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Improvements of operator reverse AM-GM inequality involving positive linear maps

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Abstract

In this paper, we shall present some reverse arithmetic-geometric mean operator inequalities for unital positive linear maps. These inequalities improve some corresponding results due to Xue (J. Inequal. Appl. 2017:283, 2017).

MSC: 47A63; 47A30

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1 Introduction

Let m, m', m'_1, m'_2, m'_3, M, M', M'_1, M'_2 and M'_3 be scalars, I be the identity operator and the other capital letters be used to represent general elements of the C^*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators acting on a Hilbert space$(\mathcal{H}, \langle\cdot,\cdot\rangle)$. The operator norm is denoted by $\|\cdot\|$. An operator A is said to be positive if $\langle Ax,x\rangle \geq 0$ for all $x \in \mathcal{H}$ and we write it as $A \geq 0$, it is said to be strictly positive if $\langle Ax,x\rangle > 0$ for all $x \in \mathcal{H} \setminus \{0\}$ and we write it as $A > 0$. A linear map Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. For $A, B > 0$ the μ-weighted arithmetic mean and μ-weighted geometric mean of A and B are defined, respectively, by

$$A \triangledown_\mu B = (1-\mu)A + \mu B, \qquad A \sharp_\mu B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\mu A^{1/2},$$

where $\mu \in [0,1]$, when $\mu = 1/2$, we write $A\nabla B$ and $A\sharp B$ for brevity for $A\nabla_{1/2}B$ and $A\sharp_{1/2}B$, respectively.

For $0 < m \leq A, B \leq M$, Tominaga [2] proved that the following operator reverse AM-GM inequality holds:

$$\frac{A+B}{2} \leq S(h)A\sharp B, \tag{1.1}$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$ is called Specht's ratio with $h = \frac{M}{m}$.

The inequality (1.1) can be regarded as a counterpart of the following AM-GM inequality:

$$\frac{A+B}{2} \geq A\sharp B. \tag{1.2}$$

Lin [3, (3.3)] observed that

$$S(h) \leq K(h) \leq S^2(h) \quad (h \geq 1), \tag{1.3}$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$. The constant $K(t,2) = \frac{(t+1)^2}{4t}$ $(t > 0)$ is called the Kantorovich constant, which is simply represented by $K(t)$ satisfying the following properties:

$K(1,2) = 1$,

$K(t,2) = K(\frac{1}{t},2) \geq 1$ $(t > 0)$,

$K(t,2)$ is monotone increasing on $[1,\infty)$ and monotone decreasing on $(0,1]$.

By inequalities (1.1) and (1.3), we have

$$\frac{A+B}{2} \leq K(h)A\sharp B. \tag{1.4}$$

Because Φ is order preserving, (1.4) implies that

$$\Phi\left(\frac{A+B}{2}\right) \leq K(h)\Phi(A\sharp B). \tag{1.5}$$

For a positive linear map Φ and $A, B \geq 0$. Ando [4] has proved the following inequality:

$$\Phi(A\sharp B) \leq \bigl(\Phi(A)\sharp\Phi(B)\bigr). \tag{1.6}$$

Then, by (1.5) and (1.6), we have

$$\Phi\left(\frac{A+B}{2}\right) \leq K(h)\bigl(\Phi(A)\sharp\Phi(B)\bigr). \tag{1.7}$$

The studies of squaring operator inequalities start with [3, 5] and continued by a number of authors [6–10]. Lin [3] revealed that inequalities (1.5) and (1.7) can be squared as follows:

$$\Phi^2\left(\frac{A+B}{2}\right) \leq K^2(h)\Phi^2(A\sharp B), \tag{1.8}$$

$$\Phi^2\left(\frac{A+B}{2}\right) \leq K^2(h)\bigl(\Phi(A)\sharp\Phi(B)\bigr)^2. \tag{1.9}$$

