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The existence of solutions for nonlinear fractional boundary value problem and its Lyapunov-type inequality involving conformable variable-order derivative

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Abstract

We consider a nonlinear fractional boundary value problem involving conformable variable-order derivative with Dirichlet conditions. We prove the existence of solutions to the considered problem by using the upper and lower solutions' method with Schauder's fixed-point theorem. In addition, under some assumptions on the nonlinear term, a new Lyapunov-type inequality is given for the corresponding boundary value problem. The obtained inequality provides a necessary condition for the existence of nontrivial solutions to the considered problem and a method to prove uniqueness for the nonhomogeneous boundary value problem. These new results are illustrated through examples.

Keywords: Lyapunov-type inequality; Conformable variable-order derivative; Green's function; Boundary value problem; Upper and lower solutions' method

1 Introduction

Fractional calculus is considered to be an extremely powerful tool in describing complex systems due to applications, see [1, 2]. This field has attracted many authors working on generalizations of existing results including definitions, theorems, models, and many more. The conventional fractional derivatives, such as Riemann–Liouville and Caputo, have found numerous applications in science and engineering on account of their nonlocality and heredity effects. However, the two typical fractional derivatives do not obey the product, quotient, and chain rules, which presents us with some difficulties and inconvenience in mathematical handling. To overcome these difficulties, an interesting new well-behaved fractional derivative, called the conformable derivative, was introduced in [3, 4], where authors proved many similar properties as in the classical calculus. The conformable derivative is regarded as a natural extension of the classical derivative. Its physical interpretation given in [5] is a modification of the classical derivative in direction and magnitude, which indicates its potential applications in physics and engineering. Recently, there have been some works on conformable derivatives and their applications in various fields. In [6], the fractional Newtonian mechanics with conformable derivative was discussed. In [7], exact solutions of Wu–Zhang system with conformable time-fractional derivative were constructed. In [8], Sturm–Liouville eigenvalue problems in the framework of conformable derivative were studied. In [9], the distributions δ^r and $(\delta')^r$ for any $r \in \mathbb{R}$ with conformable derivative were found. On the other hand, in [10–12], authors proved the existence of solutions to nonlinear conformable fractional differential equations by using some fixed-point theorems.

The classical inequalities and their applications play an essential role in the theory of differential equations and applied mathematics. A large number of generalizations of classical inequalities by means of fractional operators are established in [9, 10, 13–29] and the references therein. The Lyapunov inequality was presented and proved by a Russian mathematician A.M. Lyapunov in [30], where he stated the fact that if the boundary value problem (BVP)

$$\begin{cases} x''(t) + q(t)x(t) = 0, \quad t \in (a,b), \\ x(a) = x(b) = 0, \end{cases} \tag{1.1}$$

has a nontrival solution, then

$$\int_a^b \left|q(t)\right| dt > \frac{4}{b-a}, \tag{1.2}$$

where $q : [a,b] \to \mathbb{R}$ is a continuous function. Generalizations of Lyapunov inequality (1.2) to fractional boundary value problems have been the interest of some researchers in the last few years due to their applications in the study of qualitative properties of solutions for differential equations (see, e.g., [10, 17–19, 21–25] and the references therein). In [10], Khaldi et al. gave a Lyapunov inequality for a boundary value problem involving conformable derivative. In [24, 25, 31], some new Lyapunov-type inequalities related to conformable fractional derivative were presented.

In [32], Zhang et al. defined a new conformable variable-order derivative. It was a generalization of the conformable constant-order derivative introduced in [3, 4]. A few properties of conformable variable-order derivative were presented in [32]. However, the progress in this direction is still at its earliest stage. Motivated by the above cited excellent works, in this paper we prove the existence of solutions and give a new Lyapunov-type inequality for the following boundary value problem involving conformable variable-order derivative:

$$\begin{cases} T^a_{p(t)}x(t) + \varphi(t,x(t)) = 0, \quad t \in (a,b), \\ x(a) = x(b) = 0, \end{cases} \tag{1.3}$$

where $T^a_{p(t)}$ defined in Sect. 2 is the conformable variable-order fractional derivative, $p : [a,b] \to (1,2]$ is a continuous function, and $\varphi : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a given function.

