambda(a,b) \leq \frac{a+b}{2} - r_\lambda(a+b-2\sqrt{ab}).$$

First, observe that $1-2r_\lambda = |2\lambda-1|$. If we take $a = 1-t$ and $b = t$ in the latter inequalities, we get, after simple manipulations,

$$-\frac{|2\lambda-1|}{2} + 2(1-r_\lambda)\sqrt{t(1-t)} \leq \mathrm{HZ}_\lambda(1-t,t) \leq \frac{|2\lambda-1|}{2} + 2r_\lambda\sqrt{t(1-t)}.$$

Substituting the result into (4.4), we obtain

$$L.H.S \leq \frac{\pi}{\sin(\lambda\pi)}\big(\mathcal{A}(f,g)(x) - \mathcal{HZ}_\lambda(f,g)(x)\big) \leq R.H.S,$$

with

$$L.H.S = \int_0^1 \left(\frac{-|2\lambda-1|}{2} + 2(1-r_\lambda)\sqrt{t(1-t)}\right)\frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t(1-t)}\,dt$$

and

$$R.H.S = \int_0^1 \left(\frac{|2\lambda-1|}{2} + 2r_\lambda\sqrt{t(1-t)}\right)\frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t(1-t)}\,dt.$$

It is not hard to see that

$$L.H.S = -\frac{|2\lambda-1|}{2}\mathcal{J}(f,g)(x) + 2(1-r_\lambda)\int_0^1 \frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{\sqrt{t(1-t)}}\,dt,$$

which, with (3.3), immediately yields the left inequality of (4.21).

Similarly, we show the right inequality of (4.21). □

Theorem 4.12 *Let $(f,g) \in \mathcal{D}(H)$ and $\lambda \in (0,1)$. Let $x \in H$ be such that $\partial f(x) \neq \emptyset$ and $\partial g(x) \neq \emptyset$. Then the following inequalities*:

$$\begin{aligned}
&\frac{\sin(\lambda\pi)}{2\pi}\big\{\mathcal{J}(f,g)(x) - \lambda(1-\lambda)\big(\mathcal{F}_g\big(x,x^*\big) + \mathcal{F}_f\big(x,z^*\big)\big)\big\} \\
&\quad \leq \mathcal{A}(f,g)(x) - \mathcal{HZ}_\lambda(f,g)(x) \\
&\quad \leq \frac{\sin(\lambda\pi)}{2\pi}\big\{8\lambda(1-\lambda)\pi\big(\mathcal{A}(f,g)(x) - \mathcal{G}(f,g)(x)\big) + (2\lambda-1)^2\mathcal{J}(f,g)(x)\big\}
\end{aligned} \tag{4.22}$$

hold for any $x^* \in \partial f(x)$ and $z^* \in \partial g(x)$.

Proof If in (1.7) we take $a = 1-t$ and $b = t$ then we obtain

$$\frac{1}{2} - \frac{1}{2}\lambda(1-\lambda)(1-2t)\log\frac{1-t}{t} \leq \mathrm{HZ}_\lambda(1-t,t) \leq \frac{1}{2} - 2\lambda(1-\lambda)\big(1-2\sqrt{t(1-t)}\big).$$

Substituting the result into (4.4), we get

$$L.H.S \leq \mathcal{A}(f,g)(x) - \mathcal{HZ}_\lambda(f,g)(x) \leq R.H.S,$$

where

$$L.H.S = \frac{\sin(\lambda\pi)}{\pi}\int_0^1 \left\{\frac{1}{2} - \frac{1}{2}\lambda(1-\lambda)(1-2t)\log\frac{1-t}{t}\right\}\frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t(1-t)}\,dt$$

and

$$R.H.S = \frac{\sin(\lambda\pi)}{\pi}\int_0^1 \left\{\frac{1}{2} - 2\lambda(1-\lambda)\big(1 - 2\sqrt{t(1-t)}\big)\right\}\frac{\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)}{t(1-t)}\,dt.$$

