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On weak sharp solutions in $(\rho, \mathbf{b}, \mathbf{d})$-variational inequalities

Savin Treanţă[1*]

*Correspondence:
savin_treanta@yahoo.com
[1]Faculty of Applied Sciences, Department of Applied Mathematics, University Politehnica of Bucharest, Bucharest, Romania

Abstract

In this paper, weak sharp solutions are investigated for a variational-type inequality governed by $(\rho, \mathbf{b}, \mathbf{d})$-convex path-independent curvilinear integral functional. Moreover, an equivalence between the *minimum principle sufficiency property* and the *weak sharpness property* of the solution set associated with the considered variational-type inequality is established.

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Keywords: Weak sharp solutions; Variational-type inequality; $(\rho, \mathbf{b}, \mathbf{d})$-convex path-independent curvilinear integral functional

1 Introduction

Based on the works of Burke and Ferris [3], Patriksson [11] and following Marcotte and Zhu [10], the concept of *weak sharp solution* associated with variational-type inequalities has attracted the attention of many researchers (see, for instance, Hu and Song [7], Liu and Wu [9], Zhu [17] and Jayswal and Singh [8]). Recently, by using gap-type functions, in accordance with Ferris and Mangasarian [5] and following Hiriart-Urruty and Lemaréchal [6], Alshahrani et al. [1] studied the minimum and maximum principle sufficiency properties associated with nonsmooth variational inequalities.

In this paper, motivated and inspired by the ongoing research in this field and by using some variational techniques developed in Ansari [2], Clarke [4] and Treanţă [12–16], we investigate a new class of variational-type inequalities governed by $(\rho, \mathbf{b}, \mathbf{d})$-*convex path-independent curvilinear integral functionals* (a new concept introduced in Treanţă [16]). The extended concept of a *normal cone* (see Treanţă [16]), firstly introduced by Marcotte and Zhu [10], plays a crucial role in our investigations. More precisely, under some working assumptions and using a dual gap-type functional, the weak sharpness property of the solution set for the considered variational-type inequality is studied. In this regard, two characterization results are formulated and proved.

The present paper is organized as follows. Section 2 contains notations, problem description and some auxiliary results. The main results of this paper are included in Sect. 3. Concretely, weak sharp solutions are investigated for an extended variational-type inequality involving $(\rho, \mathbf{b}, \mathbf{d})$-convex path-independent curvilinear integral functional. Finally, Sect. 4 concludes this study.

2 Preliminaries

In this paper, in order to introduce our study, consider the following notations and mathematical objects:

- ► $\Theta \subset \mathbb{R}^m$ is a compact domain and the point $\Theta \ni t = (t^\beta)$, $\beta = \overline{1,m}$, is considered as a multiple parameter of evolution;
- ► consider $\Theta \supset \Gamma : t = t(\tau)$, $\tau \in [a,b]$, a piecewise smooth curve joining the different points $t_1 = (t_1^1, \dots, t_1^m)$, $t_2 = (t_2^1, \dots, t_2^m)$ in Θ;
- ► let $\overline{\mathcal{L}}$ be the space of piecewise smooth functions $x : \Theta \to \mathbb{R}^n$, endowed with the Euclidean inner product

$$\begin{aligned}\langle x, y\rangle &= \int_\Gamma x(t)\cdot y(t)\,dt^\beta = \int_\Gamma \sum_{i=1}^n x^i(t)y^i(t)\,dt^\beta \\ &= \int_\Gamma \sum_{i=1}^n x^i(t)y^i(t)\,dt^1 + \cdots + \int_\Gamma \sum_{i=1}^n x^i(t)y^i(t)\,dt^m, \quad \forall x, y \in \overline{\mathcal{L}}\end{aligned}$$

  and the induced norm;
- ► denote by $\mathcal{L}$ a nonempty, closed and convex subset of $\overline{\mathcal{L}}$, defined as

$$\mathcal{L} = \bigl\{x \in \overline{\mathcal{L}} : x(t) \in E \subset \mathbb{R}^n, x(t_1) = x_1 = \text{given}, x(t_2) = x_2 = \text{given}\bigr\};$$

- ► throughout this paper, the summation over the repeated indices is assumed and x, x_α are the simplified notations for $x(t)$, $x_\alpha(t)$ and $x_\alpha(t) = \frac{\partial x}{\partial t^\alpha}(t)$;
- ► consider the real-valued continuously differentiable functions (closed Lagrange 1-form densities)

$$f_\beta, g_\beta, h_\beta : J^1\bigl(\mathbb{R}^m, \mathbb{R}^n\bigr) \to \mathbb{R}, \quad \beta = \overline{1,m},$$

  (see $J^1(\mathbb{R}^m, \mathbb{R}^n)$ as the first-order jet bundle associated to $\mathbb{R}^m$ and $\mathbb{R}^n$) which generate the following path-independent curvilinear integral functionals:

$$\begin{aligned}&F : \overline{\mathcal{L}} \to \mathbb{R}, \quad F(x) = \int_\Gamma f_\beta(t, x, x_\alpha)\,dt^\beta, \\ &G : \overline{\mathcal{L}} \to \mathbb{R}, \quad G(x) = \int_\Gamma g_\beta(t, x, x_\alpha)\,dt^\beta, \\ &H : \overline{\mathcal{L}} \to \mathbb{R}, \quad H(x) = \int_\Gamma h_\beta(t, x, x_\alpha)\,dt^\beta.\end{aligned}$$

Let ρ be a real number, $\mathbf{b}(x,y)$ a symmetric positive real-valued functional on $\overline{\mathcal{L}} \times \overline{\mathcal{L}}$ and $\mathbf{d}(x,y)$ a real-valued functional on $\overline{\mathcal{L}} \times \overline{\mathcal{L}}$.