The paper is organized as follows. In Sect. 2, we recall some basic definitions on conformable variable-order derivative, define its higher-order analogue, and establish some preliminary results. In Sect. 3, we prove the existence of solutions to BVP (1.3) by using the lower and upper solutions' method with Schauder's fixed-point theorem. In Sect. 4, under some assumptions on the nonlinear term, a new Lyapunov-type inequality is given and some special cases are discussed.

2 Preliminaries

In this section, we give several definitions related to conformable variable-order fractional derivatives which can be found in [19]. Furthermore, conformable higher variable-order derivative is defined and some basic results are presented.

Definition 2.1 ([32]) The (left) conformable variable-order fractional derivative starting from a of a function $f : [a,\infty) \to \mathbb{R}$ of order $p : [a,\infty) \to (0,1]$ is defined by

$$\left(T^a_{p(t)}f\right)(t) = \lim_{\epsilon\to 0}\frac{f(t+\epsilon(t-a)^{1-p(t)}) - f(t)}{\epsilon}, \quad t > a,$$

when $a = 0$, we write $T_{p(t)}$. If $(T^a_{p(t)}f)(t)$ exists on (a,∞), then $(T^a_{p(t)}f)(a) = \lim_{t\to a^+}(T^a_{p(t)}f)(t)$.

Remark 2.1 ([32]) If the fractional derivative of f of order $p(t) \in (0,1]$ for all $t \in [a,\infty)$ exists, then we simply say that f is $p(t)$-differentiable.

Lemma 2.1 ([32]) *Let $p : [a,\infty) \to (0,1]$. If a function $f : [a,\infty) \to \mathbb{R}$ is $p(t)$- differentiable at $t_0 > a$, then f is continuous at t_0.*

Lemma 2.2 ([32]) *Let $p : [a,\infty) \to (0,1]$. If f is differentiable at a point $t > a$, then $(T^a_{p(t)}f)(t) = (t-a)^{1-p(t)}f'(t)$.*

In the higher-order case, we can generalize to the following:

Definition 2.2 Let $p : [a,\infty) \to (n,n+1]$ and $q(t) = p(t) - n$. Then the (left) conformable variable-order derivative starting from a of a function $f : [a,\infty) \to \mathbb{R}$ of order $p(t)$, where $f^{(n)}(t)$ exists, is defined by

$$\left(T^a_{p(t)}f\right)(t) = \left(T^a_{q(t)}f^{(n)}\right)(t), \quad t > a, \tag{2.1}$$

when $a = 0$, we write $T_{p(t)}$. If $(T^a_{p(t)}f)(t)$ exists on (a,∞), then $(T^a_{p(t)}f)(a) = \lim_{t\to a^+}(T^a_{p(t)}f)(t)$.

Note that if $p(t) = \alpha$, where $\alpha \in (n,n+1]$ is a constant, then the definition of conformable variable-order derivative coincides with the definition in [4]. Also when $n = 0$ (or $0 < p(t) \le 1$), then $q(t) = p(t)$ and the definition coincides with Definition 2.1.

Remark 2.2 If $n < p(t) \le n+1$, $q(t) = p(t) - n$, and $f^{(n+1)}(t)$ exists, then we have

$$\left(T^a_{p(t)}f\right)(t) = \left(T^a_{q(t)}f^{(n)}\right)(t) = (t-a)^{1-q(t)}f^{(n+1)}(t) = (t-a)^{n+1-p(t)}f^{(n+1)}(t), \quad t > a. \tag{2.2}$$

Lemma 2.3 *Let $p : [a,b] \to (1,2]$. If a function $f \in C^1[a,b] \cap C^2(a,b)$ attains a global minimum (respectively maximum) at some point $\xi \in (a,b)$, then $(T^a_{p(t)}f)(\xi) \ge 0$ (respectively $(T^a_{p(t)}f)(\xi) \le 0$).*

Proof Since $f \in C^2(a,b)$ attains a global minimum at $\xi \in (a,b)$, we have

$$f''(\xi) \ge 0.$$

According to Remark 2.2, we have

$$\left(T^a_{p(t)}f\right)(\xi) = (\xi - a)^{2-p(\xi)} f''(\xi) \geq 0. \qquad \square$$

Lemma 2.4 *If both* $p : [a,b] \to (1,2]$ *and* $h : [a,b] \to \mathbb{R}$ *are continuous functions, then*

$$\left| \int_a^b (s-a)^{p(s)-2} h(s)\, ds \right| < \infty.$$

Proof Since

$$\left| \int_a^b (s-a)^{p(s)-2} h(s)\, ds \right| \leq \|h\| \int_a^b (s-a)^{p(s)-2}\, ds,$$

where $\|h\| = \max_{s\in[a,b]} |h(s)|$, we just need to demonstrate $\int_a^b (s-a)^{p(s)-2}\, ds < \infty$.