According to (4.1), with the fact that $(1-2t)\log((1-t)/t) \geq 0$ for any $t \in (0,1)$, it is easy to see that

$$L.H.S \geq \frac{\sin(\lambda\pi)}{2\pi}\big\{\mathcal{J}(f,g)(x) - \lambda(1-\lambda)\alpha\big(\mathcal{F}_g(x,x^*) + \mathcal{F}_f(x,z^*)\big)\big\},$$

for any $x^* \in \partial f(x)$ and $z^* \in \partial g(x)$, where we set

$$\alpha := \int_0^1 (1-2t)\log\frac{1-t}{t}\,dt.$$

A simple integration by parts leads to $\alpha = 1$, and so the left inequality of (4.22) is obtained.

Otherwise, it is not hard to see that

$$R.H.S = \frac{\sin(\lambda\pi)}{2\pi}\Big\{\big(1 - 4\lambda(1-\lambda)\big)\mathcal{J}(f,g)(x) + 8\lambda(1-\lambda)\int_0^1 \frac{1}{\sqrt{t(1-t)}}\big(\mathcal{A}_t(f,g)(x) - \mathcal{H}_t(f,g)(x)\big)\,dt\Big\}.$$

This, with (3.3) and the fact that $1 - 4\lambda(1-\lambda) = (2\lambda-1)^2$, immediately yields the right inequality of (4.22). The proof of the theorem is finished. □

Theorem 4.12 implies an interesting result which may be recited in the following.

Corollary 4.13 *Let $(f,g) \in \mathcal{D}(H)$ and let $x \in H$ be as in Theorem 4.12. Then we have*

$$\lim_{\lambda\downarrow 0}\frac{\mathcal{A}(f,g)(x) - \mathcal{HZ}_\lambda(f,g)(x)}{\lambda} = \frac{1}{2}\mathcal{J}(f,g)(x).$$

Proof It is immediate from (4.22). The details are simple and therefore omitted. □

Another interesting result that gives a reverse of (4.20) is given in what follows.

Corollary 4.14 *Let $(f,g) \in \mathcal{D}(H)$ and let $x \in H$ be as in Theorem 4.12. Then the following inequalities hold*:

$$\frac{\sin(\lambda\pi)}{2\pi}\big\{\mathcal{J}(f,g)(x) - 2\lambda(1-\lambda)\big(\mathcal{A}(f,g)(x) - \mathcal{A}(f\diamond g, g\diamond f)(x)\big)\big\} \leq \mathcal{A}(f,g)(x) - \mathcal{HZ}_\lambda(f,g)(x), \tag{4.23}$$

$$\mathcal{J}(f,g)(x) \leq 2\pi\big(\mathcal{A}(f,g)(x) - \mathcal{G}(f,g)(x)\big) + \frac{1}{2}\big(\mathcal{A}(f,g)(x) - \mathcal{A}(f\diamond g, g\diamond f)(x)\big). \tag{4.24}$$

Proof According to (4.12), the left inequality of (4.22) yields (4.23) after a simple manipulation. If we take $\lambda = 1/2$ in (4.23), we immediately obtain (4.24). □

We end this section by stating the following remark which may be of interest.

Remark 4.2

(i) We mention that if $\lambda = 1/2$ then the inequalities in (4.21) are reduced to a trivial equality. Otherwise, if we take $\lambda = 0$ or $\lambda = 1$ in (4.21) and use Corollary 4.13, we obtain (4.20).

However, (4.15), (4.19), and (4.22) are reduced to a trivial equality when $\lambda = 0$ and $\lambda = 1$.

(ii) We leave to the reader the routine task of formulating the operator versions of Theorems 4.11 and 4.12, as well as Corollary 4.14, in a similar manner as previously.

5 Applications for functional entropies

In this section we will apply the previous results for obtaining some refined upper bounds for the relative entropy and Tsallis entropy of two (convex) functionals. The operator versions will be immediately obtained. We first need to recall some basic notions.