Definition 2.1

(i) The scalar functional $F : \overline{\mathcal{L}} \to \mathbb{R}$, $F(x) = \int_\Gamma f_\beta(t, x, x_\alpha)\,dt^\beta$, is called $(\rho, \mathbf{b}, \mathbf{d})$-*convex* on $\mathcal{L}$ if, for any $x, y \in \mathcal{L}$,

$$\begin{aligned}F(x) - F(y) \geq \mathbf{b}(x,y) &\int_\Gamma \left[\frac{\partial f_\beta}{\partial x}(t, y, y_\alpha)(x-y) + \frac{\partial f_\beta}{\partial x_\alpha}(t, y, y_\alpha)D_\alpha(x-y)\right]dt^\beta \\ &+ \rho\mathbf{b}(x,y)\mathbf{d}(x,y),\end{aligned}$$

where D_α denotes the total derivative operator.

(ii) The functional F is said to be *strongly* $\mathbf{b}$*-convex*, $\mathbf{b}$*-convex*, or *weakly* $\mathbf{b}$*-convex* on $\mathcal{L}$, according to $\rho\mathbf{d} > 0$, $\rho\mathbf{d} = 0$, or $\rho\mathbf{d} < 0$.

Definition 2.2 The *variational (functional) derivative* $\frac{\delta_\beta F}{\delta x}$ of the path-independent curvilinear integral functional $F : \overline{\mathcal{L}} \to \mathbb{R}$, $F(x) = \int_\Gamma f_\beta(t, x, x_\alpha)\, dt^\beta$, is defined as

$$\frac{\delta_\beta F}{\delta x} = \frac{\partial f_\beta}{\partial x}(t, x, x_\alpha) - D_\alpha \frac{\partial f_\beta}{\partial x_\alpha}(t, x, x_\alpha) \in \overline{\mathcal{L}}$$

and, for any $\psi \in \overline{\mathcal{L}}$ with $\psi(t_1) = \psi(t_2) = 0$, it satisfies the following relation:

$$\left\langle \frac{\delta_\beta F}{\delta x}, \psi \right\rangle = \int_\Gamma \frac{\delta_\beta F}{\delta x}(t) \cdot \psi(t)\, dt^\beta = \lim_{\varepsilon \to 0} \frac{F(x + \varepsilon\psi) - F(x)}{\varepsilon}.$$

Throughout this paper, it is assumed that the inner product between the variational derivative associated with a path-independent curvilinear integral functional and an element $\psi \in \overline{\mathcal{L}}$ is accompanied by the condition $\psi(t_1) = \psi(t_2) = 0$.

By using the previous mathematical tools, we formulate the following *extended variational-type inequality problem*: for some given ρ, $\mathbf{b}$, $\mathbf{d}$ (introduced as above), find $y \in \mathcal{L}$ such that

$$\text{(EVIP)} \qquad \mathbf{b}(x,y) \int_\Gamma \left[\frac{\partial f_\beta}{\partial x}(t, y, y_\alpha)(x - y) + \frac{\partial f_\beta}{\partial x_\alpha}(t, y, y_\alpha) D_\alpha (x - y) \right] dt^\beta$$
$$+ \rho \mathbf{b}(x,y)\mathbf{d}(x,y) \geq 0,$$

for any $x \in \mathcal{L}$. The *dual extended variational-type inequality problem* associated to (EVIP) is formulated as follows: for some given ρ, $\mathbf{b}$, $\mathbf{d}$ (introduced as above), find $y \in \mathcal{L}$ such that

$$\text{(DEVIP)} \qquad \mathbf{b}(x,y) \int_\Gamma \left[\frac{\partial f_\beta}{\partial x}(t, x, x_\alpha)(x - y) + \frac{\partial f_\beta}{\partial x_\alpha}(t, x, x_\alpha) D_\alpha (x - y) \right] dt^\beta$$
$$+ \rho \mathbf{b}(x,y)\mathbf{d}(x,y) \geq 0,$$

for any $x \in \mathcal{L}$.

Denote by $\mathcal{L}^*$ and $\mathcal{L}_*$ the solution set associated with (EVIP) and (DEVIP), respectively, and assume they are nonempty.

Remark 2.1 As can be easily seen, the above extended variational-type inequality problems can be reformulated as follows: *for some given* ρ, $\mathbf{b}$, $\mathbf{d}$ *(introduced as above), find* $y \in \mathcal{L}$ *such that*

$$\text{(EVIP)} \qquad \mathbf{b}(x,y) \left[\left\langle \frac{\delta_\beta F}{\delta y}, x - y \right\rangle + \rho \mathbf{d}(x,y) \right] \geq 0, \quad \forall x \in \mathcal{L},$$

respectively: *for some given* ρ, $\mathbf{b}$, $\mathbf{d}$ *(introduced as above), find* $y \in \mathcal{L}$ *such that*

$$\text{(DEVIP)} \qquad \mathbf{b}(x,y) \left[\left\langle \frac{\delta_\beta F}{\delta x}, x - y \right\rangle + \rho \mathbf{d}(x,y) \right] \geq 0, \quad \forall x \in \mathcal{L}$$

if and only if

$$dU := D_\alpha\left[\frac{\partial f_\beta}{\partial x_\alpha}(x-y)\right]dt^\beta$$

is an exact total differential and it is satisfied the condition $U(t_1) = U(t_2)$. Throughout this paper, this working hypothesis is assumed.