Recently, Xue [1] proved that if $\sqrt{\frac{M}{m}} \leq 2.314$, then the following refinement of the inequality(1.4) holds:

$$\left(\frac{A+B}{2}\right) \leq K^{\frac{1}{2}}(h)(A\sharp B). \tag{1.10}$$

Inspired by Lin's idea [3] , Xue [1] also proved that if $0 < m \leq A, B \leq M$ and $\sqrt{\frac{M}{m}} \leq 2.314$, then

$$\left(\frac{A+B}{2}\right)^2 \leq K(h)(A\sharp B)^2, \tag{1.11}$$

$$\Phi^2\left(\frac{A+B}{2}\right) \leq K(h)\Phi^2(A\sharp B), \tag{1.12}$$

and

$$\Phi^2\left(\frac{A+B}{2}\right) \leq K(h)\big(\Phi(A)\sharp\Phi(B)\big)^2, \tag{1.13}$$

inequalities (1.12) and (1.13) are refinements of the inequalities (1.8) and (1.9), respectively.

Moreover, she proved Lin's conjecture [3] as follows:

$$\Phi^2\left(\frac{A+B}{2}\right) \leq S^2(h)\Phi^2(A\sharp B), \tag{1.14}$$

$$\Phi^2\left(\frac{A+B}{2}\right) \leq S^2(h)\big(\Phi(A)\sharp\Phi(B)\big)^2. \tag{1.15}$$

Recently, Ali et al. obtained more refinements of the results presented by Xue [1] by using the relation (1.2), for comprehensive study, the reader is referred to [11]. In this article, in Sect. 2, we shall refine the inequalities (1.10)–(1.15), when $\sqrt{\frac{M}{m}} \leq 2.314$, with the help of the Kantorovich constant.

2 Main results

We begin this section with the following lemmas.

Lemma 2.1 ([12]) *Let $A, B > 0$. Then the following norm inequality holds*:

$$\|AB\| \leq \frac{1}{4}\|A+B\|^2. \tag{2.1}$$

Remark 2.2 Lemma 2.1 is proved by Bhatia and Kittaneh in [12] for the finite dimensional case. However, all technical results used to prove this result for operator norm are also true for the infinite dimensional case. Here also, we mention that if A, B are compact operators, then a stronger result can be found in [13].

Lemma 2.3 ([14]) *Let $A > 0$. Then, for every positive unital linear map Φ,*

$$\Phi^{-1}(A) \leq \Phi\big(A^{-1}\big). \tag{2.2}$$

Lemma 2.4 ([15]) *Suppose that two operators A, B and positive real numbers m, m', M, M' satisfy either of the following conditions*:

(1) $0 < m \leq A \leq m' < M' \leq B \leq M$,

(2) $0 < m \leq B \leq m' < M' \leq A \leq M$.

Then

$$K^r(h')\left(A^{-1}\sharp_\mu B^{-1}\right) \le A^{-1}\nabla_\mu B^{-1}, \tag{2.3}$$

for all $\mu \in [0,1]$, $r = \min[\mu, 1-\mu]$, $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$.

Now, we prove the first main result in the following theorem.

Theorem 2.5 *Let* $0 < m \le M$ *and* $\sqrt{\frac{M}{m}} \le 2.314$, *we have*

(1) *If* $0 < m \le A \le m_1' < M_1' \le B \le \frac{M+m}{2}$, *then*

$$\left(\frac{A+B}{2}\right)^2 \le \frac{K(h)}{K(h_1')}(A\sharp B)^2, \tag{2.4}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_1') = \frac{(h_1'+1)^2}{4h_1'}$, $h = \frac{M}{m}$ *and* $h_1' = \frac{M_1'}{m_1'}$.

(2) *If* $0 < \frac{M+m}{2} \le A \le m_2' < M_2' \le B \le M$, *then*

$$\left(\frac{A+B}{2}\right)^2 \le \frac{K(h)}{K(h_2')}(A\sharp B)^2, \tag{2.5}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_2') = \frac{(h_2'+1)^2}{4h_2'}$, $h = \frac{M}{m}$ *and* $h_2' = \frac{M_2'}{m_2'}$.