Because $p : [a,b] \to (1,2]$ is a continuous function, then there exist two constants p_1, $p_2 \in (1,2]$ such that

$$\min_{s\in[a,b]} p(s) = p_1, \qquad \max_{s\in[a,b]} p(s) = p_2.$$

Therefore, if $0 < s-a < 1$, then

$$\int_a^b (s-a)^{p(s)-2}\, ds \leq \int_a^b (s-a)^{p_1-2}\, ds = \frac{(b-a)^{p_1-1}}{p_1-1};$$

if $s-a = 1$, then

$$\int_a^b (s-a)^{p(s)-2}\, ds = b-a;$$

if $s-a > 1$, then

$$\int_a^b (s-a)^{p(s)-2}\, ds \leq \int_a^b (s-a)^{p_2-2}\, ds = \frac{(b-a)^{p_2-1}}{p_2-1}.$$

In conclusion, $|\int_a^b (s-a)^{p(s)-2} h(s)\, ds| < \infty$. $\square$

Lemma 2.5 *Assume that* $h \in C[a,b]$ *and* $p : [a,b] \to (1,2]$ *is a continuous function. A function* $x \in C^1[a,b] \cap C^2(a,b)$ *is a solution of the following BVP with conformable variable-order derivative:*

$$\begin{cases} T^a_{p(t)} x(t) + h(t) = 0, \quad t \in (a,b), \\ x(a) = x(b) = 0 \end{cases} \tag{2.3}$$

if and only if it satisfies the integral equation

$$x(t) = \int_a^b H(t,s) h(s)\, ds, \tag{2.4}$$

where H is the Green function given by

$$H(t,s)=\begin{cases}\frac{(b-t)(s-a)^{p(s)-1}}{b-a}, & a\le s\le t\le b,\\ \frac{(t-a)(b-s)(s-a)^{p(s)-2}}{b-a}, & a\le t\le s\le b.\end{cases} \tag{2.5}$$

Proof If $x\in C^1[a,b]\cap C^2(a,b)$ is a solution of BVP (2.3), then using Remark 2.2, we have

$$x''(t)=-(t-a)^{p(t)-2}h(t). \tag{2.6}$$

Integrating twice both sides of equation (2.6) and taking into account the boundary condition $x(a)=x(b)=0$, we obtain

$$x(t)=\int_a^b H(t,s)h(s)\,ds.$$

On the other hand, if

$$x(t)=\int_a^b H(t,s)h(s)\,ds,$$

using the branches of (2.5), we have

$$x(t)=\int_a^t \frac{b-t}{b-a}(s-a)^{p(s)-1}h(s)\,ds+\int_t^b \frac{(s-a)^{p(s)-2}(t-a)(b-s)}{b-a}h(s)\,ds.$$

Checking the boundary value conditions, we see that $x(a)=0$, $x(b)=0$. Taking second-order derivative yields

$$x''(t)=-(t-a)^{p(t)-2}h(t).$$

According to Remark 2.2,

$$T^a_{p(t)}x(t)+h(t)=0,$$

which is what we set out to prove. □

Lemma 2.6 *The Green function H given by* (2.5) *has the following properties*:
(i) $H(t,s)\ge 0$ *for all* $s,t\in[a,b]$;
(ii) $\max_{t\in[a,b]}H(t,s)=H(s,s)$ *for any* $a\le s\le b$;
(iii) *There exists a point* $s^*\in(a,b)$ *such that* $H(s^*,s^*)=\max_{s\in[a,b]}H(s,s)$. *In particular, if* $p'(t)$ *exists, then* s^* *is the solution of equation* $p'(s)\ln(s-a)+\frac{p(s)-1}{s-a}=\frac{1}{b-s}$.

