Bakery and Elmatty *Journal of Inequalities and Applications* (2020) 2020:55
https://doi.org/10.1186/s13660-020-02318-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some properties of pre-quasi norm on Orlicz sequence space

Awad A. Bakery[1,2*] and Afaf R. Abou Elmatty[2]

*Correspondence: awad_bakery@yahoo.com
[1]Department of Mathematics, College of Science and Arts at Khulis, University of Jeddah, Jeddah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt

Abstract

In this article, we introduce the concept of pre-quasi norm on E (Orlicz sequence space), which is more general than the usual norm, and give the conditions on E equipped with the pre-quasi norm to be Banach space. We give the necessity and sufficient conditions on E equipped with the pre-quasi norm such that the multiplication operator defined on E is a bounded, approximable, invertible, Fredholm, and closed range operator. The components of pre-quasi operator ideal formed by the sequence of s-numbers and E is strictly contained for different Orlicz functions are determined. Furthermore, we give the sufficient conditions on E equipped with a pre-modular such that the pre-quasi Banach operator ideal constructed by s-numbers and E is simple and its components are closed. Finally the pre-quasi operator ideal formed by the sequence of s-numbers and E is strictly contained in the class of all bounded linear operators, whose sequence of eigenvalues belongs to E.

Keywords: Pre-quasi norm; Orlicz sequence space; Multiplication operator; Fredholm operator; Approximable operator; Simple Banach space

1 Introduction

Throughout the paper, we denote the space of all bounded linear operators from a Banach space X into a Banach space Y by $L(X,Y)$, and if $X = Y$, we write $L(X)$, the space of all real sequences is denoted by w, the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, $\mathbb{N} = \{0,1,2,\ldots\}$, the space of null sequences by C_0, and the space of bounded sequences by ℓ_∞. In operator theory, the multiplication operators on L_p-spaces are related to the composition operators; this means that the properties of composition operators on L_p-spaces can be stated by the properties of multiplication operators. Singh and Kumar [28] proved that a composition operator on $L_p(X;\mathbb{C})$ is compact if and only if the multiplication operator T_α is compact, where $\alpha = \frac{d\mu T^{-1}}{d\mu}$ is the Radon–Nikodym derivative of the measure μT^{-1} with respect to the measure μ. In the theory of Hilbert space, every normal operator on a separable Hilbert space is unitarily equivalent to a multiplication operator. In the spectral theory, multiplication operators have their roots in the spectral theory. For more details on multiplication operators, see [1, 26, 27, 29–31]. On sequence spaces, Mursaleen and Noman in [17, 18] studied the compact operators on some difference sequence spaces; Komal and Gupta [10] studied the multiplication operators on Orlicz spaces equipped with the Luxemburg norm, and Komal et al. [11] examined the multiplication operators on Cesáro sequence spaces. The theory of operator ideal goals possesses an uncommon essentialness in useful examination. Some of operator ideals in the class of Banach spaces or Hilbert spaces are defined by different scalar sequence spaces. For example the ideal of compact operators is defined by the space C_0 of null sequence and Kolmogorov numbers. Pietsch [24] examined the quasi-ideals formed by the approximation numbers and classical sequence space ℓ^p $(0 < p < \infty)$. He proved that the ideals of nuclear operators and of Hilbert–Schmidt operators between Hilbert spaces are defined by ℓ^1 and ℓ^2, respectively. He proved that the class of all finite rank operators is dense in the Banach quasi-ideal, and the algebra $L(\ell^p)$, where $(1 \leq p < \infty)$, contains one and only one nontrivial closed ideal. Pietsch [23] showed that the quasi Banach operator ideal formed by the sequence of approximation numbers is small. Makarov and Faried [14] proved that the quasi-operator ideal formed by the sequence of approximation numbers is strictly contained for different powers, i.e., for any infinite dimensional Banach spaces X, Y and for any $q > p > 0$, it is true that $S^{\mathrm{app}}_{\ell^p}(X,Y) \subsetneqq S^{\mathrm{app}}_{\ell^q}(X,Y) \subsetneqq L(X,Y)$. In [8], Faried and Bakery studied the operator ideals constructed by approximation numbers, generalized Cesáro and Orlicz sequence spaces ℓ_M. In [9], Faried and Bakery introduced the concept of pre-quasi operator ideal, which is more general than the usual classes of operator ideals. They studied the operator ideals constructed by s-numbers, generalized Cesáro and Orlicz sequence spaces ℓ_M, and proved that the operator ideal formed by the previous sequence spaces and approximation numbers is small under certain conditions. The aim of this paper to study the concept of pre-quasi norm on E (Orlicz sequence space), which is more general than the usual norm, and give the conditions for E equipped with the pre-quasi norm to be Banach space. We give the necessity and sufficient conditions on E equipped with the pre-quasi norm such that the multiplication operator defined on E is a bounded, approximable, invertible, Fredholm, and closed range operator. The components of pre-quasi operator ideal formed by the sequence of s-numbers and E is strictly contained for different Orlicz functions are determined. Furthermore, we give the sufficient conditions on E equipped with a pre-modular such that the pre-quasi Banach operator ideal constructed by s-numbers and E is simple and its components are closed. Finally the pre-quasi operator ideal formed by the sequence of s-numbers and E is strictly contained in the class of all bounded linear operators, whose sequence of eigenvalues belongs to E.