For $A, B \in \mathcal{B}^{+*}(H)$ and $\lambda \in (0,1)$, the relative operator entropy $S(A|B)$ and the Tsallis relative operator entropy $T_\lambda(A|B)$ are defined by (see [6, 7], for instance)

$$S(A|B) := A^{1/2}\log\bigl(A^{-1/2}BA^{-1/2}\bigr)A^{1/2},$$
$$T_\lambda(A|B) := \frac{A\sharp_\lambda B - A}{\lambda}.$$

It is well known that $T_\lambda(A|B)$ extends $S(A|B)$ in the sense that the following equality:

$$\lim_{\lambda\downarrow 0} T_\lambda(A|B) = S(A|B)$$

holds for all $A, B \in \mathcal{B}^{+*}(H)$, where the limit is taken in the strong operator topology.

The operators $S(A|B)$ and $T_\lambda(A|B)$ have been extended from the case where the variables are invertible positive operators to the case where the variables are (convex) functionals (see [20], for instance) as

$$\mathcal{E}(f|g) := \int_0^1 \frac{\mathcal{H}_t(f,g) - f}{t}\,dt, \qquad \mathcal{R}_\lambda(f|g) := \frac{\mathcal{G}_\lambda(f,g) - f}{\lambda}. \tag{5.1}$$

The following relationship:

$$\lim_{\lambda\downarrow 0} \mathcal{R}_\lambda(f|g) = \mathcal{E}(f|g), \tag{5.2}$$

has been proved in [20]. Here, the limit is taken for the pointwise topology.

Some inequalities giving lower and upper bounds of $\mathcal{E}(f|g)$ and $\mathcal{R}_\lambda(f|g)$ can be found in [20]. In particular, the following inequalities:

$$\mathcal{R}_\lambda(f|g)(x) \le (g-f)(x) \quad \text{and} \quad \mathcal{E}(f|g)(x) \le (g-f)(x) \tag{5.3}$$

hold for any $x \in \operatorname{dom} f$.

In what follows, applying some results of the previous section, we will give some reverses and refinements of (5.3). The related operator versions will be immediately deduced without any additional tools.

Let us first state the following result which is useful and will be needed in the sequel.

Proposition 5.1 *Let $(f,g) \in \mathcal{D}(H)$ and $\lambda \in (0,1)$. Let $x \in \operatorname{dom} f \cap \operatorname{dom} g$. Then we have*

$$\mathcal{HZ}_\lambda(f,g)(x) - \mathcal{A}(f,g)(x) = \frac{\lambda}{2}\big(\mathcal{R}_\lambda(f|g)(x) + \mathcal{R}_\lambda(g|f)(x)\big). \tag{5.4}$$

In particular, the following equality:

$$\lim_{\lambda\downarrow 0} \frac{\mathcal{HZ}_\lambda(f,g)(x) - \mathcal{A}(f,g)(x)}{\lambda} = \frac{1}{2}\big(\mathcal{E}(f|g)(x) + \mathcal{E}(g|f)(x)\big) \tag{5.5}$$

holds for any $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

Proof If $x \in \operatorname{dom} f$ then (5.1) implies that

$$\mathcal{G}_\lambda(f,g)(x) = f(x) + \lambda\mathcal{R}_\lambda(f|g)(x). \tag{5.6}$$

We infer that if $x \in \operatorname{dom} g$ then

$$\mathcal{G}_{1-\lambda}(f,g)(x) = \mathcal{G}_\lambda(g,f)(x) = g(x) + \lambda\mathcal{R}_\lambda(g|f)(x). \tag{5.7}$$

If $x \in \operatorname{dom} f \cap \operatorname{dom} g$, we can add (5.6) and (5.7). We then obtain (5.4). Equality (5.5) follows from (5.4) when combined with (5.2). The proof is finished. □

Combining Corollary 4.13 and Proposition 5.1, we immediately deduce the following result which contains a relationship between the relative functional entropy and the functional $\mathcal{J}(f,g)$ previously defined.