(3) *If* $0 < m \le A \le m_3' < \frac{M+m}{2} \le B \le M$, *then*

$$\left(\frac{A+B}{2}\right)^2 \le \frac{K(h)}{K(h_3')}(A\sharp B)^2, \tag{2.6}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_3') = \frac{(h_3'+1)^2}{4h_3'}$, $h = \frac{M}{m}$ *and* $h_3' = \frac{M+m}{2m_3'}$.

(4) *If* $0 < m \le A \le \frac{M+m}{2} < M_3' \le B \le M$, *then*

$$\left(\frac{A+B}{2}\right)^2 \le \frac{K(h)}{K(h_4')}(A\sharp B)^2, \tag{2.7}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_4') = \frac{(h_4'+1)^2}{4h_4'}$, $h = \frac{M}{m}$ *and* $h_4' = \frac{2M_3'}{M+m}$.

Proof The operator inequality (2.4) is equivalent to

$$\left\|\frac{A+B}{2}(A\sharp B)^{-1}\right\| \le \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_1')}.$$

If $0 < m \le A \le m_1' < M_1' \le B \le \frac{M+m}{2}$, we get

$$A + \frac{M+m}{2}mA^{-1} \le \frac{M+m}{2} + m \tag{2.8}$$

and

$$B + \frac{M+m}{2}mB^{-1} \le \frac{M+m}{2} + m. \tag{2.9}$$

Compute

$$\begin{aligned}
&\left\| \frac{A+B}{2}\frac{M+m}{2}.mK^{\frac{1}{2}}\left(h_1'\right)(A\sharp B)^{-1}\right\| \\
&\quad\le \frac{1}{4}\left\| \frac{A+B}{2}+\frac{M+m}{2}.mK^{\frac{1}{2}}\left(h_1'\right)(A\sharp B)^{-1}\right\|^2 \quad \left(\text{by (2.1)}\right) \\
&\quad= \frac{1}{4}\left\| \frac{A+B}{2}+\frac{M+m}{2}.mK^{\frac{1}{2}}\left(h_1'\right)\left(A^{-1}\sharp B^{-1}\right)\right\|^2 \\
&\quad\le \frac{1}{4}\left\| \frac{A+B}{2}+\frac{M+m}{2}.m\frac{A^{-1}+B^{-1}}{2}\right\|^2 \quad \left(\text{by (2.3)}\right) \\
&\quad\le \frac{1}{4}\left(\frac{M+m}{2}+m\right)^2 \quad \left(\text{by (2.8), (2.9)}\right).
\end{aligned}$$

That is,

$$\left\| \frac{A+B}{2}(A\sharp B)^{-1}\right\| \le \frac{(\frac{M+m}{2}+m)^2}{4\frac{M+m}{2}.mK^{\frac{1}{2}}(h_1')}.$$

Since $1 \le \sqrt{\frac{M}{m}} \le 2.314$, it follows that

$$\left(\sqrt{\frac{M}{m}}-1\right)^2\left[\left(\sqrt{\frac{M}{m}}\right)^3-\frac{2M}{m}+\sqrt{\frac{M}{m}}-4\right] \le 0. \tag{2.10}$$

It is easy to see that $\frac{(\frac{M+m}{2}+m)^2}{4\frac{M+m}{2}.m} \le \frac{M+m}{2\sqrt{Mm}}$ is equivalent to (2.10).
Thus

$$\left\| \frac{A+B}{2}(A\sharp B)^{-1}\right\| \le \frac{M+m}{2\sqrt{Mm}K^{\frac{1}{2}}(h_1')} = \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_1')}.$$

If $0 < \frac{M+m}{2} \le A \le m_2' < M_2' \le B \le M$, we get

$$A+\frac{M+m}{2}MA^{-1} \le \frac{M+m}{2}+M, \tag{2.11}$$

$$B+\frac{M+m}{2}MB^{-1} \le \frac{M+m}{2}+M. \tag{2.12}$$

Similarly, we have

$$\left\| \frac{A+B}{2}(A\sharp B)^{-1}\right\| \le \frac{(\frac{M+m}{2}+M)^2}{4\frac{M+m}{2}.MK^{\frac{1}{2}}(h_2')}. \tag{2.13}$$