2 Definitions and preliminaries

Definition 2.1 ([24]) A finite rank operator is a bounded linear operator whose dimension of the range space is finite. The space of all finite rank operators on E is denoted by $F(E)$.

Definition 2.2 ([24]) A bounded linear operator $A : E \to E$ (where E is a Banach space) is called approximable if there are $S_n \in F(E)$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \|A - S_n\| = 0$. The space of all approximable operators on E is denoted by $\Psi(E)$, and the space of all approximable operators from E to F is denoted by $\Psi(E,F)$.

Lemma 2.3 ([24]) *Let $T \in L(X,Y)$. If T is not approximable, then there are operators $G \in L(X,X)$ and $B \in L(Y,Y)$ such that $BTGe_k = e_k$ for all $k \in \mathbb{N}$.*

Definition 2.4 ([24]) A Banach space E is called simple if the algebra $L(E)$ contains one and only one nontrivial closed ideal.

Definition 2.5 ([24]) A bounded linear operator $A : E \to E$ (where E is a Banach space) is called compact if $A(B_1)$ has compact closure, where B_1 denotes the closed unit ball of E. The space of all compact operators on E is denoted by $L_c(E)$.

Theorem 2.6 ([24]) *If E is infinite dimensional Banach space, we have*

$$F(E) \subsetneqq \Psi(E) \subsetneqq L_c(E) \subsetneqq L(E).$$

Definition 2.7 ([16]) A bounded linear operator $A : E \to E$ is called Fredholm if A has closed range, $\dim(\ker A)$ and co-dim(range A) are finite.

Definition 2.8 ([12]) An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$, which is convex, continuous, and nondecreasing with $M(0) = 0$, $M(u) > 0$ for $u > 0$ and $M(u) \to \infty$, as $u \to \infty$.

Definition 2.9 ([22]) An Orlicz function M is said to satisfy Δ_2-condition for all values of $x \geq 0$, if there exists a constant $k > 0$ such that $M(2x) \leq kM(x)$. The Δ_2-condition is equivalent to $M(lx) \leq klM(x)$ for all values of x and for $l > 1$.

Lindenstrauss and Tzafriri [13] utilized the idea of an Olicz function to define Orlicz sequence space:

$$\ell_M = \{u \in \omega : \rho(\beta u) < \infty \text{ for some } \beta > 0\} \quad \text{where } \rho(u) = \sum_{k=0}^{\infty} M(|u_k|),$$

$(\ell_M, \|\cdot\|)$ is a Banach space with the Luxemburg norm:

$$\|u\| = \inf\left\{\beta > 0 : \rho\left(\frac{u}{\beta}\right) \leq 1\right\}.$$

Every Orlicz sequence space contains a subspace that is isomorphic to c_0 or ℓ^q for some $1 \leq q < \infty$.