Further, in order to investigate the solution set $\mathcal{L}^*$, we introduce the following gap functionals.

Definition 2.3 For $x \in \overline{\mathcal{L}}$, the *primal gap functional* associated to (EVIP) is defined as

$$\begin{aligned}G(x) = \max_{y\in\mathcal{L}}\bigg\{&\mathbf{b}(x,y)\int_\Gamma\left[\frac{\partial f_\beta}{\partial x}(t,x,x_\alpha)(x-y) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x,x_\alpha)D_\alpha(x-y)\right]dt^\beta\\ &+ \rho\mathbf{b}(x,y)\mathbf{d}(x,y)\bigg\}\end{aligned}$$

and, similarly, the *dual gap functional* associated to (EVIP) is defined as

$$\begin{aligned}H(x) = \max_{y\in\mathcal{L}}\bigg\{&\mathbf{b}(x,y)\int_\Gamma\left[\frac{\partial f_\beta}{\partial x}(t,y,y_\alpha)(x-y) + \frac{\partial f_\beta}{\partial x_\alpha}(t,y,y_\alpha)D_\alpha(x-y)\right]dt^\beta\\ &+ \rho\mathbf{b}(x,y)\mathbf{d}(x,y)\bigg\}.\end{aligned}$$

From now onwards, for $x \in \overline{\mathcal{L}}$, consider the following notations:

$$\begin{aligned}Q(x) = \bigg\{&z \in \mathcal{L} : G(x) = \mathbf{b}(x,z)\int_\Gamma\left[\frac{\partial f_\beta}{\partial x}(t,x,x_\alpha)(x-z) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x,x_\alpha)D_\alpha(x-z)\right]dt^\beta\\ &+ \rho\mathbf{b}(x,z)\mathbf{d}(x,z)\bigg\},\\ R(x) = \bigg\{&z \in \mathcal{L} : H(x) = \mathbf{b}(x,z)\int_\Gamma\left[\frac{\partial f_\beta}{\partial x}(t,z,z_\alpha)(x-z) + \frac{\partial f_\beta}{\partial x_\alpha}(t,z,z_\alpha)D_\alpha(x-z)\right]dt^\beta\\ &+ \rho\mathbf{b}(x,z)\mathbf{d}(x,z)\bigg\}.\end{aligned}$$

Remark 2.2 By using the previous notations, we can observe the following:

(i)

$$\begin{aligned}G(x) &= \max_{y\in\mathcal{L}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta x}, x-y\right\rangle + \rho\mathbf{d}(x,y)\right]\right\},\\ H(x) &= \max_{y\in\mathcal{L}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta y}, x-y\right\rangle + \rho\mathbf{d}(x,y)\right]\right\};\end{aligned}$$

(ii) $Q(x) = \arg\max_{y\in\mathcal{L}}\{\mathbf{b}(x,y)[\langle\frac{\delta_\beta F}{\delta x}, x-y\rangle + \rho\mathbf{d}(x,y)]\}$, where

$$\arg\max_{y\in\mathcal{L}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta x}, x-y\right\rangle + \rho\mathbf{d}(x,y)\right]\right\}$$

denotes the (possibly empty) solution set of

$$\max_{y\in\mathcal{L}}\left\{\mathbf{b}(x,y)\left[\left\langle\frac{\delta_\beta F}{\delta x},x-y\right\rangle+\rho\mathbf{d}(x,y)\right]\right\};$$

(iii) $R(x)=\arg\max_{y\in\mathcal{L}}\{\mathbf{b}(x,y)[\langle\frac{\delta_\beta F}{\delta y},x-y\rangle+\rho\mathbf{d}(x,y)]\}$;

(iv) if $Q(x)=\emptyset$, then $G(x)=\sup_{y\in\mathcal{L}}\{\mathbf{b}(x,y)[\langle\frac{\delta_\beta F}{\delta x},x-y\rangle+\rho\mathbf{d}(x,y)]\}$; similarly, if $R(x)=\emptyset$, then $H(x)=\sup_{y\in\mathcal{L}}\{\mathbf{b}(x,y)[\langle\frac{\delta_\beta F}{\delta y},x-y\rangle+\rho\mathbf{d}(x,y)]\}$.

In order to formulate and prove the main results of this paper, in accordance with Marcotte and Zhu [10], we introduce the following relevant concepts.

Definition 2.4 The *polar set* $\mathcal{L}^\circ$ associated to $\mathcal{L}$ is defined as follows:

$$\mathcal{L}^\circ=\left\{y\in\overline{\mathcal{L}}:\langle y,x\rangle\le 0,\forall x\in\mathcal{L}\right\}.$$

Definition 2.5 The *projection* of a point $x\in\overline{\mathcal{L}}$ onto the set $\mathcal{L}$ is defined as

$$\operatorname{proj}_{\mathcal{L}}x=\arg\min_{y\in\mathcal{L}}\|x-y\|.$$

Definition 2.6 The *normal cone to* $\mathcal{L}$ *at* $x\in\overline{\mathcal{L}}$, *with respect to* ρ, $\mathbf{b}$ *and* $\mathbf{d}$ (introduced as above), is defined as

$$N_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(x)=\left\{y\in\overline{\mathcal{L}}:\mathbf{b}(z,x)\left[\langle y,z-x\rangle-\rho\mathbf{d}(z,x)\right]\le 0,\forall z\in\mathcal{L}\right\},\quad x\in\mathcal{L},$$
$$N_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(x)=\emptyset,\quad x\notin\mathcal{L}$$

and the *tangent cone to* $\mathcal{L}$ *at* $x\in\overline{\mathcal{L}}$, *with respect to* ρ, $\mathbf{b}$ *and* $\mathbf{d}$ (introduced as above), is $T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(x)=[N_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(x)]^\circ$.