Proof (i) It is clear that $H(t,s)\ge 0$ for all $s,t\in[a,b]$.
(ii) Set

$$h_1(t,s)=\frac{(b-t)(s-a)^{p(s)-1}}{b-a},\quad a\le s\le t\le b,$$

and

$$h_2(t,s) = \frac{(t-a)(b-s)(s-a)^{p(s)-2}}{b-a}, \quad a \le t \le s \le b.$$

For every fixed s, $h_1(t,s)$ is a decreasing function in t and $h_2(t,s)$ is an increasing function in t. Hence, $\max_{t\in[a,b]} H(t,s) = H(s,s)$ for any $a \le s \le b$.

(iii) Let $g(s) = H(s,s) = \frac{(b-s)(s-a)^{p(s)-1}}{b-a}$, then $g(s) \ge 0$. Since $g(s) \in C[a,b]$ and $g(a) = g(b) = 0$, we conclude that there must exist a point $s^* \in (a,b)$ such that the maximum of function $g(s)$ is achieved at s^*, denoted as $H(s^*,s^*) = \max_{s\in[a,b]} H(s,s)$. If $p'(t)$ exists, then $g'(s) = \frac{(s-a)^{p(s)-1}}{b-a}[(b-s)(p'(s)\ln(s-a) + \frac{p(s)-1}{s-a}) - 1]$. From $g'(s^*) = 0$, we conclude that $p'(s^*)\ln(s^* - a) + \frac{p(s^*)-1}{s^*-a} = \frac{1}{b-s^*}$. □

3 Existence of solutions

In the section, our aim is to prove the existence of solutions for BVP (1.3) by using the lower and upper solutions' method. Therefore, we need to give the definitions of lower and upper solutions of BVP (1.3) as follows.

Definition 3.1 A function $\underline{u} \in C^1[a,b] \cap C^2(a,b)$ is said to be a lower solution of BVP (1.3) if we have

$$\begin{cases} T^a_{p(t)}\underline{u}(t) + \varphi(t,\underline{u}(t)) \ge 0, & t \in [a,b], \\ \underline{u}(a) \le 0, \quad \underline{u}(b) \le 0. \end{cases}$$

Similarly, a function $\overline{u}(t) \in C^1[a,b] \cap C^2(a,b)$ is said to be an upper solution of BVP (1.3) if we have

$$\begin{cases} T^a_{p(t)}\overline{u}(t) + \varphi(t,\overline{u}(t)) \le 0, & t \in [a,b], \\ \overline{u}(a) \ge 0, \quad \overline{u}(b) \ge 0. \end{cases}$$

Theorem 3.1 *Assume that the following hypotheses hold*:

(i) $\underline{u}$ *and* $\overline{u}$ *are lower and upper solutions of BVP* (1.3), *respectively, such that* $\underline{u} \le \overline{u}$,

(ii) *Define* $E = \{(t,y) | (t,y) \in [a,b] \times \mathbb{R}, \underline{u} \le y \le \overline{u}\}$, *and assume* $\varphi(t,y)$ *is continuous on* E.

Then BVP (1.3) *has at least one solution* $x \in C^1[a,b] \cap C^2(a,b)$ *such that*

$$\underline{u}(t) \le x(t) \le \overline{u}(t), \quad a \le t \le b.$$

Proof Consider the following modified problem:

$$\begin{cases} T^a_{p(t)}x(t) + \phi(t,x(t)) = 0, & t \in (a,b), \\ x(a) = x(b) = 0, \end{cases} \tag{3.1}$$

where $\phi(t,y)$ is given by

$$\phi(t,y) = \begin{cases} \varphi(t,\overline{u}(t)) + \frac{\overline{u}(t)-y}{y-\overline{u}(t)+1}, & y > \overline{u}(t), \\ \varphi(t,y), & \underline{u}(t) \le y \le \overline{u}(t), \\ \varphi(t,\underline{u}(t)) + \frac{\underline{u}(t)-y}{\underline{u}(t)-y+1}, & y < \underline{u}(t). \end{cases}$$

Since $\varphi(t,y)$ is continuous on E, from the definition of ϕ, we deduce that $\phi(t,y)$ is continuous on $[a,b]\times\mathbb{R}$ and there exists $M=\max_{(t,x)\in E}|\varphi(t,x)|$ such that

$$\bigl|\phi(t,y)\bigr|\le M+1,\quad \forall(t,y)\in[a,b]\times\mathbb{R}. \tag{3.2}$$