As of late, different classes of sequences have been studied using Orlicz functions by Et et al. [7], Mursaleen et al. [19–21], Alotaibi et al. [2–4], and Mohiuddine et al. [15].

Definition 2.10 ([6]) A class of linear sequence spaces $\mathbb{E}$ is called a special space of sequences (sss) if

(1) $e_i \in \mathbb{E}$ for all $i \in \mathbb{N}$;
(2) if $u = (u_i) \in w$, $v = (v_i) \in \mathbb{E}$, and $|u_i| \leq |v_i|$ for every $i \in \mathbb{N}$, then $u \in \mathbb{E}$, "i.e., $\mathbb{E}$ is solid";
(3) if $(u_i)_{i=0}^{\infty} \in \mathbb{E}$, then $(u_{[\frac{i}{2}]})_{i=0}^{\infty} \in \mathbb{E}$, wherever $[\frac{i}{2}]$ means the integral part of $\frac{i}{2}$.

Theorem 2.11 ([9]) *ℓ_M is a (sss) if M is an Orlicz function satisfying Δ_2-condition.*

Definition 2.12 ([6]) A subclass of the special space of sequences is called a pre-modular (sss) if there is a function $\varrho : \mathbb{E} \to [0,\infty[$ satisfying the following conditions:

(i) $\varrho(u) \geq 0$ for each $u \in \mathbb{E}$ and $\varrho(u) = 0 \Leftrightarrow u = \theta$, where θ is the zero element of $\mathbb{E}$;
(ii) There exists $L \geq 1$ such that $\varrho(\beta u) \leq L|\beta|\varrho(u)$ for all $u \in \mathbb{E}$ and for any scalar β;
(iii) For some $K \geq 1$, $\varrho(u + v) \leq K(\varrho(u) + \varrho(v))$ for every $u, v \in \mathbb{E}$;

(iv) If $|u_i| \leq |v_i|$ for all $i \in \mathbb{N}$, then $\varrho((u_i)) \leq \varrho((v_i))$;
(v) For some $K_0 \geq 1$, $\varrho((u_i)) \leq \varrho((u_{[\frac{i}{2}]})) \leq K_0\varrho((u_i))$;
(vi) The set of all finite sequences is ϱ-dense in $\mathbb{E}$. This means that, for each $u = (u_i)_{i=o}^{\infty} \in \mathbb{E}$ and for each $\varepsilon > 0$, there exists $s \in \mathbb{N}$ such that $\varrho((u_i)_{i=s}^{\infty}) < \varepsilon$;
(vii) There exists a constant $\xi > 0$ such that $\varrho(\beta, 0, 0, 0, \ldots) \geq \xi|\beta|\varrho(1, 0, 0, 0, \ldots)$ for any $\beta \in \mathbb{R}$.

Theorem 2.13 ([9]) *ℓ_M is a pre-modular (sss) if M is an Orlicz function satisfying Δ_2-condition.*

Definition 2.14 ([24]) Let L be a class of all bounded linear operators between any arbitrary Banach spaces. A sub class U of L is called an operator ideal if each element $U(X,Y) = U \cap L(X,Y)$ fulfills the following conditions:
(i) $I_F \in U$ wherever F represents a Banach space of one dimension.
(ii) The space $U(X,Y)$ is linear over $\mathbb{R}$.
(iii) If $T \in L(X_0,X)$, $V \in U(X,Y)$, and $R \in L(Y,Y_0)$, then $RVT \in U(X_0,Y_0)$.

Closed ideal means an ideal which contains its limit points.

The concept of pre-quasi operator ideal is more general than the usual classes of operator ideals.