Since $\frac{(\frac{M+m}{2}+M)^2}{4\frac{M+m}{2}.M} \le \frac{(\frac{M+m}{2}+m)^2}{4\frac{M+m}{2}.m} \le \frac{M+m}{2\sqrt{Mm}}$, so (2.13) becomes

$$\left\| \frac{A+B}{2}(A\sharp B)^{-1}\right\| \le \frac{M+m}{2\sqrt{Mm}K^{\frac{1}{2}}(h_2')} = \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_2')}.$$

If $0 < m \leq A \leq m_3' < \frac{M+m}{2} \leq B \leq M$, then we compute

$$\begin{aligned}
&\left\| \frac{A+B}{2} \frac{M+m}{2} . K^{\frac{1}{2}}(h_3') \sqrt{mM} (A \sharp B)^{-1} \right\| \\
&\quad \leq \frac{1}{4} \left\| \frac{A+B}{2} + \frac{M+m}{2} . K^{\frac{1}{2}}(h_3') \sqrt{mM} (A \sharp B)^{-1} \right\|^2 \quad (\text{by}(2.1)) \\
&\quad = \frac{1}{4} \left\| \frac{A+B}{2} + \frac{M+m}{2} . K^{\frac{1}{2}}(h_3') \sqrt{mM} (A^{-1} \sharp B^{-1}) \right\|^2 \\
&\quad = \frac{1}{4} \left\| \frac{A+B}{2} + \frac{M+m}{2} . K^{\frac{1}{2}}(h_3') (mA^{-1} \sharp MB^{-1}) \right\|^2 \\
&\quad \leq \frac{1}{4} \left\| \frac{A+B}{2} + \frac{M+m}{2} . \frac{mA^{-1} + MB^{-1}}{2} \right\|^2 \quad (\text{by } (2.3)) \\
&\quad \leq \frac{1}{4} (M+m)^2 \quad (\text{by } (2.8), (2.12)),
\end{aligned} \tag{2.14}$$

so we have

$$\left\| \frac{A+B}{2} (A \sharp B)^{-1} \right\| \leq \frac{(M+m)^2}{4 \frac{M+m}{2} \sqrt{Mm} K^{\frac{1}{2}}(h_3')} = \frac{M+m}{2\sqrt{Mm} K^{\frac{1}{2}}(h_3')} = \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_3')}.$$

If $0 < m \leq A \leq \frac{M+m}{2} < M_3' \leq B \leq M$, similarly, by (2.1), (2.3), (2.8) and (2.12), we have

$$\left\| \frac{A+B}{2} (A \sharp B)^{-1} \right\| \leq \frac{M+m}{2\sqrt{Mm} K^{\frac{1}{2}}(h_4')} = \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_4')}.$$

This completes the proof. □

Remark 2.6 Because $\frac{K(h)}{K(h_1')} < K(h)$, $\frac{K(h)}{K(h_2')} < K(h)$, $\frac{K(h)}{K(h_3')} < K(h)$ and $\frac{K(h)}{K(h_4')} < K(h)$, so Theorem 2.5 is a refinement of the inequality (1.11).

Remark 2.7 Since t^p is operator monotone function for $0 \leq p \leq 1$, so, by taking power $\frac{1}{2}$ both sides of (2.4), (2.5), (2.6) and (2.7), respectively, we can easily get a refinement of the inequality (1.10) for the condition $\sqrt{\frac{M}{m}} \leq 2.314$.

Theorem 2.8 *Let* $0 < m \leq M$ *and* $\sqrt{\frac{M}{m}} \leq 2.314$, *we have*

(1) *If* $0 < m \leq A \leq m_1' < M_1' \leq B \leq \frac{M+m}{2}$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{K(h)}{K(h_1')} \Phi^2(A \sharp B), \tag{2.15}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_1') = \frac{(h_1'+1)^2}{4h_1'}$, $h = \frac{M}{m}$ *and* $h_1' = \frac{M_1'}{m_1'}$.