Corollary 5.2 *The following equality*:

$$\mathcal{E}(f|g)(x) + \mathcal{E}(g|f)(x) = -\mathcal{J}(f,g)(x) \tag{5.8}$$

holds for any $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

By using (4.17), it is easy to see that the operator version of (5.8) is given by

$$S(B|A) = -BA^{-1}S(A|B).$$

Now, we are in the position to state the following result which gives some reverses of (5.3).

Theorem 5.3 *Let $(f,g) \in \mathcal{D}(H)$ and $\lambda \in (0,1)$. Let $x \in H$ be as in Theorem 4.1. Then the following inequalities*:

$$0 \le (g-f)(x) - \mathcal{R}_\lambda(f|g)(x) \le \frac{1-\lambda}{2}\big(\mathcal{F}_g(x,x^*) + \mathcal{F}_f(x,z^*)\big), \tag{5.9}$$

$$0 \le (g-f)(x) - \mathcal{E}(f|g)(x) \le \frac{1}{2}\big(\mathcal{F}_g(x,x^*) + \mathcal{F}_f(x,z^*)\big) \tag{5.10}$$

hold for any $x^* \in \partial f(x)$ *and* $z^* \in \partial g(x)$.

Proof First, for any $x \in \operatorname{dom} f$, it is not hard to see that

$$\begin{aligned} \mathcal{A}_\lambda(f,g)(x) - \mathcal{G}_\lambda(f,g)(x) &= \lambda(g-f)(x) + f(x) - \mathcal{G}_\lambda(f,g)(x) \\ &= \lambda\big((g-f)(x) - \mathcal{R}_\lambda(f|g)(x)\big). \end{aligned} \tag{5.11}$$

Substituting this into (4.7) and then dividing by $\lambda > 0$, we obtain (5.9).

Letting $\lambda \to 0$ in (5.9) and using (5.2), we immediately get (5.10). □

The operator version of Theorem 5.3 is recited in the following.

Corollary 5.4 *Let* $A, B \in \mathcal{B}^{+*}(H)$ *and* $\lambda \in (0,1)$. *Then we have*

$$0 \le B - A - T_\lambda(A|B) \le \frac{1-\lambda}{2}\big(AB^{-1}A + BA^{-1}B - A - B\big),$$

$$0 \le B - A - S(A|B) \le \frac{1}{2}\big(AB^{-1}A + BA^{-1}B - A - B\big).$$

We now state another result which gives a refinement and reverse of (5.3).

Theorem 5.5 *Let* $(f,g) \in \mathcal{D}(H)$ *and* $\lambda, s \in (0,1)$. *The following inequalities*:

$$\begin{aligned} &\frac{\sin(\lambda\pi)}{\lambda\pi}\big(\Lambda_s^\lambda + \Lambda_{1-s}^{1-\lambda}\big)\big(\mathcal{A}_s(f,g)(x) - \mathcal{H}_s(f,g)(x)\big) \\ &\quad \le (g-f)(x) - \mathcal{R}_\lambda(f|g)(x) \le \frac{\sin(\lambda\pi)}{\lambda\pi}\big((1-\lambda)\sharp_\lambda\lambda\big)\mathcal{J}(f,g)(x) \end{aligned} \tag{5.12}$$

and

$$\frac{\ln(s)}{s-1}\big(\mathcal{A}_s(f,g)(x) - \mathcal{H}_s(f,g)(x)\big) \le (g-f)(x) - \mathcal{E}(f|g)(x) \le \mathcal{J}(f,g)(x) \tag{5.13}$$

hold for any $x \in \operatorname{dom} f$.

Proof Substituting (5.11) into the middle part of (4.15) and then dividing by $\lambda > 0$, we obtain the desired inequalities (5.12).