Definition 2.15 ([6]) A function $g : \Omega \to [0,\infty)$ is said to be a pre-quasi norm on the ideal Ω if the following conditions hold:
(1) For all $T \in \Omega(X,Y)$, $g(T) \geq 0$ and $g(T) = 0$ if and only if $T = 0$;
(2) There exists a constant $M \geq 1$ such that $g(\lambda T) \leq M|\lambda|g(T)$ for all $T \in \Omega(X,Y)$ and $\lambda \in \mathbb{R}$;
(3) There exists a constant $K \geq 1$ such that $g(T_1 + T_2) \leq K[g(T_1) + g(T_2)]$ for all $T_1, T_2 \in \Omega(X,Y)$;
(4) There exists a constant $C \geq 1$ such that if $T \in L(X_0,X)$, $P \in \Omega(X,Y)$, and $R \in L(Y,Y_0)$, then $g(RPT) \leq C\|R\|g(P)\|T\|$, where X_0 and Y_0 are normed spaces.

Theorem 2.16 ([9]) *The function $g(P) = \varrho(s_i(P))_{i=0}^{\infty}$ is a pre-quasi norm on $S_{\mathbb{E}_\varrho}$, where $\mathbb{E}_\varrho$ is a pre-modular (sss).*

Definition 2.17 ([25]) An s-number function is a map defined on $L(X,Y)$ which associates with each operator $T \in L(X,Y)$ a nonnegative scaler sequence $(s_n(T))_{n=0}^{\infty}$ assuming that the taking after states are verified:
(a) $\|T\| = s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for $T \in L(X,Y)$;
(b) $s_{m+n-1}(T_1 + T_2) \leq s_m(T_1) + s_n(T_2)$ for all $T_1, T_2 \in L(X,Y)$, $m, n \in \mathbb{N}$;
(c) Ideal property: $s_n(RVT) \leq \|R\|s_n(V)\|T\|$ for all $T \in L(X_0,X)$, $V \in L(X,Y)$, and $R \in L(Y,Y_0)$, where X_0 and Y_0 are arbitrary Banach spaces;
(d) If $G \in L(X,Y)$ and $\lambda \in \mathbb{R}$, we obtain $s_n(\lambda G) = |\lambda|s_n(G)$;
(e) Rank property: If $\operatorname{rank}(T) \leq n$, then $s_n(T) = 0$ for each $T \in L(X,Y)$;
(f) Norming property: $s_{r\geq n}(I_n) = 0$ or $s_{r<n}(I_n) = 1$, where I_n represents the unit operator on the n-dimensional Hilbert space ℓ_2^n.

Notations 2.18 ([9])

$$S_E := \{S_E(X,Y); X \text{ and } Y \text{ are Banach spaces}\}, \quad \text{where}$$
$$S_E(X,Y) := \{T \in L(X,Y) : ((s_i(T))_{i=0}^{\infty} \in E\}.$$

Throughout this paper, we define $e_n = \{0,0,\dots,1,0,0,\dots\}$ where 1 appears at the nth place for all $n \in \mathbb{N}$, the sequence (q_i) is a bounded sequence of positive numbers, and the following well-known inequality [5]: $|a_i + b_i|^{q_i} \leq H(|a_i|^{q_i} + |b_i|^{q_i})$, where $H = 2^{h-1}$, $h = \sup_i q_i$, and $q_i \geq 1$ for every $i \in \mathbb{N}$, is used.

3 Main results

In this part, we give the concept of pre-quasi norm on Orlicz sequence space, which is more general than the usual norm, and give the conditions for Orlicz sequence space equipped with the pre-quasi norm to be a Banach space.

Definition 3.1 Let $\mathbb{E}$ be special space of sequences (sss). If there is a function $\varrho : \mathbb{E} \to [0,\infty[$ fulfilling the following conditions:

(i) $\varrho(x) \geq 0$ for each $x \in \mathbb{E}$ and $\varrho(x) = 0 \Leftrightarrow x = \theta$, where θ is the zero element of $\mathbb{E}$;
(ii) There exists $L \geq 1$ such that $\varrho(\lambda x) \leq L|\lambda|\varrho(x)$ for all $x \in \mathbb{E}$ and for any scalar λ;
(iii) For some $K \geq 1$, we have $\varrho(x+y) \leq K(\varrho(x) + \varrho(y))$ for every $x, y \in \mathbb{E}$.