Remark 2.3 Taking into account the definition of normal cone at $x\in\overline{\mathcal{L}}$, we notice that: $x^*\in\mathcal{L}^*\iff-\frac{\delta_\beta F}{\delta x^*}\in N_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(x^*)$.

Further, we establish some working assumptions and auxiliary results.

Working hypotheses

(i) The equalities

$$\mathbf{d}\left(x^1,x^2\right)=-\mathbf{d}\left(x^2,x^1\right),\quad\forall x^1,x^2\in\mathcal{L}^*,$$
$$\mathbf{d}\left(z,x^*\right)=-\mathbf{d}\left(x^*,z\right),\quad\forall z\in\mathcal{L},\forall x^*\in\mathcal{L}^*,$$

are fulfilled.

(ii) For any $y\in R(x)$ and $x,z\in\overline{\mathcal{L}}$, the relations

$$\mathbf{b}(z,y)(z-y)-\mathbf{b}(x,y)(x-y)=z-x,\qquad\mathbf{b}(z,y)\mathbf{d}(z,y)-\mathbf{b}(x,y)\mathbf{d}(x,y)=\mathbf{d}(z,x)$$

are true.

(iii) For any $x, \nu \in \overline{\mathcal{L}}$ and $\lambda > 0$, there exists

$$\lim_{\lambda \to 0} \frac{\mathbf{d}(x + \lambda \nu, x)}{\lambda}.$$

(iv) For any $z \in R(x^*)$, $\bar{x} \in Q(x^*)$, $x^* \in \mathcal{L}^*$ and $x \in \mathcal{L}$, the relations

$$\mathbf{b}(x,z) = \mathbf{b}(x,x^*) = \mathbf{b}(z,x^*) = \mathbf{b}(\bar{x},x^*) \quad [= 1],$$
$$\mathbf{d}(x,z) = \mathbf{d}(x,x^*), \qquad \mathbf{d}(z,x) = \mathbf{d}(x^*,x) = \mathbf{d}(x^*,\bar{x})$$

are satisfied.

Proposition 2.1 (Treanţă [16]) *Assume the functional* $F(x) = \int_\Gamma f_\beta(t,x,x_\alpha)\,dt^\beta$ *is* $(\rho,\mathbf{b},\mathbf{d})$-*convex on* $\mathcal{L}$. *Then*:

(i) *for any* $x^1, x^2 \in \mathcal{L}^*$, *it follows*

$$\mathbf{b}(x^1,x^2) \int_\Gamma \left[\frac{\partial f_\beta}{\partial x}(t,x^2,x^2_\alpha)(x^1 - x^2) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x^2,x^2_\alpha) D_\alpha (x^1 - x^2) \right] dt^\beta$$
$$+ \rho \mathbf{b}(x^1,x^2)\mathbf{d}(x^1,x^2) = 0;$$

(ii) *the inclusion* $\mathcal{L}^* \subset \mathcal{L}_*$ *is true*.

Remark 2.4 The continuity property of the variational derivative $\frac{\delta_\beta F}{\delta x}$ implies $\mathcal{L}_* \subset \mathcal{L}^*$. By Proposition 2.1, we conclude $\mathcal{L}^* = \mathcal{L}_*$. As well, the solution set $\mathcal{L}_*$ associated to (DEVIP) is convex and, consequently, the solution set $\mathcal{L}^*$ associated to (EVIP) is a convex set.

3 Main results

In this section, weak sharp solutions are investigated for the considered extended variational-type inequality governed by $(\rho,\mathbf{b},\mathbf{d})$-convex path-independent curvilinear integral functional. In accordance with Ferris and Mangasarian [5], following Marcotte and Zhu [10], the weak sharpness property of the solution set $\mathcal{L}^*$ for (EVIP) is studied. In this regard, two characterization results are formulated and proved.

Definition 3.1 The solution set $\mathcal{L}^*$ associated to (EVIP) is called *weakly sharp* if

$$-\frac{\delta_\beta F}{\delta x^*} \in \operatorname{int}\left(\bigcap_{u \in \mathcal{L}^*} \left[T^{\rho,\mathbf{b},\mathbf{d}}_{\mathcal{L}}(u) \cap N^{\rho,\mathbf{b},\mathbf{d}}_{\mathcal{L}^*}(u) \right]^\circ \right), \quad \forall x^* \in \mathcal{L}^*,$$

or, equivalently, there exists a positive number $\gamma > 0$ such that

$$\gamma B \subset \frac{\delta_\beta F}{\delta x^*} + \left[T^{\rho,\mathbf{b},\mathbf{d}}_{\mathcal{L}}(x^*) \cap N^{\rho,\mathbf{b},\mathbf{d}}_{\mathcal{L}^*}(x^*) \right]^\circ, \quad \forall x^* \in \mathcal{L}^*,$$

where $\operatorname{int}(S)$ stands for interior of the set S and B denotes the open unit ball in $\overline{\mathcal{L}}$.

