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Approximate Benson efficient solutions for set-valued equilibrium problems

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Abstract

In locally convex Hausdorff topological vector spaces, the approximate Benson efficient solution is proposed for set-valued equilibrium problems and its relationship to the Benson efficient solution is discussed. Under the assumption of generalized convexity, by using a separation theorem for convex sets, Kuhn–Tucker-type and Lagrange-type optimality conditions for set-valued equilibrium problems are established, respectively.

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1 Introduction

The vector equilibrium problem is a broad problem in many practical fields. It covers many typical mathematical problems, for instance, vector optimization, variational inequality, vector Nash equilibrium, vector complementarity, and so on. It is widely used in investment decision-making, quantitative economy, optimal control, and engineering technology. Because of the universality and unity of the problems involved and the profundity of solving them, vector equilibrium has become a hot issue in the field of nonlinear analysis and operational research [1–6]. In Banach spaces, Feng et al. [1] established Kuhn–Tucker-like conditions for weakly efficient solutions of vector equilibrium problems with constraints by using the Gerstewitz's functional, and obtained sufficient conditions of weakly efficient solutions under the assumption of generalized invexity. You et al. [2] established Lagrangian-type sufficient optimality conditions for general constrained vector optimization problems by applying Gerstewitz's function, and, under suitable restriction qualifications, by virtue of Clarke subdifferentials, they obtained Karush–Kuhn–Tucker necessary conditions. Luu et al. [3] derived necessary conditions for efficient solutions to vector equilibrium problems with equality and inequality constraints. Under the assumption of cone-convexity, Gong [4] obtained necessary and sufficient optimality conditions for several efficient solutions to constrained vector equilibrium problems. By using asymptotic analysis, Iusem et al. [5] studied vector equilibrium problems and noncoercive pseudomonotone equilibrium problems.

In recent years, approximate solutions of the set-valued optimization problem have attracted people's attention [6–8]. In real ordered linear spaces, Zhou et al. [6, 7] studied several kinds of approximate properly efficient solutions of set-valued optimization problems, including ϵ-weakly, ϵ-global, ϵ-Benson, ϵ-super properly efficient solutions, and derived the relationship between ϵ-Benson properly efficient solutions and ϵ-global properly efficient solutions. Dhingra et al. [8] established existence and scalarization using a generalized Gerstewitz's function for approximate solutions.

On the other hand, convexity is vital for studying the vector equilibrium problem. Sach [10] proposed a new type of convexity named ic-cone-convexness in 2005. Yang et al. [9] introduced another type of convexity called near cone-subconvexlikeness in 2001, which is a generalization of cone-subconvexlikeness and cone-convexness. Xu et al. [11] certified that near cone-subconvexlikeness is also an extension of ic-coneconvexness in 2011. So far, near cone-subconvexlikeness is regarded as the most universal convexity property.

The above discussions motivate the aim of this paper—discussing the relationship between approximate Benson efficient solutions and Benson efficient solutions, and establishing Lagrange-type and Kuhn–Tucker-type optimality conditions for approximate Benson efficient solutions.

2 Preliminaries

Throughout this paper, let X be a real topological vector space; let Y and Z be real locally convex Hausdorff topological vector spaces, respectively, let $S \subset Y$ and $K \subset Z$ be pointed closed convex cones with nonempty interiors. Let X_0 be a nonempty subset of X, and $\Upsilon : X_0 \times X_0 \to 2^Y$ and $G : X_0 \to 2^Z$ be maps. Furthermore, 0_Y denotes the zero element in Y; Y^* and Z^* denote the topological dual space of Y and Z, respectively; S^* and S^{*i} denote the positive dual cone and strictly positive dual cone of S, respectively, that is,

$$S^* = \{\phi \in Y^* : \phi(s) \geq 0, \forall s \in S\},$$
$$S^{*i} = \{\phi \in Y^* : \phi(s) > 0, \forall s \in S \backslash \{0_Y\}\}.$$

Definition 2.1 ([12]) The map $F : X_0 \to 2^Y$ is called generalized S-subconvexlike on X_0 if and only if there exists $\theta \in \operatorname{int} S$ such that, for all $x_1, x_2 \in X_0$, $\lambda \in [0,1]$, and $\alpha > 0$, there exist $x_3 \in X_0$ and $\rho > 0$ such that

$$\alpha\theta + \lambda F(x_1) + (1-\lambda)F(x_2) \subset \rho F(x_3) + S.$$

Definition 2.2 ([9]) The map $F : X_0 \to 2^Y$ is called nearly S-subconvexlike on X_0 iff $\operatorname{clcone}(F(X_0) + S)$ is convex.

Lemma 2.1 ([13]) *Let C and D be two cones in Y, $C \cap D = \{0_Y\}$. If D is closed and C has a compact base, then there exists a pointed convex cone M such that $C \backslash \{0_Y\} \subset \operatorname{int} M$ and $M \cap D = \{0_Y\}$.*

Lemma 2.2 ([14]) *If $f \in S^* \backslash \{0_{Y^*}\}$, $s \in \operatorname{int} S$, then $f(s) > 0$.*

Let $\Omega \subset X_0$. Consider the following constrained set-valued equilibrium problem (for short, Υ-SEPC): find $\hat{x} \in \Omega$ such that

$$\Upsilon(\hat{x}, x) \cap (-H) = \emptyset, \quad \forall x \in \Omega,$$

where $H \cup \{0\}$ is a convex cone in Y.

Definition 2.3 A vector $\hat{x} \in \Omega$ is called a Benson efficient solution to (Υ-SEPC) if

$$\operatorname{clcone}\bigl(\Upsilon(\hat{x}, \Omega) + S\bigr) \cap (-S) = \{0_Y\}.$$

The set of all Benson efficient solutions to (Υ-SEPC) is denoted by $X_{\mathrm{Ben}}(\Upsilon, \Omega)$.

Definition 2.4 Let $\varepsilon \in S$. A vector $\hat{x} \in \Omega$ is said to be an ε-Benson efficient solution to (Υ-SEPC) if

$$\operatorname{clcone}\bigl(\Upsilon(\hat{x}, \Omega) + \varepsilon + S\bigr) \cap (-S) = \{0_Y\}.$$

The set of all ε-Benson efficient solutions to (Υ-SEPC) is denoted by ε-$X_{\mathrm{Ben}}(\Upsilon, \Omega)$.

In what follows, we discuss the relationship between Benson and ε-Benson efficient solution sets to constrained set-valued equilibrium problems.

Proposition 2.1 *For any $\varepsilon \in S$, we have $X_{\mathrm{Ben}}(\Upsilon, \Omega) \subset \varepsilon$-$X_{\mathrm{Ben}}(\Upsilon, \Omega)$.*

Proof Let $x \in X_{\mathrm{Ben}}(\Upsilon, \Omega)$, then

$$\operatorname{clcone}\bigl(\Upsilon(x, \Omega) + S\bigr) \cap (-S) = \{0_Y\}. \tag{2.1}$$

Since $\varepsilon \in S$ and S is a cone, we have $\varepsilon + S \subset S + S \subset S$. Then

$$\Upsilon(x, \Omega) + \varepsilon + S \subset \Upsilon(x, \Omega) + S.$$

Hence, $0_Y \in \operatorname{clcone}(\Upsilon(x, \Omega) + \varepsilon + S) \subset \operatorname{clcone}(\Upsilon(x, \Omega) + S)$. Together with (2.1), this yields $\operatorname{clcone}(\Upsilon(x, \Omega) + \varepsilon + S) \cap (-S) = \{0_Y\}$. Then, $x \in \varepsilon$-$X_{\mathrm{Ben}}(\Upsilon, \Omega)$. Hence we obtain $X_{\mathrm{Ben}}(\Upsilon, \Omega) \subset \varepsilon$-$X_{\mathrm{Ben}}(\Upsilon, \Omega)$. □

Remark 2.1 From Proposition 2.1, it follows that

$$X_{\mathrm{Ben}}(\Upsilon, \Omega) \subset \bigcap_{\varepsilon \in S \setminus \{0_Y\}} \varepsilon\text{-}X_{\mathrm{Ben}}(\Upsilon, \Omega).$$

However, the reversed inclusion is not necessarily true. The following example illustrates the case.