The space $\mathbb{E}$ with ϱ is called pre-quasi normed (sss) and is denoted by $\mathbb{E}_\varrho$, which gives a class more general than the quasi normed space. If the space $\mathbb{E}$ is complete with ϱ, then $\mathbb{E}_\varrho$ is called a pre-quasi Banach (sss).

Theorem 3.2 *Every quasi-norm is pre-quasi norm.*

Theorem 3.3 *If M is an Orlicz function satisfying Δ_2-condition, then $(\ell_M)_\varrho$ is a pre-quasi Banach (sss), where $\varrho(x) = \sum_{k=0}^{\infty} M(|x_k|)$ for all $x \in \ell_M$.*

Proof The function $\varrho : l_M \to [0,\infty[$, where $\varrho(x) = \sum_{k=0}^{\infty} M(|x_k|)$ for all $x \in \ell_M$, satisfies the following conditions:

(i) $\varrho(x) \geq 0$ for each $x \in \ell_M$ and $\varrho(x) = 0 \Leftrightarrow x = \theta$, where θ is the zero element of ℓ_M.

(ii) Assume $\lambda \in \mathbb{R}$, $x \in l_M$, and since M satisfies Δ_2-condition, we get a number $a > 0$ such that

$$\varrho(\lambda x) = \sum_{n=0}^{\infty} M(|\lambda x_n|) \leq |\lambda| a \sum_{n=0}^{\infty} M(|x_n|) = L|\lambda|\varrho(x),$$

where $L = \max\{1, a\}$.

(iii) Let $x, y \in l_M$. Since M is nondecreasing, convex, and satisfying Δ_2-condition, then there exists a number $a > 0$ such that

$$\begin{aligned}\varrho(x+y) &= \sum_{n=0}^{\infty} M(|x_n + y_n|) \leq \sum_{n=0}^{\infty} M(|x_n| + |y_n|) \leq \frac{1}{2}\sum_{n=0}^{\infty}\bigl[M(2|x_n|) + M(2|y_n|)\bigr] \\ &\leq \frac{a}{2}\left(\sum_{n=0}^{\infty} M(|x_n|) + \sum_{n=0}^{\infty} M(|y_n|)\right) = K\bigl(\varrho(x) + \varrho(y)\bigr)\end{aligned}$$

for some $K = \max\{1, \frac{a}{2}\}$. Hence $(\ell_M)_\varrho$ is a pre-quasi normed (sss). Since M is continuous and nondecreasing, hence M^{-1} exists. To prove that $(\ell_M)_\varrho$ is a pre-quasi Banach (sss), suppose $x^n = (x_k^n)_{k=0}^\infty$ to be a Cauchy sequence in $(\ell_M)_\varrho$, then for every $\varepsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, one has

$$\begin{aligned} &\varrho\left(x^n - x^m\right) = \sum_{k=0}^\infty M\left(\left|x_k^n - x_k^m\right|\right) < \varepsilon \\ &\quad\Rightarrow \quad M\left(\left|x_k^n - x_k^m\right|\right) < \varepsilon \quad \Rightarrow \quad \left|x_k^n - x_k^m\right| < M^{-1}(\varepsilon). \end{aligned}$$