Lemma 3.1 *There exists a positive number* $\gamma > 0$ *such that*

$$\gamma B \subset \frac{\delta_\beta F}{\delta y} + \left[T^{\rho,\mathbf{b},\mathbf{d}}_{\mathcal{L}}(y) \cap N^{\rho,\mathbf{b},\mathbf{d}}_{\mathcal{L}^*}(y) \right]^\circ, \quad \forall y \in \mathcal{L}^*, \tag{3.1}$$

if and only if

$$\left\langle \frac{\delta_\beta F}{\delta y}, z \right\rangle \geq \gamma \|z\|, \quad \forall z \in T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(y) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(y), z(t_1) = z(t_2) = 0. \tag{3.2}$$

Proof Relation (3.1) is equivalent with

$$\gamma b - \frac{\delta_\beta F}{\delta y} \in \left[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(y) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(y)\right]^\circ, \quad \forall y \in \mathcal{L}^*, \forall b \in B,$$

or

$$\left\langle \gamma b - \frac{\delta_\beta F}{\delta y}, z \right\rangle \leq 0, \quad \forall y \in \mathcal{L}^*, \forall b \in B, \forall z \in T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(y) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(y), z(t_1) = z(t_2) = 0.$$

Considering $B \ni b = \frac{z}{\|z\|}$, $z \neq 0$, the previous inequality becomes (3.2).

Conversely, if Eq. (3.2) holds, then there exists a positive number $\gamma > 0$ such that

$$\begin{aligned} \left\langle \gamma b - \frac{\delta_\beta F}{\delta y}, z \right\rangle &= \langle \gamma b, z \rangle - \left\langle \frac{\delta_\beta F}{\delta y}, z \right\rangle \\ &\leq \gamma \|z\| - \gamma \|z\| = 0, \\ &\quad \forall y \in \mathcal{L}^*, \forall b \in B, \forall z \in T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(y) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(y), z(t_1) = z(t_2) = 0, \end{aligned}$$

that is,

$$\left\langle \gamma b - \frac{\delta_\beta F}{\delta y}, z \right\rangle \leq 0, \quad \forall y \in \mathcal{L}^*, \forall b \in B, \forall z \in T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(y) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(y), z(t_1) = z(t_2) = 0,$$

or, equivalently,

$$\gamma b - \frac{\delta_\beta F}{\delta y} \in \left[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(y) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(y)\right]^\circ, \quad \forall y \in \mathcal{L}^*, \forall b \in B,$$

which implies (3.1) and the proof is complete. □

Theorem 3.1 *Assume the scalar functional $H(x)$ is differentiable on $\mathcal{L}^*$ and the scalar functional $F(x)$ is strongly* **b***-convex on $\mathcal{L}$. Also, for any $x^* \in \mathcal{L}^*$, $v \in \overline{\mathcal{L}}$, $z \in R(x^*)$, the implication*

$$\left\langle \frac{\delta_\beta H}{\delta x^*}, v \right\rangle \geq \left\langle \frac{\delta_\beta F}{\delta z}, v \right\rangle \quad \Longrightarrow \quad \frac{\delta_\beta H}{\delta x^*} = \frac{\delta_\beta F}{\delta z}$$

is true, with $v(t_1) = v(t_2) = 0$, and $\frac{\delta_\beta F}{\delta x^*}$ is constant on $\mathcal{L}^*$. Then $\mathcal{L}^*$ is weakly sharp if and only if there exists a positive number $\gamma > 0$ such that

$$H(x) \geq \gamma d\left(x, \mathcal{L}^*\right), \quad \forall x \in \mathcal{L},$$

where $d(x, \mathcal{L}^*) = \min_{y \in \mathcal{L}^*} \|x - y\|$.

Proof "$\Longrightarrow$" Consider $\mathcal{L}^*$ is weakly sharp. Consequently, by Definition 3.1, it follows

$$-\frac{\delta_\beta F}{\delta y} \in \operatorname{int}\left(\bigcap_{u\in\mathcal{L}^*}\left[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(u)\cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(u)\right]^\circ\right), \quad \forall y\in\mathcal{L}^*,$$

or, by Lemma 3.1, there exists a positive number $\gamma > 0$ such that (3.1) (or (3.2)) is fulfilled.

Further, taking into account the convexity property of the solution set $\mathcal{L}^*$ associated to (EVIP) (see Remark 2.4), it results

$$\operatorname{proj}_{\mathcal{L}^*}(x) = \hat{y} \in \mathcal{L}^*, \quad \forall x \in \mathcal{L}$$

and, following Hiriart-Urruty and Lemaréchal [6], we get $x - \hat{y} \in T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(\hat{y}) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(\hat{y})$. By the hypothesis and Lemma 3.1, we get

$$\left\langle \frac{\delta_\beta F}{\delta \hat{y}}, x-\hat{y}\right\rangle \geq \gamma \|x-\hat{y}\| = \gamma d\big(x,\mathcal{L}^*\big),$$

or, equivalently,

$$\int_\Gamma \left[\frac{\partial f_\beta}{\partial x}(t,\hat{y},\hat{y}_\alpha)(x-\hat{y}) + \frac{\partial f_\beta}{\partial x_\alpha}(t,\hat{y},\hat{y}_\alpha)D_\alpha(x-\hat{y})\right]dt^\beta \geq \gamma d\big(x,\mathcal{L}^*\big), \quad \forall x\in\mathcal{L}. \tag{3.3}$$