(2) *If* $0 < \frac{M+m}{2} \leq A \leq m_2' < M_2' \leq B \leq M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{K(h)}{K(h_2')} \Phi^2(A \sharp B), \tag{2.16}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_2') = \frac{(h_2'+1)^2}{4h_2'}$, $h = \frac{M}{m}$ *and* $h_2' = \frac{M_2'}{m_2'}$.

(3) *If* $0 < m \leq A \leq m_3' < \frac{M+m}{2} \leq B \leq M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{K(h)}{K(h_3')}\Phi^2(A\sharp B), \tag{2.17}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_3') = \frac{(h_3'+1)^2}{4h_3'}$, $h = \frac{M}{m}$ *and* $h_3' = \frac{M+m}{2m_3'}$.

(4) *If* $0 < m \leq A \leq \frac{M+m}{2} < M_3' \leq B \leq M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{K(h)}{K(h_4')}\Phi^2(A\sharp B), \tag{2.18}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_4') = \frac{(h_4'+1)^2}{4h_4'}$, $h = \frac{M}{m}$ *and* $h_4' = \frac{2M_3'}{M+m}$.

Proof Inequality (2.15) is equivalent to

$$\left\|\Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A\sharp B)\right\| \leq \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_1')}.$$

If $0 < m \leq A \leq m_1' < M_1' \leq B \leq \frac{M+m}{2}$, then we compute

$$\begin{aligned}
&\left\|\Phi\left(\frac{A+B}{2}\right)\frac{M+m}{2}.mK^{\frac{1}{2}}(h_1')\Phi^{-1}(A\sharp B)\right\| \\
&\quad\leq \frac{1}{4}\left\|\Phi\left(\frac{A+B}{2}\right)+\frac{M+m}{2}.mK^{\frac{1}{2}}(h_1')\Phi^{-1}(A\sharp B)\right\|^2 \quad \bigl(\text{by (2.1)}\bigr) \\
&\quad\leq \frac{1}{4}\left\|\Phi\left(\frac{A+B}{2}\right)+\frac{M+m}{2}.mK^{\frac{1}{2}}(h_1')\Phi\bigl((A\sharp B)^{-1}\bigr)\right\|^2 \quad \bigl(\text{by (2.2)}\bigr) \\
&\quad= \frac{1}{4}\left\|\Phi\left(\frac{A+B}{2}+\frac{M+m}{2}.mK^{\frac{1}{2}}(h_1')\bigl(A^{-1}\sharp B^{-1}\bigr)\right)\right\|^2 \\
&\quad\leq \frac{1}{4}\left\|\Phi\left(\frac{A+B}{2}+\frac{M+m}{2}.m\frac{A^{-1}+B^{-1}}{2}\right)\right\|^2 \quad \bigl(\text{by (2.3)}\bigr) \\
&\quad\leq \frac{1}{4}\left(\frac{M+m}{2}+m\right)^2 \quad \bigl(\text{by (2.8), (2.9)}\bigr),
\end{aligned} \tag{2.19}$$

that is,

$$\left\|\Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A\sharp B)\right\| \leq \frac{(\frac{M+m}{2}+m)^2}{4\frac{M+m}{2}.mK^{\frac{1}{2}}(h_1')}. \tag{2.20}$$

By $1 \leq \sqrt{\frac{M}{m}} \leq 2.314$ and (2.10), we have

$$\left\|\Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A\sharp B)\right\| \leq \frac{M+m}{2\sqrt{Mm}K^{\frac{1}{2}}(h_1')} = \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_1')}.$$

If $0 < \frac{M+m}{2} \leq A \leq m_2' < M_2' \leq B \leq M$, similarly, by (2.1), (2.2), (2.3), (2.11), (2.12), $\frac{(\frac{M+m}{2}+M)^2}{4\frac{M+m}{2}M} \leq \frac{(\frac{M+m}{2}+m)^2}{4\frac{M+m}{2}m}$ and (2.10), we obtain

$$\left\| \Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A\sharp B)\right\| \leq \frac{M+m}{2\sqrt{Mm}K^{\frac{1}{2}}(h_2')} = \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_2')}.$$