So (x_k^m) is a Cauchy sequence in $\mathbb{R}$ for fixed $k \in \mathbb{N}$, this gives $\lim_{m\to\infty} x_k^m = x_k^0$ for fixed $k \in \mathbb{N}$. Hence $\varrho(x^n - x^0) < \varepsilon$. Finally, to prove that $x^0 \in \ell_M$, we have

$$\begin{aligned} \varrho\left(x^0\right) &= \sum_{k=0}^\infty M\left(\left|x_k^0 - x_k^n + x_k^n\right|\right) \leq K\left(\sum_{k=0}^\infty M\left(\left|x_k^0 - x_k^n\right|\right) + \sum_{k=0}^\infty M\left(\left|x_k^n\right|\right)\right) \\ &= K\left(\varrho\left(x^n - x^0\right) + \varrho\left(x^n\right)\right) < \infty, \end{aligned}$$

so $x^0 \in \ell_M$. This means that $(\ell_M)_\varrho$ is a pre-quasi Banach (sss). □

Corollary 3.4 $(\ell^p)_\varrho$, *where* $\varrho(x) = \sum_{i=0}^\infty |x_i|^p$ *for all* $x \in \ell^p$ *is a pre-quasi Banach* (*sss*), *if* $0 < p < \infty$.

4 Multiplication operator on pre-quasi normed (sss)

In this part, we define a multiplication operator on Orlicz sequence space with a pre-quasi norm and give the necessity and sufficient conditions on Orlicz sequence space equipped with the pre-quasi norm such that the multiplication operator defined on Orlicz sequence space is a bounded, approximable, invertible, Fredholm, and closed range operator.

Definition 4.1 Let $\alpha : \mathbb{N} \to \mathbb{C}$ be a bounded sequence and E_ϱ be a pre-quasi normed (sss), the multiplication operator is defined as $T_\alpha : E \to E$, where $T_\alpha x = \alpha x = (\alpha_k x_k)_{k=0}^\infty$ for all $x \in E$. If T_α is continuous, we call it a multiplication operator induced by α.

Theorem 4.2 *If* $\alpha : \mathbb{N} \to \mathbb{C}$ *is a mapping and* M *is an Orlicz function satisfying* Δ_2-*condition, then* $\alpha \in \ell_\infty$ *if and only if* $T_\alpha \in L((\ell_M)_\varrho)$, *where* $\varrho(x) = \sum_{k=0}^\infty M(|x_k|)$ *for all* $x \in \ell_M$.

Proof Let $\alpha \in \ell_\infty$. Then there exists $C > 0$ such that $|\alpha_n| \leq C$ for all $n \in \mathbb{N}$. For $x \in (\ell_M)_\varrho$, since M is nondecreasing and satisfying Δ_2-condition, we have

$$\begin{aligned} \varrho(T_\alpha x) &= \varrho(\alpha x) = \varrho\left((\alpha_k x_k)_{k=0}^\infty\right) = \sum_{k=0}^\infty M\left(|\alpha_k||x_k|\right) \leq \sum_{k=0}^\infty M\left(C|x_k|\right) \\ &\leq D\sum_{k=0}^\infty M\left(|x_k|\right) = D\varrho(x), \end{aligned}$$

where D is a constant depending on C, which implies that $T_\alpha \in L((\ell_M)_\varrho)$.

Conversely, suppose that $T_\alpha \in L((\ell_M)_\varrho)$. We prove that $\alpha \in \ell_\infty$. For, if α is not a bounded function, then for every $n \in \mathbb{N}$, there exists some $i_n \in \mathbb{N}$ such that $\alpha_{i_n} > n$. Since M is nondecreasing, we obtain

$$\begin{aligned}\varrho(T_\alpha e_{i_n}) = \varrho(\alpha e_{i_n}) = \varrho\big(\big(\alpha_k (e_{i_n})_k\big)_{k=0}^{\infty}\big) = \sum_{k=0}^{\infty} M\big(|\alpha_k|\big|(e_{i_n})_k\big|\big) \\ = M\big(|\alpha_{i_n}|\big) > M(n) = M(n)\varrho(e_{i_n}).\end{aligned}$$

This proves that T_α is not a bounded operator. Hence, α must be a bounded function. □

Theorem 4.3 *Let $\alpha : \mathbb{N} \to \mathbb{C}$ be a mapping and $(\ell_M)_\varrho$ be a pre-quasi normed (sss), where $\varrho(x) = \sum_{k=0}^{\infty} M(|x_k|)$ for all $x \in \ell_M$. $|\alpha_n| = 1$ for all $n \in \mathbb{N}$ if and only if T_α is an isometry.*