Example 2.1 Let $X = Y = \mathbb{R}^2, X_0 = \mathbb{R}^2, S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}, \Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq x_1^2\}$. Consider the set-valued map $\Upsilon : X_0 \times X_0 \to 2^Y$ defined by $\Upsilon(x, u) = u - x$. If

$\hat{x} = (0,0)$, then $\Upsilon(\hat{x}, \Omega) = \Omega$. Then for any $\varepsilon \in S\backslash\{0_Y\}$, one has

$$\operatorname{clcone}\big(\Upsilon(\hat{x},\Omega) + S\big) \cap (-S) = \big\{(x_1, 0) : x_1 \leq 0\big\} \neq \{0_Y\} \quad \Longrightarrow \quad \hat{x} \notin X_{\mathrm{Ben}}(\Upsilon, \Omega),$$

however,

$$\operatorname{clcone}\big(\Upsilon(\hat{x},\Omega) + \varepsilon + S\big) \cap (-S) = \{0_Y\} \quad \Longrightarrow \quad \hat{x} \in \varepsilon\text{-}X_{\mathrm{Ben}}(\Upsilon, \Omega).$$

Hence,

$$\bigcap_{\varepsilon \in S\backslash\{0_Y\}} \varepsilon\text{-}X_{\mathrm{Ben}}(\Upsilon, \Omega) \not\subset X_{\mathrm{Ben}}(\Upsilon, \Omega).$$

Proposition 2.2 *For any $\varepsilon_1, \varepsilon_2 \in S$, if $\varepsilon_2 - \varepsilon_1 \in S$, then*

$$\varepsilon_1\text{-}X_{\mathrm{Ben}}(\Upsilon, \Omega) \subset \varepsilon_2\text{-}X_{\mathrm{Ben}}(\Upsilon, \Omega).$$

Proof Similar to the proofs of Proposition 2.1 and [16], Proposition 3.2. □

In the above, $\Omega \subset X_0$ is the normal constraint set in the constrained set-valued equilibrium problem. In the following, we give a specific constraint set, that is, $E = \{x \in X_0 : G(x) \cap (-K) \neq \emptyset\}$.

Let $F : X_0 \to 2^Y$. Consider the following set-valued optimization problem:

$$\begin{aligned} &(\mathrm{SOP}) \quad \min F(x), \\ &\qquad\quad \text{s.t. } x \in E = \big\{x \in X_0 : G(x) \cap (-K) \neq \emptyset\big\}. \end{aligned}$$

Definition 2.5 ([15]) A vector $\hat{x} \in E$ is called a Benson efficient solution to (SOP) if there exists $\hat{y} \in F(\hat{x})$ such that

$$\operatorname{clcone}\big(F(E) - \hat{y} + S\big) \cap (-S) = \{0_Y\}.$$

In this case, $(\hat{x}, \hat{y})$ is called a Benson efficient pair to (SOP).

3 Kuhn–Tucker-type optimality conditions

In this part, under the assumption of near cone-subconvexlikeness, we present Kuhn–Tucker-type optimality conditions for ε-Benson efficient solutions to constrained set-valued equilibrium problems.

If $\emptyset \neq Q \subset Y$, $\emptyset \neq W \subset Y$, $\psi \in Y^*$, then

$$\psi(W) \geq \psi(Q) \quad \text{implies} \quad \psi(w) \geq \psi(q), \quad \forall w \in W, q \in Q.$$

Definition 3.1 ([16]) Let $\hat{x} \in X_0$ and define an ordered pair map $\varphi : X_0 \to 2^{Y \times Z}$ as follows:

$$\varphi(x) = \big(\Upsilon(\hat{x}, x) + \varepsilon, G(x)\big), \quad \forall x \in X_0.$$

Remark 3.1 ([16]) Note that φ is nearly $S\times K$-subconvexlike on X_0 iff $\operatorname{clcone}(\varphi(X_0)+S\times K)$ is convex, where $\varphi(X_0)=\bigcup_{x\in X_0}\varphi(x)=\bigcup_{x\in X_0}(\Upsilon(\hat{x},x)+\varepsilon,G(x))$.

Theorem 3.1 *Assume that $\hat{x}\in E$, S has a compact base, φ is nearly $S\times K$-subconvexlike on X_0, and that there exists $x'\in X_0$ such that $G(x')\cap(-\operatorname{int}K)\neq\emptyset$. If $\hat{x}$ is an ε-Benson efficient solution to* (Υ-SEPC), *then there exist $\gamma^*\in S^{*i}$, $\omega^*\in K^*$ such that*

$$\gamma^*(y)+\gamma^*(\varepsilon)+\omega^*(z)\geq 0,\quad \forall x\in X_0, y\in\Upsilon(\hat{x},x), z\in G(x).$$

Proof Since $\hat{x}$ is an ε-Benson efficient solution to (Υ-SEPC), we have

$$\operatorname{clcone}\bigl(\Upsilon(\hat{x},E)+\varepsilon+S\bigr)\cap(-S)=\{0_Y\}.$$

Since $\operatorname{clcone}(\Upsilon(\hat{x},E)+\varepsilon+S)$ is a closed cone, and S has a compact base, from Lemma 2.1 we deduce that there exists a pointed convex cone P such that $-S\backslash\{0_Y\}\subset-\operatorname{int}P$ and

$$\operatorname{clcone}\bigl(\Upsilon(\hat{x},E)+\varepsilon+S\bigr)\cap(-P)=\{0_Y\}.$$

From $0_Y\notin-\operatorname{int}P$, we have $\operatorname{int}P\subset P\backslash\{0_Y\}$, hence

$$\operatorname{clcone}\bigl(\Upsilon(\hat{x},E)+\varepsilon+S\bigr)\cap(-\operatorname{int}P)=\emptyset. \tag{3.1}$$

Next, we prove

$$\operatorname{clcone}\bigl(\varphi(X_0)+S\times K\bigr)\cap\bigl(-(\operatorname{int}P\times\operatorname{int}K)\bigr)=\emptyset,$$

where $\varphi(X_0)=\bigcup_{x\in X_0}\varphi(x)=\bigcup_{x\in X_0}(\Upsilon(\hat{x},x)+\varepsilon,G(x))$.