According to Lemma 2.5, the modified BVP (3.1) is transformed into a fixed-point problem. Consider the operator $A:C[a,b]\to C[a,b]$ defined by

$$Ax(t)=\int_a^b H(t,s)\phi\bigl(s,x(s)\bigr)\,ds.$$

Set $\delta=(M+1)(b-a)H(s^*,s^*)$, where $H(s^*,s^*)$ is given by Lemma 2.6, and $\Omega=\{y\in C[a,b]:\|y\|\le\delta\}$. Clearly, Ω is a closed, bounded, and convex subset of $C[a,b]$, and A maps Ω into Ω. We shall show that A satisfies the assumptions of Schauder's fixed-point theorem. The proof will be given in several steps.

Step 1. *A is continuous.* Indeed, consider any $x_n\in\Omega$ such that $x_n\to x$ in Ω. Then

$$\begin{aligned}\bigl|Ax_n(t)-Ax(t)\bigr|&=\biggl|\int_a^b H(t,s)\bigl(\phi\bigl(s,x_n(s)\bigr)-\phi\bigl(s,x(s)\bigr)\bigr)\,ds\biggr|\\&\le H\bigl(s^*,s^*\bigr)(b-a)\bigl\|\phi\bigl(\cdot,x_n(\cdot)\bigr)-\phi\bigl(\cdot,x(\cdot)\bigr)\bigr\|.\end{aligned}$$

Since ϕ is a continuous function, we have

$$\|Ax_n-Ax\|\le H\bigl(s^*,s^*\bigr)(b-a)\bigl\|\phi\bigl(\cdot,x_n(\cdot)\bigr)-\phi\bigl(\cdot,x(\cdot)\bigr)\bigr\|\to 0,\quad n\to\infty.$$

Step 2. *$A(\Omega)$ is uniformly bounded.* For any $x\in\Omega$, using Lemma 2.6 and (3.2), we get

$$\bigl|Ax(t)\bigr|\le H\bigl(s^*,s^*\bigr)\int_a^b\bigl|\phi\bigl(s,x(s)\bigr)\bigr|\,ds\le(M+1)(b-a)H\bigl(s^*,s^*\bigr)=\delta.$$

Therefore, $A(\Omega)$ is uniformly bounded and $A(\Omega)\subset\Omega$.

Step 3. *$A(\Omega)$ is equicontinuous.* Consider arbitrary $t_1,t_2\in[a,b],x\in\Omega$, and, without loss of generality, let $t_1<t_2$. Then

$$\begin{aligned}\bigl|Ax(t_2)-Ax(t_1)\bigr|&\le(M+1)\biggl[(t_2-t_1)\int_a^{t_1}(s-a)^{p(s)-2}\,ds+2(t_2-t_1)\int_{t_1}^{t_2}(s-a)^{p(s)-2}\,ds\\&\quad+(t_2-t_1)\int_{t_2}^b(s-a)^{p(s)-2}\,ds\biggr]\\&\le 4(M+1)(t_2-t_1)\int_a^b(s-a)^{p(s)-2}\,ds\to 0,\end{aligned}$$

when $t_1\to t_2$.

As a consequence of Steps 1–3, together with the Arzela–Ascoli theorem, we can conclude that $A:\Omega\to\Omega$ is completely continuous. From the Schauder's fixed-point theorem, we deduce that A has a fixed point $x\in\Omega$ which is a solution of the modified BVP (3.1).

Step 4. The solution x of BVP (3.1) satisfies

$$\underline{u}(t)\le x(t)\le\overline{u}(t),\quad a\le t\le b.$$

First, we prove that

$$\underline{u}(t) \leq x(t), \quad a \leq t \leq b.$$

Assume on the contrary and set $v(t) = x(t) - \underline{u}(t)$, then there exists $t_0 \in [a,b]$ such that

$$\min_{t\in[a,b]} v(t) = v(t_0) = x(t_0) - \underline{u}(t_0) < 0.$$

We conclude that $x(t_0) < \underline{u}(t_0)$. Then we distinguish the following cases.