If $0 < m \leq A \leq m_3' < \frac{M+m}{2} \leq B \leq M$, then we compute

$$\begin{aligned}
&\left\| \Phi\left(\frac{A+B}{2}\right)\frac{M+m}{2}.K^{\frac{1}{2}}(h_3')\sqrt{mM}\Phi^{-1}(A\sharp B)\right\| \\
&\quad \leq \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2}\right) + \frac{M+m}{2}.K^{\frac{1}{2}}(h_3')\sqrt{mM}\Phi^{-1}(A\sharp B)\right\|^2 \quad (\text{by } (2.1)) \\
&\quad \leq \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2}\right) + \frac{M+m}{2}.K^{\frac{1}{2}}(h_3')\sqrt{mM}\Phi\left((A\sharp B)^{-1}\right)\right\|^2 \quad (\text{by } (2.2)) \\
&\quad = \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2} + \frac{M+m}{2}.K^{\frac{1}{2}}(h_3')\sqrt{mM}\left(A^{-1}\sharp B^{-1}\right)\right)\right\|^2 \\
&\quad = \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2} + \frac{M+m}{2}.K^{\frac{1}{2}}(h_3')\left(mA^{-1}\sharp MB^{-1}\right)\right)\right\|^2 \\
&\quad \leq \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2} + \frac{M+m}{2}.\frac{mA^{-1}+MB^{-1}}{2}\right)\right\|^2 \quad (\text{by } (2.3)) \\
&\quad \leq \frac{(M+m)^2}{4} \quad (\text{by } (2.8), (2.12)),
\end{aligned} \tag{2.21}$$

that is, we have

$$\left\| \Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A\sharp B)\right\| \leq \frac{(M+m)^2}{4\frac{M+m}{2}\sqrt{Mm}K^{\frac{1}{2}}(h_3')} = \frac{M+m}{2\sqrt{Mm}K^{\frac{1}{2}}(h_3')} = \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_3')}.$$

If $0 < m \leq A \leq \frac{M+m}{2} < M_3' \leq B \leq M$, similarly, by (2.1), (2.2), (2.3), (2.8) and (2.12), we have

$$\left\| \Phi\left(\frac{A+B}{2}\right)\Phi^{-1}(A\sharp B)\right\| \leq \frac{M+m}{2\sqrt{Mm}K^{\frac{1}{2}}(h_4')} = \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_4')}.$$

It completes the proof. □

Remark 2.9 Obviously, Theorem 2.8 is a refinement of (1.12).

Theorem 2.10 *Let* $0 < m \leq M$ *and* $\sqrt{\frac{M}{m}} \leq 2.314$, *we have*

(1) *If* $0 < m \leq A \leq m_1' < M_1' \leq B \leq \frac{M+m}{2}$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{K(h)}{K(h_1')}\left(\Phi(A)\sharp\Phi(B)\right)^2, \tag{2.22}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_1') = \frac{(h_1'+1)^2}{4h_1'}$, $h = \frac{M}{m}$ *and* $h_1' = \frac{M_1'}{m_1'}$.

(2) *If* $0 < \frac{M+m}{2} \leq A \leq m_2' < M_2' \leq B \leq M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{K(h)}{K(h_2')}\big(\Phi(A)\sharp\Phi(B)\big)^2, \tag{2.23}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_2') = \frac{(h_2'+1)^2}{4h_2'}$, $h = \frac{M}{m}$ *and* $h_2' = \frac{M_2'}{m_2'}$.

(3) *If* $0 < m \leq A \leq m_3' < \frac{M+m}{2} \leq B \leq M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{K(h)}{K(h_3')}\big(\Phi(A)\sharp\Phi(B)\big)^2, \tag{2.24}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_3') = \frac{(h_3'+1)^2}{4h_3'}$, $h = \frac{M}{m}$ *and* $h_3' = \frac{M+m}{2m_3'}$.