If not, then there exists $(\tilde{y},\tilde{z})\in Y\times Z$ such that

$$(\tilde{y},\tilde{z})\in\operatorname{clcone}\bigl(\varphi(X_0)+S\times K\bigr)\cap\bigl(-(\operatorname{int}P\times\operatorname{int}K)\bigr).$$

Thus, $\tilde{y}\in-\operatorname{int}P$, $\tilde{z}\in-\operatorname{int}K$, and there exist $\lambda_n>0$, $x_n\in X_0$, $(y_n,z_n)\in(\Upsilon(\hat{x},x_n),G(x_n))$ and $(s_n,k_n)\in S\times K$ such that

$$\tilde{y}=\lim_{n\to\infty}\lambda_n(y_n+\varepsilon+s_n),$$

and

$$\tilde{z}=\lim_{n\to\infty}\lambda_n(z_n+k_n).$$

Since $-\operatorname{int}K$ is open and K is a cone, we have $z_n+k_n\in-K$, and from $k_n\in K$, we get $z_n\in-K$. Thus, $z_n\in G(x_n)\cap(-K)$, therefore, $x_n\in E$, combining with $\tilde{y}\in-\operatorname{int}P$, we obtain $\tilde{y}\in\operatorname{clcone}(\Upsilon(\hat{x},E)+\varepsilon+S)\cap(-\operatorname{int}P)$, which contradicts (3.1). Hence $\operatorname{clcone}(\varphi(X_0)+S\times K)\cap(-(\operatorname{int}P\times\operatorname{int}K))=\emptyset$. From Remark 3.1, we know $\operatorname{clcone}(\varphi(X_0)+S\times K)$ is convex. By the separation theorem for convex sets, there exists $(\gamma^*,\omega^*)\in Y^*\times Z^*\backslash\{(0_{Y^*},0_{Z^*})\}$ such that

$$\bigl(\gamma^*,\omega^*\bigr)\bigl(\operatorname{clcone}\bigl(\varphi(X_0)+S\times K\bigr)\bigr)\geq\gamma^*(-\operatorname{int}P)+\omega^*(-\operatorname{int}K). \tag{3.2}$$

Since $\operatorname{clcone}(\varphi(X_0)+S\times K)$ is a cone and on which (γ^*,ω^*) has a lower bound, we conclude

$$(\gamma^*,\omega^*)\big(\operatorname{clcone}\big(\varphi(X_0)+S\times K\big)\big)\geq 0. \tag{3.3}$$

It follows from $(0_Y,0_Z)\in S\times K$ and (3.3) that $(\gamma^*,\omega^*)(\varphi(X_0))\geq 0$. In other words,

$$\gamma^*(y)+\gamma^*(\varepsilon)+\omega^*(z)\geq 0,\quad \forall x\in X_0, y\in \Upsilon(\hat{x},x), z\in G(x). \tag{3.4}$$

In view of $(0_Y,0_Z)\in \operatorname{clcone}(\varphi(X_0)+S\times K)$ and (3.2), one has

$$\gamma^*(-\operatorname{int}P)+\omega^*(-\operatorname{int}K)\leq 0. \tag{3.5}$$

From (3.3), we conclude that for any $x\in X_0$, $y\in \Upsilon(\hat{x},x)$, $z\in G(x)$, $\beta_1,\beta_2\geq 0$, $s\in S$ and $k\in K$, one has

$$\gamma^*(y+\varepsilon+\beta_1 s)+\omega^*(z+\beta_2 k)\geq 0. \tag{3.6}$$

(i) Firstly, we prove that $\omega^*\in K^*$, i.e., $\omega^*(k)\geq 0,\forall k\in K$.

If not, then there exists $k_0\in K$ such that $\omega^*(k_0)<0$. When β_2 is large enough, there exist $x_2\in X_0$, $y_2\in\Upsilon(\hat{x},x_2)$, $z_2\in G(x_2)$, $\beta_1'\geq 0$, and $s_2\in S$ such that

$$\omega^*(\beta_2 k_0)=\beta_2\omega^*(k_0)<-\gamma^*\big(y_2+\varepsilon+\beta_1' s_2\big)-\omega^*(z_2),$$

which contradicts (3.6). Hence we obtain $\omega^*(k)\geq 0,\forall k\in K$.

(ii) Next, we prove that $\gamma^*\neq 0_{Y^*}$.

If not, then $\gamma^*=0_{Y^*}$. Since $(\gamma^*,\omega^*)\neq(0_{Y^*},0_{Z^*})$, we have $\omega^*\neq 0_{Z^*}$, and then $\omega^*\in K^*\backslash\{0_{Z^*}\}$. From $\gamma^*=0_{Y^*}$ and (3.4), we get

$$\omega^*\big(G(x)\big)\geq 0,\quad \forall x\in X_0. \tag{3.7}$$

On the other hand, from $G(x')\cap(-\operatorname{int}K)\neq\emptyset$, there exists $z'\in G(x')$ such that $z'\in-\operatorname{int}K$, hence combining with Lemma 2.2, we have $\omega^*(z')<0$, which contradicts (3.7). Hence we get $\gamma^*\neq 0_{Y^*}$.

(iii) Finally, we prove that $\gamma^*\in S^{*i}$.

From (3.5) we derive $\gamma^*(\operatorname{int}P)\geq\omega^*(-\operatorname{int}K)$. Since $\operatorname{int}P$ is a cone on which γ^* has a lower bound, we conclude that $\gamma^*(\operatorname{int}P)\geq 0$. Since P is a convex cone, we have $P\subset \operatorname{cl}P=\operatorname{cl}(\operatorname{int}P)$. Then, for any $p\in P$, there exists a net $\{p_\alpha\}\subset\operatorname{int}P$ such that $p=\lim p_\alpha$. Thus $\gamma^*(p)=\gamma^*(\lim p_\alpha)=\lim\gamma^*(p_\alpha)\geq 0$, which implies $\gamma^*(P)\geq 0$, therefore $\gamma^*\in P^*$. It follows from (ii) that $\gamma^*\in P^*\backslash\{0_{Y^*}\}$. From Lemma 2.2, we have $\gamma^*(\operatorname{int}P)>0$. By $S\backslash\{0_Y\}\subset\operatorname{int}P$, we get $\gamma^*(S\backslash\{0_Y\})>0$. Thus, $\gamma^*\in S^{*i}$. □

Corollary 3.1 *Assume that $\hat{x}\in E$, $0\in\Upsilon(\hat{x},\hat{x})$, S has a compact base, $(\Upsilon(\hat{x},\cdot),G(\cdot))$ is nearly $S\times K$-subconvexlike on X_0, and that there exists $x'\in X_0$ such that $G(x')\cap(-\operatorname{int}K)\neq\emptyset$. If $\hat{x}$ is a Benson efficient solution to (Υ-SEPC), then there exist $\gamma^*\in S^{*i}$, $\omega^*\in K^*$ such that* $\min\omega^*(G(\hat{x}))=0$ *and*

$$\min\big\{\gamma^*(y)+\omega^*(z):x\in X_0,y\in\Upsilon(\hat{x},x),z\in G(x)\big\}=0.$$

Proof In Theorem 3.1, letting $\varepsilon = 0$, we see that there exist $\gamma^* \in S^{*i}$, $\omega^* \in K^*$ such that

$$\gamma^*(y) + \omega^*(z) \geq 0, \quad \forall x \in X_0, y \in \Upsilon(\hat{x}, x), z \in G(x). \tag{3.8}$$