Case 1. If $t_0 \in (a,b)$, then $T^a_{p(t)}v(t_0) \geq 0$ is obtained from Lemma 2.3. Using the fact that $\underline{u}$ is a lower solution of BVP (1.3) and x is a solution of BVP (3.1), we get

$$\begin{aligned} T^a_{p(t)}v(t_0) &= T^a_{p(t)}x(t_0) - T^a_{p(t)}\underline{u}(t_0) \\ &= -\phi\bigl(t, x(t_0)\bigr) - T^a_{p(t)}\underline{u}(t_0) \\ &= -\varphi\bigl(t_0, \underline{u}(t_0)\bigr) - \frac{\underline{u}(t_0) - x(t_0)}{\underline{u}(t_0) - x(t_0) + 1} - T^a_{p(t)}\underline{u}(t_0) < 0, \end{aligned}$$

which is a contradiction to $T^a_{p(t)}v(t_0) \geq 0$.

Case 2. If $t_0 = a$, since $x(a) = 0$, we get

$$\underline{u}(a) > 0.$$

On the other hand, $\underline{u}(a) \leq 0$ thanks to the fact that $\underline{u}$ is a lower solution of BVP (1.3). This is a contradiction.

Case 3. If $t_0 = b$, we obtain a contradiction as in the second case. Thus we have

$$\underline{u}(t) \leq x(t), \quad a \leq t \leq b.$$

Analogously, we can prove that $x(t) \leq \overline{u}(t), a < t < b$. This shows that the modified BVP (3.1) has a solution $x \in C^1[a,b] \cap C^2(a,b)$ satisfying $\underline{u}(t) \leq x(t) \leq \overline{u}(t)$ which is solution of BVP (1.3). The proof is completed. □

Example 3.1 Consider the following fractional boundary value problem:

$$\begin{cases} T_{p(t)}x(t) - tln(1+x) = 0, \quad t \in (0,1), \\ x(0) = x(1) = 0. \end{cases} \tag{3.3}$$

It can be easily seen that $\underline{u} = 0$ is a lower solution of BVP (3.3) and $\overline{u} = 2$ is an upper solution of BVP (3.3). According to Theorem 3.1, BVP (3.3) has at least one solution $x \in C^1[0,1] \cap C^2(0,1)$ such that $0 \leq x(t) \leq 2$.

4 Lyapunov-type inequality

Letting $\varphi(t, x(t)) = q(t)f(x(t))$, BVP (1.3) becomes

$$\begin{cases} T^a_{p(t)}x(t) + q(t)f(x) = 0, \quad t \in (a,b), \\ x(a) = x(b) = 0. \end{cases} \tag{4.1}$$

Now we are ready to give a Lyapunov-type inequality of BVP (4.1) under the following assumptions:

(i) The function $f:\mathbb{R}\to\mathbb{R}$ is continuous and monotone,

(ii) $q(t)\in C[a,b]$.

Theorem 4.1 *If $x(t)\in C^1[a,b]\cap C^2(a,b)$ is a nontrivial solution of BVP (4.1), then*

$$\int_a^b \left|q(t)\right| dt > \frac{\eta}{H(s^*,s^*)\max\{|f(\eta)|,|f(\gamma)|\}}, \tag{4.2}$$

where $\eta=\|x\|=\max_{t\in[a,b]}|x(t)|$, $\gamma=\min_{t\in[a,b]}x(t)$, and $H(s^*,s^*)$ is given by Lemma 2.6.

Proof According to Lemma 2.5, if $x(t)\in C^1[a,b]\cap C^2(a,b)$ is a solution of BVP (4.1), then

$$x(t)=\int_a^b H(t,s)q(s)f\bigl(x(s)\bigr)\,ds.$$

Since $x(t)$ is a nontrivial solution, taking the norm of $x(t)$ and making use of the properties of Green function $H(t,s)$ given by Lemma 2.6, we obtain that

$$\begin{aligned}\|x\| &= \max_{t\in[a,b]}\left|\int_a^b H(t,s)q(s)f\bigl(x(s)\bigr)\,ds\right| \\ &< H\bigl(s^*,s^*\bigr)\int_a^b \left|q(s)\right|\left|f\bigl(x(s)\bigr)\right| ds \\ &\le H\bigl(s^*,s^*\bigr)\max\bigl\{\left|f(\eta)\right|,\left|f(\gamma)\right|\bigr\}\int_a^b \left|q(s)\right| ds,\end{aligned}$$