Proof Let $|\alpha_n| = 1$ for all $n \in \mathbb{N}$. Then

$$\varrho(T_\alpha x) = \varrho(\alpha x) = \varrho\big((\alpha_k x_k)_{k=0}^{\infty}\big) = \sum_{k=0}^{\infty} M\big(|\alpha_k||x_k|\big) = \sum_{k=0}^{\infty} M\big(|x_k|\big) = \varrho(x)$$

for all $x \in (\ell_M)_\varrho$. Hence T_α is an isometry.

Conversely, suppose that $|\alpha_n| < 1$ for some $n = n_0$. Since M is nondecreasing, we have

$$\varrho(T_\alpha e_{n_0}) = \varrho(\alpha e_{n_0}) = \varrho\big(\big(\alpha_k (e_{n_0})_k\big)_{k=0}^{\infty}\big) = \sum_{k=0}^{\infty} M\big(|\alpha_k|\big|(e_{n_0})_k\big|\big) = M(\alpha_{n_0}) < \varrho(e_{n_0}).$$

Similarly, if $|\alpha_{n_0}| > 1$, then we can show that $\varrho(T_\alpha e_{n_0}) > \varrho(e_{n_0})$. In both cases, we get contradiction. Hence, $|\alpha_n| = 1$ for all $n \in \mathbb{N}$. □

Theorem 4.4 *Let $\alpha : \mathbb{N} \to \mathbb{C}$ be a mapping, M be an Orlicz function satisfying Δ_2-condition, and $T_\alpha \in L((\ell_M)_\varrho)$, where $\varrho(x) = \sum_{k=0}^{\infty} M(|x_k|)$ for all $x \in \ell_M$. Then $T_\alpha \in \Psi((\ell_M)_\varrho)$ if and only if $(\alpha_n)_{n=0}^{\infty} \in C_0$.*

Proof Suppose that T_α is an approximable operator, hence T_α is a compact operator. We show that $\lim_{n\to\infty} \alpha_n = 0$. For if this were not true, then there exists $\delta > 0$ such that the set $B_\delta = \{r \in \mathbb{N} : |\alpha_r| \geq \delta\}$ is an infinite set. Let $d_1, d_2, \ldots, d_n, \ldots$ be in B_δ. Then $\{e_{d_n} : d_n \in B_\delta\}$ is an infinite bounded set in $(\ell_M)_\varrho$. Consider

$$\begin{aligned}\varrho(T_\alpha e_{d_n} - T_\alpha e_{d_m}) &= \varrho(\alpha e_{d_n} - \alpha e_{d_m}) = \varrho\big(\big(\alpha_k\big((e_{d_n})_k - (e_{p_m})_k\big)\big)_{k=0}^{\infty}\big) \\ &= \sum_{k=0}^{\infty} M\big(\big|\alpha_k\big((e_{d_n})_k - (e_{p_m})_k\big)\big|\big) \\ &\geq \sum_{k=0}^{\infty} M\big(\delta\big|\big((e_{d_n})_k - (e_{p_m})_k\big)\big|\big) = \varrho(\delta e_{d_n} - \delta e_{d_m})\end{aligned}$$

for all $d_n, d_m \in B_\delta$. This proves that $\{e_{d_n} : d_n \in B_\delta\}$ is a bounded sequence which cannot have a convergent subsequence under T_α. This shows that T_α cannot be compact, hence it is not an approximable operator, which is a contradiction. Hence, $\lim_{n\to\infty}\alpha_n = 0$. Conversely, suppose $\lim_{n\to\infty}\alpha_n = 0$. Then, for every $\delta > 0$, the set $B_\delta = \{n \in \mathbb{N} : |\alpha_n| \geq \delta\}$ is a finite set. Then