According to $\hat{x} \in E$, we obtain $G(\hat{x}) \cap (-K) \neq \emptyset$. Thus there exists $\hat{z} \in G(\hat{x})$ such that $\hat{z} \in -K$, and since $\omega^* \in K^*$ we know

$$\omega^*(\hat{z}) \leq 0. \tag{3.9}$$

In equation (3.8), letting $x = \hat{x}$, from $0 \in \Upsilon(\hat{x}, \hat{x})$ we get

$$\omega^*(z) \geq 0, \quad \forall z \in G(\hat{x}). \tag{3.10}$$

Since $\hat{z} \in G(\hat{x})$, we know $\omega^*(\hat{z}) \geq 0$, which, together with (3.9), implies $\omega^*(\hat{z}) = 0$. Then

$$0 \in \omega^*\bigl(G(\hat{x})\bigr). \tag{3.11}$$

It follows from (3.10) that $\min \omega^*(G(\hat{x})) = 0$. It follows from $0 \in \Upsilon(\hat{x}, \hat{x})$ and (3.11) that $0 \in \gamma^*(\Upsilon(\hat{x}, \hat{x})) + \omega^*(G(\hat{x}))$. From (3.8), we derive

$$\min\bigl\{\gamma^*(y) + \omega^*(z) : x \in X_0, y \in \Upsilon(\hat{x}, x), z \in G(x)\bigr\} = 0. \qquad \square$$

Theorem 3.2 *Suppose that*

(i) $\hat{x} \in E$;

(ii) *there exist* $\gamma^* \in S^{*i}$, $\omega^* \in K^*$ *such that*

$$\gamma^*(y) + \gamma^*(\varepsilon) + \omega^*(z) \geq 0, \quad \forall x \in E, y \in \Upsilon(\hat{x}, x), z \in G(x).$$

Then $\hat{x}$ *is an* ε*-Benson efficient solution to* (Υ-SEPC).

Proof Let $s \in \operatorname{clcone}(\Upsilon(\hat{x}, E) + \varepsilon + S) \cap (-S)$, then there exist $\lambda_n > 0$, $y_n \in \Upsilon(\hat{x}, E)$, and $s_n \in S$ such that

$$s = \lim_{n\to\infty} \lambda_n(y_n + \varepsilon + s_n).$$

Thus,

$$\gamma^*(s) = \lim_{n\to\infty} \lambda_n\bigl(\gamma^*(y_n) + \gamma^*(\varepsilon) + \gamma^*(s_n)\bigr). \tag{3.12}$$

It follows from (ii) that

$$\gamma^*\bigl(\Upsilon(\hat{x}, x)\bigr) + \gamma^*(\varepsilon) + \omega^*\bigl(G(x)\bigr) \geq 0, \quad \forall x \in E. \tag{3.13}$$

By $x \in E$, there exists $z_x \in G(x)$ such that $z_x \in -K$, combining with $\omega^* \in K^*$, we know $\omega^*(z_x) \leq 0$, hence $\omega^*(G(x)) \cap (-\infty, 0] \neq \emptyset$. Then, from (3.13) we have

$$\gamma^*\bigl(\Upsilon(\hat{x}, x)\bigr) + \gamma^*(\varepsilon) \geq 0, \quad \forall x \in E.$$

Thus $\gamma^*(y_n) + \gamma^*(\varepsilon) \geq 0$, and since $\gamma^* \in S^{*i}$, we derive $\gamma^*(s_n) \geq 0$. Hence, by (3.12) we obtain $\gamma^*(s) \geq 0$. On the other hand, from $s \in -S$ we know $\gamma^*(s) \leq 0$. Thus $\gamma^*(s) = 0$, combining with $\gamma^* \in S^{*i}$, we know $s = 0_Y$. Then,

$$\operatorname{clcone}\bigl(\Upsilon(\hat{x}, E) + \varepsilon + S\bigr) \cap (-S) = \{0_Y\}.$$

Thus, $\hat{x}$ is an ε-Benson efficient solution to (Υ-SEPC). □

Corollary 3.2 *Suppose that*

(i) $\hat{x} \in E$;

(ii) *there exist* $\gamma^* \in S^{*i}$, $\omega^* \in K^*$ *such that*

$$\gamma^*(y) + \omega^*(z) \geq 0, \quad \forall x \in E, y \in \Upsilon(\hat{x}, x), z \in G(x).$$

Then $\hat{x}$ *is a Benson efficient solution to* (Υ-SEPC).

Proof In Theorem 3.2, letting $\varepsilon = 0$, we derive that the conclusion is true. □

Remark 3.2 Let $\Upsilon(y, x) = F(x) - \hat{y}$. Since $\hat{y} \in F(\hat{x})$ and $\hat{x} \in E$, $\Upsilon(\cdot, \cdot)$ depends only on the second variable. Then Theorem 2.3 in [15] is a special case of Corollary 3.2.

Corollary 3.3 *Suppose that* $\hat{x} \in E$, *S has a compact base,* φ *is nearly* $S \times K$*-subconvexlike on* X_0, *and that there exists* $x' \in X_0$ *such that* $G(x') \cap (-\operatorname{int} K) \neq \emptyset$. *Then* $\hat{x}$ *is an* ε*-Benson efficient solution to* (Υ-SEPC) *if and only if there exist* $\gamma^* \in S^{*i}$, $\omega^* \in K^*$ *such that*

$$\gamma^*(y) + \gamma^*(\varepsilon) + \omega^*(z) \geq 0, \quad \forall x \in X_0, y \in \Upsilon(\hat{x}, x), z \in G(x).$$

Proof It follows from Theorems 3.1 and 3.2 that the conclusion is true. □

Corollary 3.4 *Suppose that* $\hat{x} \in E$, $0 \in \Upsilon(\hat{x}, \hat{x})$, *$S$ has a compact base,* $(\Upsilon(\hat{x}, \cdot), G(\cdot))$ *is nearly* $S \times K$*-subconvexlike on* X_0, *and that there exists* $x' \in X_0$ *such that* $G(x') \cap (-\operatorname{int} K) \neq \emptyset$. *Then* $\hat{x}$ *is a Benson efficient solution to* (Υ-SEPC) *if and only if there exist* $\gamma^* \in S^{*i}$, $\omega^* \in K^*$ *such that* $\min \omega^*(G(\hat{x})) = 0$ *and*

$$\min\bigl\{\gamma^*(y) + \omega^*(z) : x \in X_0, y \in \Upsilon(\hat{x}, x), z \in G(x)\bigr\} = 0.$$

Proof By Corollaries 3.1 and 3.2, we can easily see that the conclusions are true. □

Corollary 3.5 *Assume that*

(i) $\hat{x} \in E$;

(ii) *there exist* $\hat{y} \in F(\hat{x})$, $\gamma^* \in S^{*i}$, $\omega^* \in K^*$ *such that*

$$\gamma^*(\hat{y}) = \min\bigl\{\gamma^*(y) + \omega^*(z) : x \in E, y \in F(x), z \in G(x)\bigr\}.$$

Then $(\hat{x}, \hat{y})$ *is a Benson efficient pair to* (SOP).