We now prove (5.13). If in all sides of (5.12) we let $\lambda \downarrow 0$ and use (3.10) and (5.2), with the fact that

$$\lim_{\lambda\downarrow 0}\frac{\sin(\lambda\pi)}{\lambda\pi} = 1 \quad \text{and} \quad \lim_{\lambda\downarrow 0}\big((1-\lambda)\sharp_\lambda\lambda\big) = 1,$$

we then get the desired inequalities (5.13). □

The operator version of Theorem 5.5 is recited in the following result.

Corollary 5.6 *Let* $A, B \in \mathcal{B}^{+*}(H)$ *and* $\lambda, s \in (0,1)$. *Then we have*

$$\frac{\sin(\lambda\pi)}{\lambda\pi}\big(\Lambda_s^{\lambda} + \Lambda_{1-s}^{1-\lambda}\big)(A\nabla_s B - A!_s B) \le B - A - T_\lambda(A|B) \le \frac{\sin(\lambda\pi)}{\lambda\pi}\big((1-\lambda)\sharp_\lambda\lambda\big)J(A,B)$$

and

$$\frac{\ln(s)}{s-1}(A\nabla_s B - A!_s B) \le B - A - S(A|B) \le J(A,B),$$

where $J(A,B)$ *is given by* (4.17).

Now we will see other applications for the relative functional entropies. The first result reads as follows.

Theorem 5.7 *Let* $(f,g) \in \mathcal{D}(H)$ *and* $\lambda, s \in (0,1)$. *The following inequalities*:

$$\begin{aligned} &-\frac{2\sin(\lambda\pi)}{\lambda\pi}\big((1-\lambda)\sharp_\lambda\lambda\big)\mathcal{J}(f,g)(x) \\ &\quad\le \mathcal{R}_\lambda(f|g)(x) + \mathcal{R}_\lambda(g|f)(x) \\ &\quad\le -\frac{2\sin(\lambda\pi)}{\lambda\pi}\big(\Lambda_s^{\lambda} + \Lambda_{1-s}^{1-\lambda}\big)\big(\mathcal{A}(f,g)(x) - \Theta_s(f,g)(x)\big) \end{aligned} \tag{5.14}$$

hold for any $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

Proof From (4.19) we can write, for any $x \in \operatorname{dom} f \cap \operatorname{dom} g$,

$$\begin{aligned} &-\frac{\sin(\lambda\pi)}{\pi}\big((1-\lambda)\sharp_\lambda\lambda\big)\mathcal{J}(f,g)(x) \\ &\quad\le \mathcal{HZ}_\lambda(f,g)(x) - \mathcal{A}(f,g)(x) \\ &\quad\le -\frac{\sin(\lambda\pi)}{\pi}\big(\Lambda_s^{\lambda} + \Lambda_{1-s}^{1-\lambda}\big)\big(\mathcal{A}(f,g)(x) - \Theta_s(f,g)(x)\big). \end{aligned} \tag{5.15}$$

Substituting (5.4) in the middle part of these inequalities and then dividing by $\lambda/2$, we get (5.14). □

Corollary 5.8 *Let* $(f,g) \in \mathcal{D}(H)$ *and* $s \in (0,1)$. *The following inequality*:

$$\mathcal{E}(f|g)(x) + \mathcal{E}(g|f)(x) \le \frac{2\ln(s)}{1-s}\big(\mathcal{A}(f,g)(x) - \Theta_s(f,g)(x)\big) \tag{5.16}$$

holds for any $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

Proof Letting $\lambda \to 0$ in (5.14) with the help of (5.2) and (3.10), we obtain (5.16). □

The operator versions of Theorem 5.7 and Corollary 5.8 are given in the following result.

Corollary 5.9 *Let $A, B \in \mathcal{B}^{+*}(H)$ and $\lambda, s \in (0,1)$. Then we have*

$$-\frac{2\sin(\lambda\pi)}{\lambda\pi}\big((1-\lambda)\sharp_\lambda\lambda\big)J(A,B) \le T_\lambda(A|B) + T_\lambda(B|A)$$
$$\le -\frac{2\sin(\lambda\pi)}{\lambda\pi}\big(\Lambda_s^\lambda + \Lambda_{1-s}^{1-\lambda}\big)(A\nabla B - A\flat_s B)$$

and

$$S(A|B) + S(B|A) \le \frac{2\ln(s)}{1-s}(A\nabla B - A\flat_s B),$$

where we set

$$A\flat_s B := \frac{A!_s B + A!_{1-s}B}{2}.$$

Another main result is given in the following.