Since

$$H(x) \geq \mathbf{b}(x,\hat{y})\int_\Gamma \left[\frac{\partial f_\beta}{\partial x}(t,\hat{y},\hat{y}_\alpha)(x-\hat{y}) + \frac{\partial f_\beta}{\partial x_\alpha}(t,\hat{y},\hat{y}_\alpha)D_\alpha(x-\hat{y})\right]dt^\beta$$
$$+ \rho\mathbf{b}(x,\hat{y})\mathbf{d}(x,\hat{y}), \quad \forall x\in\mathcal{L},$$

by the strong $\mathbf{b}$-convexity on $\mathcal{L}$ of the scalar functional $F(x)$ and the *Working hypotheses*, it results

$$H(x) \geq \int_\Gamma \left[\frac{\partial f_\beta}{\partial x}(t,\hat{y},\hat{y}_\alpha)(x-\hat{y}) + \frac{\partial f_\beta}{\partial x_\alpha}(t,\hat{y},\hat{y}_\alpha)D_\alpha(x-\hat{y})\right]dt^\beta, \quad \forall x\in\mathcal{L}.$$

Now, by using (3.3), we obtain

$$H(x) \geq \gamma d\big(x,\mathcal{L}^*\big), \quad \forall x\in\mathcal{L}.$$

"$\Longleftarrow$" Consider there exists a positive number $\gamma > 0$ such that

$$H(x) \geq \gamma d\big(x,\mathcal{L}^*\big), \quad \forall x\in\mathcal{L}.$$

Obviously, for any $y \in \mathcal{L}^*$, the case $T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(y) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(y) = \{0\}$ involves

$$\left[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(y)\cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(y)\right]^\circ = \overline{\mathcal{L}}$$

and, consequently,

$$\gamma B \subset \frac{\delta_\beta F}{\delta y} + \left[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(y)\cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(y)\right]^\circ, \quad \forall y\in\mathcal{L}^*$$

is trivial. In the following, let $0 \neq u \in T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(y) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(y)$, with $u(t_1) = u(t_2) = 0$, $\langle u, x-y\rangle \leq 0$, $x \in \mathcal{L}$, involving there exists a sequence u^k converging to u with $y + t_k u^k \in \mathcal{L}$ (for some sequence of positive numbers $\{t_k\}$ decreasing to zero), such that

$$d(y + t_k u^k, \mathcal{L}^*) \geq d(y + t_k u^k, H_u) = \frac{t_k \langle u, u^k\rangle}{\|u\|}, \tag{3.4}$$

where $H_u = \{x \in \mathcal{L} : \langle u, x-y\rangle = 0\}$ is a hyperplane passing through y and orthogonal to u. By the hypothesis and (3.4), it follows

$$H(y + t_k u^k) \geq \gamma \frac{t_k \langle u, u^k\rangle}{\|u\|},$$

or, equivalently ($H(y) = 0$, $\forall y \in \mathcal{L}^*$),

$$\frac{H(y + t_k u^k) - H(y)}{t_k} \geq \gamma \frac{\langle u, u^k\rangle}{\|u\|}. \tag{3.5}$$

Further, by taking the limit for $k \to \infty$ in (3.5) and using a classical result of functional analysis, we get

$$\lim_{\lambda \to 0} \frac{H(y + \lambda u) - H(y)}{\lambda} \geq \gamma \|u\|, \tag{3.6}$$

where $\lambda > 0$. By Definition 2.2, the inequality (3.6) can be rewritten as

$$\left\langle \frac{\delta_\beta H}{\delta y}, u \right\rangle \geq \gamma \|u\|. \tag{3.7}$$

Now, taking into account the hypothesis and (3.7), for any $b \in B$, it results

$$\left\langle \gamma b - \frac{\delta_\beta F}{\delta y}, u \right\rangle = \langle \gamma b, u\rangle - \left\langle \frac{\delta_\beta H}{\delta y}, u \right\rangle \leq \gamma \|u\| - \gamma \|u\| = 0$$

and therefore

$$\gamma B \subset \frac{\delta_\beta F}{\delta y} + \left[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(y) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(y)\right]^\circ, \quad \forall y \in \mathcal{L}^*$$

and the proof is complete. □

Remark 3.1

(i) The *weak sharpness property* of the solution set associated to the scalar variational problem

$$\min_{x \in \mathcal{L}} H(x)$$

is described by the inequality ($H(y) = 0$, $\forall y \in \mathcal{L}^*$)

$$H(x) - H(x^*) \geq \gamma d(x, \mathcal{L}^*), \quad \forall x \in \mathcal{L}, x^* \in \mathcal{L}^*,$$

formulated in Theorem 3.1.

(ii) If the condition

$$H(x) \geq \gamma d(x, \mathcal{L}^*), \quad \forall x \in \mathcal{L}$$

is fulfilled, the function H provides an error bound for the distance from a feasible point and the solution set $\mathcal{L}^*$. The positive constant γ is called the *modulus of sharpness* for the solution set $\mathcal{L}^*$.

The second characterization of weak sharpness for $\mathcal{L}^*$ implies the notion of *minimum principle sufficiency property*, introduced by Ferris and Mangasarian [5].