Proof From (ii), we have

$$\gamma^*\big(F(x) - \hat{y}\big) + \omega^*\big(G(x)\big) \geq 0, \quad \forall x \in E.$$

Letting $\Upsilon(y,x) = F(x) - \hat{y}$, then $\Upsilon(\hat{x},x) = F(x) - \hat{y}$. From $\hat{y} \in F(\hat{x})$ we get $0 \in F(\hat{x}) - \hat{y}$. Then, $0 \in \Upsilon(\hat{x},\hat{x})$ and $F(x) - \hat{y} = \Upsilon(\hat{x},x)$. Thus

$$\gamma^*\big(\Upsilon(\hat{x},x)\big) + \omega^*\big(G(x)\big) \geq 0, \quad \forall x \in E.$$

It follows from Corollary 3.2 that $\hat{x}$ is a Benson efficient solution to (Υ-SEPC). Hence, $(\hat{x},\hat{y})$ is a Benson efficient pair to (SOP). □

Remark 3.3 Comparing with Theorem 2.5 in [15], this corollary does not require $\inf \omega^*(G(\hat{x})) = 0$.

4 Lagrange-type optimality conditions

In this section, we establish Lagrange-type optimality conditions for ε-Benson efficient solutions to unconstrained set-valued equilibrium problems.

Let $L(Z,Y)$ be the space of continuous linear operators from Z to Y, and let

$$L_+(Z,Y) = \big\{T \in L(Z,Y) : T(K) \subset S\big\}.$$

Let $\Theta : X_0 \times X_0 \to 2^Y$. We consider unconstrained set-valued equilibrium problem (in brief, Θ-USEP): find $\hat{x} \in X_0$ such that

$$\Theta(\hat{x},x) \cap (-H) = \emptyset, \quad \forall x \in X_0,$$

where $H \cup \{0\}$ is a convex cone on Y.

Next, we introduce Benson and ε-Benson efficient solutions to unconstrained set-valued equilibrium problems.

Definition 4.1 A vector $\hat{x} \in X_0$ is said to be a Benson efficient solution to (Θ-USEP) if

$$\operatorname{clcone}\big(\Theta(\hat{x},X_0) + S\big) \cap (-S) = \{0_Y\}.$$

Definition 4.2 Let $\varepsilon \in S$. A vector $\hat{x} \in X_0$ is called an ε-Benson efficient solution to (Θ-USEP) if

$$\operatorname{clcone}\big(\Theta(\hat{x},X_0) + \varepsilon + S\big) \cap (-S) = \{0_Y\}.$$

Let $F : X_0 \to 2^Y$, $\hat{T} \in L_+(Z,Y)$. We consider the following unconstrained set-valued optimization problem:

$$(\mathrm{USOP})_{\hat{T}} \quad \min_{x \in X_0} \zeta(x,\hat{T}),$$

where $\zeta(x,\hat{T}) = F(x) + \hat{T}(G(x))$.

Definition 4.3 ([18]) A vector $\hat{x} \in X_0$ is called a Benson efficient solution to $(\text{USOP})_{\hat{\text{T}}}$ if there exists $\hat{y} \in F(\hat{x})$ such that

$$\operatorname{clcone}\bigl(\zeta(X_0, \hat{T}) - \hat{y} + S\bigr) \cap (-S) = \{0_Y\},$$

where $\zeta(X_0, \hat{T}) = \bigcup_{x \in X_0} \zeta(x, \hat{T}) = \bigcup_{x \in X_0}(F(x) + \hat{T}(G(x)))$. In this case, $(\hat{x}, \hat{y})$ is called a Benson efficient pair to $(\text{USOP})_{\hat{\text{T}}}$.

Definition 4.4 Let $\varepsilon \in S$. A vector $\hat{x} \in X_0$ is called an ε-Benson efficient solution to $(\text{USOP})_{\hat{\text{T}}}$ if there exists $\hat{y} \in F(\hat{x})$ such that

$$\operatorname{clcone}\bigl(\zeta(X_0, \hat{T}) - \hat{y} + \varepsilon + S\bigr) \cap (-S) = \{0_Y\},$$

where $\zeta(X_0, \hat{T}) = \bigcup_{x \in X_0} \zeta(x, \hat{T}) = \bigcup_{x \in X_0}(F(x) + \hat{T}(G(x)))$. In this case, $(\hat{x}, \hat{y})$ is called an ε-Benson efficient pair to $(\text{USOP})_{\hat{\text{T}}}$.

Theorem 4.1 *Assume that $\hat{x} \in E$, $0 \in \Upsilon(\hat{x}, \hat{x})$, S has a compact base, φ is nearly $S \times K$-subconvexlike on X_0, and that there exists $x' \in X_0$ such that $G(x') \cap (-\operatorname{int} K) \neq \emptyset$. If $\hat{x}$ is an ε-Benson efficient solution to* (Υ-SEPC), *then there exists $\hat{T} \in L_+(Z, Y)$ such that $\hat{x}$ is an ε-Benson efficient solution to* (Ψ-USEP) *and*

$$-\hat{T}\bigl(G(\hat{x}) \cap (-K)\bigr) \subset S \backslash \bigl(\varepsilon + \bigl(S \backslash \{0_Y\}\bigr)\bigr),$$

where $\Psi : X_0 \times X_0 \to 2^Y$ is defined as $\Psi(y, x) = \Upsilon(y, x) + \hat{T}(G(x))$.

Proof It follows from Theorem 3.1 that there exist $\gamma^* \in S^{*i}$, $\omega^* \in K^*$ such that

$$\gamma^*(y) + \gamma^*(\varepsilon) + \omega^*(z) \geq 0, \quad \forall x \in X_0, y \in \Upsilon(\hat{x}, x), z \in G(x). \tag{4.1}$$

From $\gamma^* \in S^{*i}$, we obtain that there exists $s_0 \in S \backslash \{0_Y\} \subset S$ such that $\gamma^*(s_0) = 1$. Define the operator $\hat{T} : Z \to Y$ as follows:

$$\hat{T}(z) = \omega^*(z) s_0, \quad \forall z \in Z.$$

Thus, $\hat{T}(K) = \omega^*(K) s_0 \subset S$, which implies $\hat{T} \in L_+(Z, Y)$. In equation (4.1), letting $x = \hat{x}$, from $0 \in \Upsilon(\hat{x}, \hat{x})$ we obtain

$$\gamma^*(\varepsilon) + \omega^*(z) \geq 0, \quad \forall z \in G(\hat{x}) \cap (-K). \tag{4.2}$$

From $z \in -K$, we know $-\hat{T}(z) = -\omega^*(z) s_0 \in S$. Then,

$$-\hat{T}\bigl(G(\hat{x}) \cap (-K)\bigr) = -\omega^*\bigl(G(\hat{x}) \cap (-K)\bigr) s_0 \subset S. \tag{4.3}$$

In the following, we prove $-\hat{T}(G(\hat{x}) \cap (-K)) \cap (\varepsilon + (S \backslash \{0_Y\})) = \emptyset$.

If not, then there exists $\hat{z} \in G(\hat{x}) \cap (-K)$ such that

$$-\hat{T}(\hat{z}) \in \varepsilon + \bigl(S \backslash \{0_Y\}\bigr). \tag{4.4}$$

Hence, $-\hat{T}(\hat{z}) - \varepsilon \in S\backslash\{0_Y\}$. It follows from (4.2) and the definition of $\hat{T}$ that

$$\gamma^*\big(-\hat{T}(\hat{z}) - \varepsilon\big) = \gamma^*\big(-\omega^*(\hat{z})s_0 - \varepsilon\big) = -\big(\gamma^*(\varepsilon) + \omega^*(\hat{z})\big) \leq 0.$$

On the other hand, from $\gamma^* \in S^{*i}$ we have $\gamma^*(S\backslash\{0_Y\}) > 0$. Then, $-\hat{T}(\hat{z}) - \varepsilon \notin S\backslash\{0_Y\}$, which contradicts (4.4). Hence, $-\hat{T}(G(\hat{x}) \cap (-K)) \cap (\varepsilon + (S\backslash\{0_Y\})) = \emptyset$, and, combining with (4.3), we have $-\hat{T}(G(\hat{x}) \cap (-K)) \subset S\backslash(\varepsilon + (S\backslash\{0_Y\}))$.