(4) *If* $0 < m \leq A \leq \frac{M+m}{2} < M_3' \leq B \leq M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{K(h)}{K(h_4')}\big(\Phi(A)\sharp\Phi(B)\big)^2, \tag{2.25}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h_4') = \frac{(h_4'+1)^2}{4h_4'}$, $h = \frac{M}{m}$ *and* $h_4' = \frac{2M_3'}{M+m}$.

Proof Inequality (2.22) is equivalent to

$$\left\|\Phi\left(\frac{A+B}{2}\right)\big(\Phi(A)\sharp\Phi(B)\big)^{-1}\right\| \leq \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_1')}.$$

If $0 < m \leq A \leq m_1' < M_1' \leq B \leq \frac{M+m}{2}$, then we compute

$$\begin{aligned}
&\left\|\Phi\left(\frac{A+B}{2}\right)\frac{M+m}{2}.mK^{\frac{1}{2}}\big(h_1'\big)\big(\Phi(A)\sharp\Phi(B)\big)^{-1}\right\| \\
&\quad\leq \frac{1}{4}\|\Phi\left(\frac{A+B}{2}\right) + \frac{M+m}{2}.mK^{\frac{1}{2}}\big(h_1'\big)\big(\Phi(A)\sharp\Phi B\big)^{-1}\|^2 \quad \big(\text{by } (2.1)\big) \\
&\quad\leq \frac{1}{4}\left\|\Phi\left(\frac{A+B}{2}\right) + \frac{M+m}{2}.mK^{\frac{1}{2}}\big(h_1'\big)\Phi^{-1}(A\sharp B)\right\|^2 \quad \big(\text{by } (1.6)\big) \\
&\quad\leq \frac{1}{4}\left(\frac{M+m}{2}+m\right)^2 \quad \big(\text{by } (2.19)\big),
\end{aligned}$$

that is,

$$\left\|\Phi\left(\frac{A+B}{2}\right)\big(\Phi(A)\sharp\Phi(B)\big)^{-1}\right\| \leq \frac{(\frac{M+m}{2}+m)^2}{4\frac{M+m}{2}.mK^{\frac{1}{2}}(h_1')}.$$

By $1 \leq \sqrt{\frac{M}{m}} \leq 2.314$ and (2.10), we have

$$\left\|\Phi\left(\frac{A+B}{2}\right)\Phi\big((A)\sharp\Phi(B)\big)^{-1}\right\| \leq \frac{M+m}{2\sqrt{Mm}K^{\frac{1}{2}}(h_1')} = \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_1')}.$$

Since $\frac{(\frac{M+m}{2}+M)^2}{4\frac{M+m}{2}M} \leq \frac{(\frac{M+m}{2}+m)^2}{4\frac{M+m}{2}m}$, by 2nd case $0 < \frac{M+m}{2} \leq A \leq m_2' < M_2' \leq B \leq M$, we can easily obtain the inequality (2.23).

If $0 < m \le A \le m_3' < \frac{M+m}{2} \le B \le M$, then we compute

$$\begin{aligned}
&\left\| \Phi\left(\frac{A+B}{2}\right)\frac{M+m}{2}.K^{\frac{1}{2}}(h_3')\sqrt{mM}\big(\Phi(A)\sharp\Phi(B)\big)^{-1}\right\| \\
&\quad\le \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2}\right)+\frac{M+m}{2}.K^{\frac{1}{2}}(h_3')\sqrt{mM}\big(\Phi(A)\sharp\Phi(B)\big)^{-1}\right\|^2 \quad \big(\text{by (2.1)}\big) \\
&\quad\le \frac{1}{4}\left\| \Phi\left(\frac{A+B}{2}\right)+\frac{M+m}{2}.K^{\frac{1}{2}}(h_3')\sqrt{mM}\Phi^{-1}(A\sharp B)\right\|^2 \quad \big(\text{by (1.6)}\big) \\
&\quad\le \frac{1}{4}(M+m)^2 \quad \big(\text{by (2.21)}\big),
\end{aligned}$$

thus, we have

$$\left\| \Phi\left(\frac{A+B}{2}\right)\big(\Phi(A)\sharp\Phi(B)\big)^{-1}\right\| \le \frac{M+m}{2\sqrt{Mm}K^{\frac{1}{2}}(h_3')} = \frac{K^{\frac{1}{2}}(h)}{K^{\frac{1}{2}}(h_3')}.$$

The proof of (2.25) is similar to (2.24), we omit the details.