Definition 3.2 It is said that (EVIP) satisfies *minimum principle sufficiency property* if $Q(x^*) = \mathcal{L}^*$, for any $x^* \in \mathcal{L}^*$.

Lemma 3.2 *The inclusion* $\arg\max_{y\in\mathcal{L}}\{\mathbf{b}(x,y)[\langle r, y-x\rangle + \rho\mathbf{d}(x,y)]\} \subset \mathcal{L}^*$ *is fulfilled for any* $(r,x) \in \operatorname{int}(\bigcap_{u\in\mathcal{L}^*}[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(u) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(u)]^\circ) \times \mathcal{L} \quad [\neq \emptyset]$.

Proof Let $y \in \mathcal{L} \setminus \mathcal{L}^*$. By convexity property of $\mathcal{L}^*$ (see Remark 2.4), it results

$$\operatorname{proj}_{\mathcal{L}^*}(y) = \hat{y} \in \mathcal{L}^*$$

and, following Hiriart-Urruty and Lemaréchal [6], we get $y-\hat{y} \in T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(\hat{y}) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(\hat{y})$. There exists a positive number $\alpha > 0$ such that

$$\langle r+v, y-x+x-\hat{y}\rangle < 0, \quad \forall v \in \alpha B,$$

for any $(r,x) \in \operatorname{int}(\bigcap_{u\in\mathcal{L}^*}[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(u) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(u)]^\circ) \times \mathcal{L}$, or, equivalently,

$$\langle r, y-x\rangle < \langle r, \hat{y}-x\rangle - \langle v, y-\hat{y}\rangle, \quad \forall v \in \alpha B,$$

for any $(r,x) \in \operatorname{int}(\bigcap_{u\in\mathcal{L}^*}[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(u) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(u)]^\circ) \times \mathcal{L}$. For $v = \alpha\frac{y-\hat{y}}{\|y-\hat{y}\|} \in \alpha B$, the previous inequality becomes

$$\langle r, y-x\rangle < \langle r, \hat{y}-x\rangle - \alpha\|y-\hat{y}\|, \tag{3.8}$$

for any $(r,x) \in \operatorname{int}(\bigcap_{u\in\mathcal{L}^*}[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(u) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(u)]^\circ) \times \mathcal{L}$. By (3.8), we conclude

$$y \notin \arg\max_{y\in\mathcal{L}}\{\mathbf{b}(x,y)[\langle r, y-x\rangle + \rho\mathbf{d}(x,y)]\},$$

that is, $\arg\max_{y\in\mathcal{L}}\{\mathbf{b}(x,y)[\langle r, y-x\rangle + \rho\mathbf{d}(x,y)]\} \subset \mathcal{L}^*$, for any

$$(r,x) \in \operatorname{int}\left(\bigcap_{u\in\mathcal{L}^*}[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(u) \cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(u)]^\circ\right) \times \mathcal{L}.$$

The proof is complete. □

Theorem 3.2 *If the solution set $\mathcal{L}^*$ associated to* (EVIP) *is weakly sharp and the scalar functional $F(x)$ is strongly* **b***-convex on $\mathcal{L}$, then* (EVIP) *satisfies the minimum principle sufficiency property.*

Proof By Definition 3.2, (EVIP) satisfies the minimum principle sufficiency property if $Q(x^*) = \mathcal{L}^*$, for any $x^* \in \mathcal{L}^*$. Since $\mathcal{L}^*$ is weakly sharp, by Definition 3.1, we get

$$-\frac{\delta_\beta F}{\delta x^*} \in \operatorname{int}\left(\bigcap_{u\in\mathcal{L}^*}\left[T_{\mathcal{L}}^{\rho,\mathbf{b},\mathbf{d}}(u)\cap N_{\mathcal{L}^*}^{\rho,\mathbf{b},\mathbf{d}}(u)\right]^\circ\right), \quad \forall x^*\in\mathcal{L}^*$$

and, according to Lemma 3.2, it results

$$\arg\max_{y\in\mathcal{L}}\left\{\mathbf{b}\left(x^*,y\right)\left[\left\langle -\frac{\delta_\beta F}{\delta x^*}, y-x^*\right\rangle + \rho\mathbf{d}\left(x^*,y\right)\right]\right\}\subset\mathcal{L}^* \quad\Longleftrightarrow\quad Q\left(x^*\right)\subset\mathcal{L}^*. \tag{3.9}$$

Further, let $z\in\mathcal{L}^*$. For $x^*\in\mathcal{L}^*$, in accordance with Proposition 2.1, we have

$$\begin{aligned}&\mathbf{b}\left(z,x^*\right)\int_\Gamma\left[\frac{\partial f_\beta}{\partial x}\left(t,x^*,x^*_\alpha\right)\left(z-x^*\right)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^*,x^*_\alpha\right)D_\alpha\left(z-x^*\right)\right]dt^\beta\\&\quad+\rho\mathbf{b}\left(z,x^*\right)\mathbf{d}\left(z,x^*\right)=0.\end{aligned} \tag{3.10}$$