Ultimately, we prove $\operatorname{clcone}(\Psi(\hat{x}, X_0) + \varepsilon + S) \cap (-S) = \{0_Y\}$.

Let $s \in \operatorname{clcone}(\Psi(\hat{x}, X_0) + \varepsilon + S) \cap (-S)$, then there exist $\lambda_n > 0$, $y_n \in \Psi(\hat{x}, X_0)$, and $s_n \in S$ such that

$$s = \lim_{n\to\infty} \lambda_n(y_n + \varepsilon + s_n).$$

Thus,

$$\gamma^*(s) = \lim_{n\to\infty} \lambda_n\big(\gamma^*(y_n) + \gamma^*(\varepsilon) + \gamma^*(s_n)\big). \tag{4.5}$$

From $\gamma^* \in S^{*i}$ we get $\gamma^*(s_n) \geq 0$. By the definition of $\hat{T}$, $\gamma^*(s_0) = 1$ and (4.1), we get that for any $x \in X_0$,

$$\begin{aligned}
\gamma^*\big(\Psi(\hat{x},x)\big) + \gamma^*(\varepsilon) &= \gamma^*\big(\Upsilon(\hat{x},x)\big) + \gamma^*\big(\hat{T}\big(G(x)\big)\big) + \gamma^*(\varepsilon) \\
&= \gamma^*\big(\Upsilon(\hat{x},x)\big) + \gamma^*\big(\omega^*\big(G(x)\big)s_0\big) + \gamma^*(\varepsilon) \\
&= \gamma^*\big(\Upsilon(\hat{x},x)\big) + \omega^*\big(G(x)\big)\gamma^*(s_0) + \gamma^*(\varepsilon) \\
&= \gamma^*\big(\Upsilon(\hat{x},x)\big) + \omega^*\big(G(x)\big) + \gamma^*(\varepsilon) \\
&\geq 0.
\end{aligned}$$

Hence, by (4.5) we obtain $\gamma^*(s) \geq 0$. On the other hand, from $s \in -S$ we know $\gamma^*(s) \leq 0$. Thus, $\gamma^*(s) = 0$, together with $\gamma^* \in S^{*i}$, this yields $s = 0_Y$. Then,

$$\operatorname{clcone}\big(\Psi(\hat{x}, X_0) + \varepsilon + S\big) \cap (-S) = \{0_Y\}.$$

Thus, $\hat{x}$ is an ε-Benson efficient solution to (Ψ-USEP). □

Corollary 4.1 *Assume that $\hat{x} \in E$, $0 \in \Upsilon(\hat{x}, \hat{x})$, S has a compact base, $(\Upsilon(\hat{x}, \cdot), G(\cdot))$ is nearly $S \times K$-subconvexlike on X_0, and that there exists $x' \in X_0$ such that $G(x') \cap (-\operatorname{int} K) \neq \emptyset$. If $\hat{x}$ is a Benson efficient solution to (Υ-SEPC), then there exists $\hat{T} \in L_+(Z, Y)$ such that $\hat{x}$ is a Benson efficient solution to (Ψ-USEP) and*

$$\hat{T}\big(G(\hat{x}) \cap (-K)\big) = \{0_Y\},$$

where $\Psi : X_0 \times X_0 \to 2^Y$ is defined as $\Psi(y, x) = \Upsilon(y, x) + \hat{T}(G(x))$.

Proof In Theorem 4.1, letting $\varepsilon = 0$, we get that the conclusions hold. □

Theorem 4.2 *Assume that*

(i) $\hat{x} \in E$;

(ii) *there exists* $\hat{T} \in L_+(Z,Y)$ *such that* $\hat{x}$ *is an* ε*-Benson efficient solution to* (Ψ-USEP), *where* $\Psi : X_0 \times X_0 \to 2^Y$ *is defined as*

$$\Psi(y,x) = \Upsilon(y,x) + \hat{T}\big(G(x)\big).$$

Then $\hat{x}$ *is an* ε*-Benson efficient solution to* (Υ-SEPC).

Proof Since $\hat{x}$ is an ε-Benson efficient solution to (Ψ-USEP), we gain

$$\operatorname{clcone}\big(\Psi(\hat{x},X_0) + \varepsilon + S\big) \cap (-S) = \{0_Y\}. \tag{4.6}$$

For any $x \in E$, we know $G(x) \cap (-K) \neq \emptyset$. Thus, there exists $z_x \in G(x)$ such that $z_x \in -K$. Since $\hat{T} \in L_+(Z,Y)$, we get $-\hat{T}(z_x) \in S$, hence $S - \hat{T}(z_x) \in S + S \subset S$, and thus $S \subset \hat{T}(z_x) + S$. It follows from $z_x \in G(x)$ that $S \subset \hat{T}(G(x)) + S$. Hence

$$\begin{aligned}
\Upsilon(\hat{x},E) + \varepsilon + S &= \bigcup_{x\in E}\big(\Upsilon(\hat{x},x) + S + \varepsilon\big) \\
&\subset \bigcup_{x\in E}\big(\Upsilon(\hat{x},x) + \hat{T}\big(G(x)\big) + S + \varepsilon\big) \\
&\subset \bigcup_{x\in X_0}\big(\Upsilon(\hat{x},x) + \hat{T}\big(G(x)\big) + S + \varepsilon\big) \\
&= \Psi(\hat{x},X_0) + \varepsilon + S.
\end{aligned}$$

Thus

$$0_Y \in \operatorname{clcone}\big(\Upsilon(\hat{x},E) + \varepsilon + S\big) \subset \operatorname{clcone}\big(\Psi(\hat{x},X_0) + \varepsilon + S\big).$$

Together with (4.6), this yields

$$\operatorname{clcone}\big(\Upsilon(\hat{x},E) + \varepsilon + S\big) \cap (-S) = \{0_Y\}.$$

Thus $\hat{x}$ is an ε-Benson efficient solution to (Υ-SEPC). □

Corollary 4.2 *Assume that*

(i) $\hat{x} \in E$;

(ii) *there exists* $\hat{T} \in L_+(Z,Y)$ *such that* $\hat{x}$ *is a Benson efficient solution to* (Ψ-USEP), *where* $\Psi : X_0 \times X_0 \to 2^Y$ *is defined as*

$$\Psi(y,x) = \Upsilon(y,x) + \hat{T}\big(G(x)\big).$$

Then $\hat{x}$ *is a Benson efficient solution to* (Υ-SEPC).

Proof In Theorem 4.2, letting $\varepsilon = 0$, we gain that the conclusion is true. □

Remark 4.1 Let $\Upsilon(y,x) = F(x) - \hat{y}$. Since $\hat{y} \in F(\hat{x})$ and $\hat{x} \in E$, $\Upsilon(\cdot,\cdot)$ depends only on the second variable. Then Theorem 5.2 in [18] is a special case of Corollary 4.2.