Theorem 5.10 *Let $(f,g) \in \mathcal{D}(H)$ and $\lambda \in (0,1)$. The following inequalities*:

$$\begin{aligned} &-4r_\lambda\pi\big(\mathcal{A}(f,g)(x) - \mathcal{G}(f,g)(x)\big) - |2\lambda-1|\mathcal{J}(f,g)(x) \\ &\quad\le \frac{\lambda\pi}{\sin(\lambda\pi)}\big(\mathcal{R}_\lambda(f|g)(x) + \mathcal{R}_\lambda(g|f)(x)\big) \\ &\quad\le -4(1-r_\lambda)\pi\big(\mathcal{A}(f,g)(x) - \mathcal{G}(f,g)(x)\big) + |2\lambda-1|\mathcal{J}(f,g)(x) \end{aligned} \tag{5.17}$$

hold for any $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

Proof Here, we use (4.21). The details are similar to those of the proof of the preceding results. □

The operator version of Theorem 5.10 is given below.

Corollary 5.11 *Let $A, B \in \mathcal{B}^{+*}(H)$ and $\lambda \in (0,1)$. Then we have*

$$-4r_\lambda\pi(A\nabla B - A\sharp B) - |2\lambda-1|J(A,B) \le \frac{\lambda\pi}{\sin(\lambda\pi)}\big(T_\lambda(A|B) + T_\lambda(B|A)\big)$$
$$\le -4(1-r_\lambda)\pi(A\nabla B - A\sharp B) + |2\lambda-1|J(A,B).$$

Finally, our last result of applications is stated in the following.

Theorem 5.12 *Let $(f,g) \in \mathcal{D}(H)$ and $\lambda \in (0,1)$. Let $x \in H$ be as in Theorem 4.1. Then the following inequalities*:

$$\begin{aligned} &-\frac{\sin(\lambda\pi)}{\lambda\pi}\big\{8\lambda(1-\lambda)\pi\big(\mathcal{A}(f,g)(x) - \mathcal{G}(f,g)(x)\big) + (2\lambda-1)^2\mathcal{J}(f,g)(x)\big\} \\ &\quad\le \mathcal{R}_\lambda(f|g)(x) + \mathcal{R}_\lambda(g|f)(x) \\ &\quad\le -\frac{\sin(\lambda\pi)}{\lambda\pi}\big\{\mathcal{J}(f,g)(x) - \lambda(1-\lambda)\big(\mathcal{F}_g(x,x^*) + \mathcal{F}_f(x,z^*)\big)\big\} \end{aligned} \tag{5.18}$$

hold for any $x^* \in \partial f(x)$ and $z^* \in \partial g(x)$.

Proof It is also similar to the previous proofs, by using (4.22). We leave to the reader the routine task of providing the details of this proof. □

The operator version of Theorem 5.12 can be immediately deduced and reads as follows.

Corollary 5.13 *Let* $A, B \in \mathcal{B}^{+*}(H)$ *and* $\lambda \in (0,1)$. *Then we have*

$$\begin{aligned} &-\frac{\sin(\lambda\pi)}{\lambda\pi}\bigl\{8\lambda(1-\lambda)\pi(A\nabla B - A\sharp B) + (2\lambda-1)^2 J(A,B)\bigr\} \\ &\leq T_\lambda(A|B) + T_\lambda(B|A) \\ &\leq -\frac{\sin(\lambda\pi)}{\lambda\pi}\bigl\{J(A,B) - \lambda(1-\lambda)\bigl(AB^{-1}A + BA^{-1}B - A - B\bigr)\bigr\}. \end{aligned}$$

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