Taking into account (3.10), for any $y\in\mathcal{L}$ and using the *Working hypotheses*, it follows

$$\begin{aligned}&\mathbf{b}(z,y)\int_\Gamma\left[\frac{\partial f_\beta}{\partial x}\left(t,x^*,x^*_\alpha\right)(z-y)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^*,x^*_\alpha\right)D_\alpha(z-y)\right]dt^\beta\\&\qquad+\rho\mathbf{b}(z,y)\mathbf{d}(z,y)\\&\quad=\mathbf{b}\left(x^*,y\right)\int_\Gamma\left[\frac{\partial f_\beta}{\partial x}\left(t,x^*,x^*_\alpha\right)\left(x^*-y\right)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^*,x^*_\alpha\right)D_\alpha\left(x^*-y\right)\right]dt^\beta\\&\qquad+\rho\mathbf{b}\left(x^*,y\right)\mathbf{d}\left(x^*,y\right).\end{aligned} \tag{3.11}$$

Since $x^*\in\mathcal{L}^*$, relation (3.11) provides

$$\begin{aligned}&\mathbf{b}(z,y)\int_\Gamma\left[\frac{\partial f_\beta}{\partial x}\left(t,x^*,x^*_\alpha\right)(z-y)+\frac{\partial f_\beta}{\partial x_\alpha}\left(t,x^*,x^*_\alpha\right)D_\alpha(z-y)\right]dt^\beta\\&\quad+\rho\mathbf{b}(z,y)\mathbf{d}(z,y)\le 0,\quad\forall y\in\mathcal{L},\end{aligned}$$

that is, $z\in Q(x^*)$ and, consequently,

$$\mathcal{L}^*\subset Q\left(x^*\right). \tag{3.12}$$

By using (3.9) and (3.12), the proof is complete. □

Theorem 3.3 *Assume the scalar functional $H(x)$ is differentiable on $\mathcal{L}^*$ and the scalar functional $F(x)$ is strongly* **b***-convex on $\mathcal{L}$. Also, for any $x^*\in\mathcal{L}^*$, $v\in\overline{\mathcal{L}}$, $z\in R(x^*)$, the implication*

$$\left\langle\frac{\delta_\beta H}{\delta x^*},v\right\rangle\ge\left\langle\frac{\delta_\beta F}{\delta z},v\right\rangle\quad\Longrightarrow\quad\frac{\delta_\beta H}{\delta x^*}=\frac{\delta_\beta F}{\delta z}$$

is true, with $v(t_1) = v(t_2) = 0$*, and* $\frac{\delta_\beta F}{\delta x^*}$ *is constant on* $\mathcal{L}^*$*. Then* (EVIP) *satisfies the minimum principle sufficiency property if and only if* $\mathcal{L}^*$ *is weakly sharp.*

Proof "$\Longrightarrow$" Let (EVIP) satisfies the minimum principle sufficiency property. In consequence, $Q(x^*) = \mathcal{L}^*$, for any $x^* \in \mathcal{L}^*$. Obviously, for $x^* \in \mathcal{L}^*$ and $x \in \overline{\mathcal{L}}$, we obtain

$$\begin{aligned} H(x) \geq \mathbf{b}(x,x^*) \int_\Gamma \left[\frac{\partial f_\beta}{\partial x}(t,x^*,x^*_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x^*,x^*_\alpha)D_\alpha(x-x^*)\right] dt^\beta \\ + \rho\mathbf{b}(x,x^*)\mathbf{d}(x,x^*), \end{aligned} \tag{3.13}$$

or, applying the strong **b**-convexity property on $\mathcal{L}$ of the scalar functional $F(x)$ and the *Working hypotheses*, we get

$$H(x) \geq \int_\Gamma \left[\frac{\partial f_\beta}{\partial x}(t,x^*,x^*_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x^*,x^*_\alpha)D_\alpha(x-x^*)\right] dt^\beta. \tag{3.14}$$

In the following, considering $P(x) = \{\mathbf{b}(x^*,x)[\langle\frac{\delta_\beta F}{\delta x^*}, x - x^*\rangle - \rho\mathbf{d}(x^*,x)]\}$, $x \in \mathcal{L}$, we have $Q(x^*)$ the solution set for $\min_{x\in\mathcal{L}} P(x)$. In accordance with Remark 3.1 and using the *Working hypotheses*, we can write

$$P(x) - P(\overline{x}) \geq \gamma d(x, Q(x^*)), \quad \forall x \in \mathcal{L}, \overline{x} \in Q(x^*),$$

or

$$\left\langle \frac{\delta_\beta F}{\delta x^*}, x - x^* \right\rangle \geq \gamma d(x,\mathcal{L}^*), \quad \forall x \in \mathcal{L},$$

or, equivalently,

$$\begin{aligned} \int_\Gamma \left[\frac{\partial f_\beta}{\partial x}(t,x^*,x^*_\alpha)(x-x^*) + \frac{\partial f_\beta}{\partial x_\alpha}(t,x^*,x^*_\alpha)D_\alpha(x-x^*)\right] dt^\beta \\ \geq \gamma d(x,\mathcal{L}^*), \quad \forall x \in \mathcal{L}. \end{aligned} \tag{3.15}$$

By (3.13)–(3.15) and Theorem 3.1, we get $\mathcal{L}^*$ is weakly sharp.

"$\Longleftarrow$" This implication is a consequence of Theorem 3.2. □

4 Conclusions

In this paper, by using a dual gap-type functional and some working hypotheses, the solution set has been studied for a new variational-type inequality involving $(\rho, \mathbf{b}, \mathbf{d})$-convex path-independent curvilinear integral functional. Moreover, weak sharp solutions for the considered variational-type inequality have been investigated. Also, under some hypotheses, an equivalence between *minimum principle sufficiency property* and *weak sharpness property* of the solution set associated with the considered variational-type inequality has been established.

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