Corollary 4.3 *Suppose that $\hat{x} \in E$, $0 \in \Upsilon(\hat{x},\hat{x})$, S has a compact base, φ is nearly $S \times K$-subconvexlike on X_0, and that there exists $x' \in X_0$ such that $G(x') \cap (-\operatorname{int} K) \neq \emptyset$. Then $\hat{x}$ is an ε-Benson efficient solution to* (Υ-SEPC) *if and only if there exists $\hat{T} \in L_+(Z,Y)$ such that $\hat{x}$ is an ε-Benson efficient solution to* (Ψ-USEP), *where $\Psi : X_0 \times X_0 \to 2^Y$ is defined as*

$$\Psi(y,x) = \Upsilon(y,x) + \hat{T}\big(G(x)\big).$$

Proof This proof follows immediately from Theorems 4.1 and 4.2. □

Corollary 4.4 *Suppose that $\hat{x} \in E$, $0 \in \Upsilon(\hat{x},\hat{x})$, S has a compact base, $(\Upsilon(\hat{x},\cdot), G(\cdot))$ is nearly $S \times K$-subconvexlike on X_0, and that there exists $x' \in X_0$ such that $G(x') \cap (-\operatorname{int} K) \neq \emptyset$. Then $\hat{x}$ is a Benson efficient solution to* (Υ-SEPC) *if and only if there exists $\hat{T} \in L_+(Z,Y)$ such that $\hat{x}$ is a Benson efficient solution to* (Ψ-USEP), *where $\Psi : X_0 \times X_0 \to 2^Y$ is defined as*

$$\Psi(y,x) = \Upsilon(y,x) + \hat{T}\big(G(x)\big).$$

Proof It follows from Corollaries 4.1 and 4.2 that the conclusion is true. □

Corollary 4.5 *Suppose that $\hat{x} \in E$, $\hat{y} \in F(\hat{x})$, S has a compact base, $(F - \hat{y}, G)$ is nearly $S \times K$-subconvexlike on X_0, and that there exists $x' \in X_0$ such that $G(x') \cap (-\operatorname{int} K) \neq \emptyset$. If $(\hat{x},\hat{y})$ is an ε-Benson efficient pair to* (SOP), *then there exists $\hat{T} \in L_+(Z,Y)$ such that $(\hat{x},\hat{y})$ is an ε-Benson efficient pair to* $(\mathrm{USOP})_{\hat{\mathrm{T}}}$ *and*

$$-\hat{T}\big(G(\hat{x}) \cap (-K)\big) \subset S \backslash \big(\varepsilon + \big(S \backslash \{0_Y\}\big)\big).$$

Proof Since $(\hat{x},\hat{y})$ is an ε-Benson efficient pair to (SOP), we have

$$\operatorname{clcone}\big(F(E) - \hat{y} + \varepsilon + S\big) \cap (-S) = \{0_Y\}.$$

Letting $\Upsilon(y,x) = F(x) - \hat{y}$, then $\Upsilon(\hat{x},x) = F(x) - \hat{y}$. From $\hat{y} \in F(\hat{x})$ we have $0 \in F(\hat{x}) - \hat{y}$. Then, $0 \in \Upsilon(\hat{x},\hat{x})$ and $F(E) - \hat{y} = \Upsilon(\hat{x},E)$. Thus

$$\operatorname{clcone}\big(\Upsilon(\hat{x},E) + \varepsilon + S\big) \cap (-S) = \{0_Y\}.$$

Therefore, $\hat{x}$ is an ε-Benson efficient solution to (Υ-SEPC). By Theorem 4.1, we can see that there exists $\hat{T} \in L_+(Z,Y)$ such that $\hat{x}$ is an ε-Benson efficient solution to (Ψ-SEPC) and $-\hat{T}(G(\hat{x}) \cap (-K)) \subset S \backslash (\varepsilon + (S \backslash \{0_Y\}))$. Then

$$\operatorname{clcone}\big(\Psi(\hat{x},X_0) + \varepsilon + S\big) \cap (-S) = \{0_Y\}.$$

Consequently,

$$\operatorname{clcone}\left(\bigcup_{x \in X_0} \big(\Upsilon(\hat{x},x) + \hat{T}\big(G(x)\big)\big) + \varepsilon + S\right) \cap (-S) = \{0_Y\}.$$

From $\Upsilon(\hat{x},x) = F(x) - \hat{y}$, we get

$$\operatorname{clcone}\left(\bigcup_{x \in X_0} \big(F(x) - \hat{y} + \hat{T}\big(G(x)\big)\big) + \varepsilon + S\right) \cap (-S) = \{0_Y\},$$

that is, $\operatorname{clcone}(\zeta(X_0, \hat{T}) - \hat{y} + \varepsilon + S) \cap (-S) = \{0_Y\}$. Hence, $(\hat{x}, \hat{y})$ is an ε-Benson efficient pair to $(\text{USOP})_{\hat{T}}$. □

Corollary 4.6 *Assume that*

(i) $\hat{x} \in E$, $\hat{y} \in F(\hat{x})$;

(ii) *there exists* $\hat{T} \in L_+(Z, Y)$ *such that* $(\hat{x}, \hat{y})$ *is an* ε*-Benson efficient pair to* $(\text{USOP})_{\hat{T}}$.

Then $\hat{x}$ *is an* ε*-Benson efficient pair to* (SOP).

Proof It follows from (ii) that $\operatorname{clcone}(\zeta(X_0, \hat{T}) - \hat{y} + \varepsilon + S) \cap (-S) = \{0_Y\}$, that is,

$$\operatorname{clcone}\left(\bigcup_{x \in X_0} \bigl(F(x) + \hat{T}\bigl(G(x)\bigr)\bigr) - \hat{y} + \varepsilon + S\right) \cap (-S) = \{0_Y\}.$$

Letting $\Upsilon(y, x) = F(x) - \hat{y}$, then $\Upsilon(\hat{x}, x) = F(x) - \hat{y}$. From $\hat{y} \in F(\hat{x})$ we have $0 \in F(\hat{x}) - \hat{y}$. Then, $0 \in \Upsilon(\hat{x}, \hat{x})$ and $F(x) - \hat{y} = \Upsilon(\hat{x}, x)$. Thus

$$\operatorname{clcone}\left(\bigcup_{x \in X_0} \bigl(\Upsilon(\hat{x}, x) + \hat{T}\bigl(G(x)\bigr)\bigr) + \varepsilon + S\right) \cap (-S) = \{0_Y\}.$$

Thus, $\operatorname{clcone}(\Psi(\hat{x}, X_0) + \varepsilon + S) \cap (-S) = \{0_Y\}$, Hence, $\hat{x}$ is an ε-Benson efficient solution to (Ψ-USEP). By Theorem 4.2, we can see that $\hat{x}$ is an ε-Benson efficient solution to (Υ-SEPC), that is, $\operatorname{clcone}(\Upsilon(\hat{x}, E) + \varepsilon + S) \cap (-S) = \{0_Y\}$. From $\Upsilon(\hat{x}, E) = F(E) - \hat{y}$, we get $\operatorname{clcone}(F(E) - \hat{y} + \varepsilon + S) \cap (-S) = \{0_Y\}$. Thus, $(\hat{x}, \hat{y})$ is an ε-Benson efficient pair to (SOP). □

Corollary 4.7 *Suppose that* $\hat{x} \in E$, $\hat{y} \in F(\hat{x})$, S *has a compact base,* $(F - \hat{y}, G)$ *is nearly* $S \times K$*-subconvexlike on* X_0*, and that there exists* $x' \in X_0$ *such that* $G(x') \cap (-\operatorname{int} K) \neq \emptyset$*. Then* $(\hat{x}, \hat{y})$ *is an* ε*-Benson efficient pair to* (SOP) *if and only if there exists* $\hat{T} \in L_+(Z, Y)$ *such that* $(\hat{x}, \hat{y})$ *is an* ε*-Benson efficient pair to* $(\text{USOP})_{\hat{T}}$.