This completes the proof. □

Remark 2.11 Clearly Theorem 2.10 is a refinement of (1.13).

By (1.3) and Theorem 2.8, we obtain the following refinement of inequality (1.14).

Corollary 2.12 *Let* $0 < m \le M$ *and* $\sqrt{\frac{M}{m}} \le 2.314$, *we have*

(1) *If* $0 < m \le A \le m_1' < M_1' \le B \le \frac{M+m}{2}$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \le \frac{S^2(h)}{K(h_1')}\Phi^2(A\sharp B),$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$, $K(h_1') = \frac{(h_1'+1)^2}{4h_1'}$, $h = \frac{M}{m}$ *and* $h_1' = \frac{M_1'}{m_1'}$.

(2) *If* $0 < \frac{M+m}{2} \le A \le m_2' < M_2' \le B \le M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \le \frac{S^2(h)}{K(h_2')}\Phi^2(A\sharp B),$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$, $K(h_2') = \frac{(h_2'+1)^2}{4h_2'}$, $h = \frac{M}{m}$ *and* $h_2' = \frac{M_2'}{m_2'}$.

(3) *If* $0 < m \le A \le m_3' < \frac{M+m}{2} \le B \le M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \le \frac{S^2(h)}{K(h_3')}\Phi^2(A\sharp B),$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$, $K(h_3') = \frac{(h_3'+1)^2}{4h_3'}$, $h = \frac{M}{m}$ *and* $h_3' = \frac{M+m}{2m_3'}$.

(4) *If* $0 < m \le A \le \frac{M+m}{2} < M_3' \le B \le M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \le \frac{S^2(h)}{K(h_4')}\Phi^2(A\sharp B),$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$, $K(h_4') = \frac{(h_4'+1)^2}{4h_4'}$, $h = \frac{M}{m}$ *and* $h_4' = \frac{2M_3'}{M+m}$.

By (1.3) and Theorem 2.10, we obtain the following refinement of the inequality (1.15).

Corollary 2.13 *Let* $0 < m \leq M$ *and* $\sqrt{\frac{M}{m}} \leq 2.314$, *we have*

(1) *If* $0 < m \leq A \leq m_1' < M_1' \leq B \leq \frac{M+m}{2}$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{S^2(h)}{K(h_1')}\big(\Phi(A)\sharp\Phi(B)\big)^2,$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$, $K(h_1') = \frac{(h_1'+1)^2}{4h_1'}$, $h = \frac{M}{m}$ *and* $h_1' = \frac{M_1'}{m_1'}$.

(2) *If* $0 < \frac{M+m}{2} \leq A \leq m_2' < M_2' \leq B \leq M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{S^2(h)}{K(h_2')}\big(\Phi(A)\sharp\Phi(B)\big)^2,$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$, $K(h_2') = \frac{(h_2'+1)^2}{4h_2'}$, $h = \frac{M}{m}$ *and* $h_2' = \frac{M_2'}{m_2'}$.

(3) *If* $0 < m \leq A \leq m_3' < \frac{M+m}{2} \leq B \leq M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{S^2(h)}{K(h_3')}\big(\Phi(A)\sharp\Phi(B)\big)^2,$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$, $K(h_3') = \frac{(h_3'+1)^2}{4h_3'}$, $h = \frac{M}{m}$ *and* $h_3' = \frac{M+m}{2m_3'}$.

(4) *If* $0 < m \leq A \leq \frac{M+m}{2} < M_3' \leq B \leq M$, *then*

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \frac{S^2(h)}{K(h_4')}(\Phi(A)\sharp\Phi(B)^2,$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$, $K(h_4') = \frac{(h_4'+1)^2}{4h_4'}$, $h = \frac{M}{m}$ *and* $h_4' = \frac{2M_3'}{M+m}$.

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