from which (4.2) follows. Then the proof is completed. □

Example 4.1 Consider Example 3.1 with $p(t)=\frac{t+3}{2}$, then $H(s,s)=(1-s)s^{\frac{s+1}{2}}$. We can calculate $H(s^*,s^*)=\max_{s\in[a,b]}H(s,s)\approx 0.320573$ when $s^*\approx 0.324077$. According to Example 3.1, BVP (3.3) has at least one solution $0\le x(t)\le 2$. Since $f(x)=\ln(1+x)$ is continuous and increasing on $[0,2]$, and, in addition, $q(t)=-t\in C[0,1]$ satisfies the conditions of Theorem 4.1, we obtain

$$\int_0^1 |-t|\,dt > \frac{\eta}{H(s^*,s^*)\ln(1+\eta)} \ge \frac{\eta}{H(s^*,s^*)\eta} = \frac{1}{H(s^*,s^*)} \approx 3.11941.$$

On the other hand, $\int_0^1|-t|\,dt=0.5<3.11941$. Hence, by Theorem 4.1, $x(t)\equiv 0$ is the only solution of Example 3.1 with $p(t)=\frac{t+3}{2}$.

If $f(x(t))=x(t)$, BVP (4.1) becomes

$$\begin{cases} T^a_{p(t)}x(t)+q(t)x(t)=0, & t\in(a,b),\\ x(a)=x(b)=0.\end{cases} \tag{4.3}$$

In this case, taking $f(x(t))=x(t)$ in Theorem 4.1, we obtain the following result.

Corollary 4.1 *If $x(t) \in C^1[a,b] \cap C^2(a,b)$ is a nontrivial solution of BVP (4.3), then*

$$\int_a^b |q(t)|\,dt > \frac{1}{H(s^*,s^*)}. \tag{4.4}$$

Remark 4.1 If $p(t) = \alpha \in (1,2]$, then $H(s,s) = \frac{b-s}{b-a}(s-a)^{\alpha-1}$. Moreover, $s^* = \frac{a+(\alpha-1)b}{\alpha}$ is such that $H(s^*,s^*) = \frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}{\alpha^\alpha}$. Thus inequality (4.4) reduces to the Lyapunov-type inequality in [24]. If $p(t) = 2$, inequality (4.4) reduces to the classical Lyapunov inequality (1.2).

Consider the following nonhomogeneous boundary value problem:

$$\begin{cases} T^a_{p(t)}x(t) + q(t)x(t) = g(t), & t \in (a,b), \\ x(a) = x(b) = 0. \end{cases} \tag{4.5}$$

Corollary 4.2 *If a solution of BVP (4.5) exists, and*

$$\int_a^b |q(t)|\,dt \le \frac{1}{H(s^*,s^*)}, \tag{4.6}$$

then BVP (4.5) has a unique solution.

Proof Assume that $x_1(t)$, $x_2(t)$ are both solutions of BVP (4.5), then $x(t) = x_1(t) - x_2(t)$ is a solution of the corresponding homogeneous BVP. According to (4.6) and Corollary 4.1, the corresponding homogeneous BVP has only zero solution. Therefore the nonhomogeneous BVP (4.5) has a unique solution. □

In the end, we consider the following conformable eigenvalue problem:

$$\begin{cases} T^a_{p(t)}x(t) + \lambda x(t) = 0, & t \in (a,b) \\ x(a) = x(b) = 0. \end{cases} \tag{4.7}$$

Corollary 4.3 *If λ is an eigenvalue of (4.7), then*

$$|\lambda| > \frac{1}{(b-a)H(s^*,s^*)}.$$

Proof If λ is an eigenvalue of (4.7),then there exists a nontrivial solution $x = x_\lambda$ to (4.7). Using Corollary 4.1 with $q(t) = \lambda$, we obtain

$$|\lambda| > \frac{1}{(b-a)H(s^*,s^*)}.$$ □

5 Conclusions

In this paper, we have proved the existence of solutions to a nonlinear boundary value problem involving conformable variable-order derivative. We have also obtained a new Lyapunov-type inequality for the considered problem. The obtained inequality provides a necessary condition for the existence of nontrivial solutions and a method to prove uniqueness for the corresponding nonhomogeneous boundary value problem. We notice that when $p(t) = 2$, the obtained inequality reduces to the classical Lyapunov inequality and when $p(t) = \alpha$, it reduces to the inequality in [24]. The new results generalize some existing results in the literature. We expect that the proposed approaches and the obtained results in this paper can be adapted to study other fractional boundary value problems.

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