Proof It follows from Corollaries 4.5 and 4.6 that the conclusion holds. □

Corollary 4.8 *Suppose that* $\hat{x} \in E$, $\hat{y} \in F(\hat{x})$, S *has a compact base,* $(F - \hat{y}, G)$ *is nearly* $S \times K$*-subconvexlike on* X_0*, and that there exists* $x' \in X_0$ *such that* $G(x') \cap (-\operatorname{int} K) \neq \emptyset$*. If* $(\hat{x}, \hat{y})$ *is a Benson efficient pair to* (SOP)*, then there exists* $\hat{T} \in L_+(Z, Y)$ *such that* $(\hat{x}, \hat{y})$ *is a Benson efficient pair to* $(\text{USOP})_{\hat{T}}$ *and*

$$\hat{T}\bigl(G(\hat{x}) \cap (-K)\bigr) = \{0_Y\}.$$

Proof In Corollary 4.5, letting $\varepsilon = 0$, we get that the conclusions are true. □

Remark 4.2 Corollary 4.8 generalizes Theorem 5.1 of [18] at the following points:

(i) If (F, G) is $S \times K$-subconvexlike on X_0, then $(F - \hat{y}, G)$ is nearly $S \times K$-subconvexlike on X_0;

(ii) Comparing with Theorem 5.1 in [18], this corollary does not require the convexity of F.

Corollary 4.9 *Suppose that* $\hat{x} \in E$, $\hat{y} \in F(\hat{x})$, S *has a compact base,* $(F - \hat{y}, G)$ *is nearly* $S \times K$*-subconvexlike on* X_0*, and that there exists* $x' \in X_0$ *such that* $G(x') \cap (-\operatorname{int} K) \neq \emptyset$*. Then* $(\hat{x}, \hat{y})$

is a Benson efficient pair to (SOP) *if and only if there exists* $\hat{T} \in L_+(Z, Y)$ *such that* $(\hat{x}, \hat{y})$ *is a Benson efficient pair to* $(\text{USOP})_{\hat{T}}$ *and*

$$\hat{T}\big(G(\hat{x}) \cap (-K)\big) = \{0_Y\}.$$

Proof It follows from Corollary 4.8 and Remark 4.1 that the conclusions are true. □

Remark 4.3 Corollary 4.9 is different from Theorem 5.1 of [19] at the following points:

(i) The vector-valued function is extended to a set-valued function;

(ii) According to Remarks 3.1 and 3.3 in [17], we know that if $(F - \hat{y}, G)$ is generalized $S \times K$-subconvexlike on X_0, then $(F - \hat{y}, G)$ is nearly $S \times K$-subconvexlike on X_0. Hence, Corollary 4.9 generalizes Theorem 5.1 in [19].

5 Conclusions

In this paper, we investigated the relationship between Benson and ε-Benson efficient solutions, and established Kuhn–Tucker-type and Lagrange-type optimality conditions to set-valued equilibrium problems. The results we obtained generalize those of Liu [15], Li [18], and Chen [19], respectively. As a mathematical topic, further research on ε-Benson efficient solutions seems to be of value and interest.

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The authors declare that they have no competing interests.

Authors' contributions

YX conceived and designed the study. SH wrote the paper. ZN reviewed and edited the manuscript. All authors read and approved the manuscript.

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References

1. Feng, Y.Y., Qiu, Q.S.: Optimality conditions for vector equilibrium problems with constraint in Banach spaces. Optim. Lett. **8**, 1931–1944 (2014)
2. You, M.X., Li, S.J.: Separation functions and optimality conditions in vector optimization. J. Optim. Theory Appl. **175**, 527–544 (2017)
3. Luu, D.V., Hang, D.D.: Efficient solutions and optimality conditions for vector equilibrium problems. Math. Methods Oper. Res. **79**, 163–177 (2014)
4. Gong, X.H.: Optimality conditions for vector equilibrium problems. J. Math. Anal. Appl. **342**, 1455–1466 (2008)
5. Iusem, A., Lara, F.: Optimality conditions for vector equilibrium problems with applications. J. Optim. Theory Appl. (2018). https://doi.org/10.1007/s10957-018-1321-6
6. Zhou, Z.A., Peng, J.W.: Scalarization of set-valued optimization problems with generalized cone subconvexlikeness in real ordered linear spaces. J. Optim. Theory Appl. **154**, 830–841 (2012)

7. Zhou, Z.A., Yang, X.M.: Scalarization of ϵ-super efficient solutions of set-valued optimization problems in real ordered linear spaces. J. Optim. Theory Appl. **162**, 680–693 (2014)
8. Dhingra, M., Lalitha, C.S.: Approximate solutions and scalarization in set-valued optimization. Optimization **66**, 1793–1805 (2017)
9. Yang, X.M., Li, D., Wang, S.Y.: Near-subconvexlikeness in vector optimization with set-valued functions. J. Optim. Theory Appl. **110**, 413–427 (2001)
10. Sach, P.H.: New generalized convexity notion for set-valued maps and application to vector optimization. J. Optim. Theory Appl. **125**, 157–179 (2005)
11. Xu, Y.H., Song, X.S.: The relationship between ic-cone-convexness and nearly cone-subconvexlikeness. Appl. Math. Lett. **24**, 1622–1624 (2011)
12. Yang, X.M., Yang, X.Q., Chen, G.Y.: Theorems of the alternative and optimization with set-valued maps. J. Optim. Theory Appl. **107**, 627–640 (2000)
13. Dauer, J.P., Saleh, O.A.: A characterization of proper minimal problem as a solution of sublinear optimization problems. J. Optim. Theory Appl. **178**, 227–246 (1993)
14. Li, Z.: A theorem of the alternative and its application to the optimization of set-valued maps. J. Optim. Theory Appl. **100**, 365–375 (1999)
15. Liu, S.Y., Sheng, B.H.: The optimality conditions and duality of nonconvex vector set-valued optimization with Benson proper efficiency. Acta Math. Appl. Sin. **26**, 337–344 (2003) (in Chinese)
16. Chen, J., Xu, Y.H., Zhang, K.: Approximate weakly efficient solutions of set-valued vector equilibrium problems. J. Inequal. Appl. **2018**, 181 (2018). https://doi.org/10.1186/s13660-018-1773-0
17. Sach, P.H.: Nearly subconvexlike set-valued maps and vector optimization problems. J. Optim. Theory Appl. **119**, 335–356 (2003)
18. Li, Z.F.: Benson proper efficiency in the vector optimization of set-valued maps. J. Optim. Theory Appl. **98**, 623–649 (1998)
19. Chen, G.Y., Rong, W.D.: Characterizations of the Benson proper efficiency for nonconvex vector optimization. J. Optim. Theory Appl. **98**, 365–384 (1998)