$$\big((\ell_M)_\varrho\big)_{B_\delta} = \big\{x = (x_n) \in \omega : n \in B_\delta\big\}$$

is a finite dimensional space for each $\delta > 0$. Therefore, $T_\alpha|((\ell_M)_\varrho)_{B_\delta}$ is a finite rank operator. For each $n \in \mathbb{N}$, define $\alpha_n : \mathbb{N} \to \mathbb{C}$ by

$$(\alpha_n)_m = \begin{cases} \alpha_m & m \in B_{\frac{1}{n}}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, T_{α_n} is a finite rank operator as the space $((\ell_M)_\varrho)_{B_{\frac{1}{n}}}$ is finite dimensional for each $n \in \mathbb{N}$. Now, since M is convex and nondecreasing, we have

$$\begin{aligned}
\varrho\big((T_\alpha - T_{\alpha_n})x\big) &= \varrho\Big(\big(\big(\alpha_m - (\alpha_n)_m\big)x_m\big)_{m=0}^{\infty}\Big) \\
&= \sum_{m=0}^{\infty} M\big(\big|\big(\alpha_m - (\alpha_n)_m\big)x_m\big|\big) \\
&= \sum_{m=0, m\in B_{\frac{1}{n}}}^{\infty} M\big(\big|\big(\alpha_m - (\alpha_n)_m\big)x_m\big|\big) + \sum_{m=0, m\notin B_{\frac{1}{n}}}^{\infty} M\big(\big|\big(\alpha_m - (\alpha_n)_m\big)x_m\big|\big) \\
&= \sum_{m=0, m\notin B_{\frac{1}{n}}}^{\infty} M\big(|\alpha_m x_m|\big) \leq \frac{1}{n}\sum_{m=0, m\notin B_{\frac{1}{n}}}^{\infty} M\big(|x_m|\big) < \frac{1}{n}\sum_{m=0}^{\infty} M\big(|x_m|\big) \\
&= \frac{1}{n}\varrho(x).
\end{aligned}$$

This proves that $\|T_\alpha - T_{\alpha_n}\| \leq \frac{1}{n}$ and that T_α is a limit of finite rank operators and hence, T_α is an approximable operator. □

Theorem 4.5 *Let $\alpha : \mathbb{N} \to \mathbb{C}$ be a mapping, M be an Orlicz function satisfying Δ_2-condition, and $T_\alpha \in L((\ell_M)_\varrho)$, where $\varrho(x) = \sum_{k=0}^{\infty} M(|x_k|)$ for all $x \in \ell_M$. Then $T_\alpha \in L_c((\ell_M)_\varrho)$ if and only if $(\alpha_n)_{n=0}^{\infty} \in C_0$.*

Proof It is easy, so omitted. □

Corollary 4.6 *Let M be an Orlicz function satisfying Δ_2-condition, we have*

$$L_c\big((\ell_M)_\varrho\big) \subsetneq L\big((\ell_M)_\varrho\big),$$

where $\varrho(x) = \sum_{k=0}^{\infty} M(|x_k|)$ for all $x \in \ell_M$.

Proof Since the identity operator I on $(\ell_M)_\varrho$ is a multiplication operator induced by the sequence $\alpha = (1, 1, \ldots)$, hence $I \notin L_c((\ell_M)_\varrho)$ and $I \in L((\ell_M)_\varrho)$. □

Theorem 4.7 *Let $(\ell_M)_\varrho$ be a pre-quasi Banach (sss), where $\varrho(x) = \sum_{k=0}^{\infty} M(|x_k|)$ for all $x \in \ell_M$ and $T_\alpha \in L((\ell_M)_\varrho)$. Then α is bounded away from zero on $\mathbb{N} \setminus \ker(\alpha) := \ker(\alpha)^c$ if and only if T_α has closed range.*

Proof Suppose that α is bounded away from zero on $\ker(\alpha)^c$. Then there exists $\epsilon > 0$ such that $|\alpha_n| \geq \epsilon$ for all $n \in \ker(\alpha)^c$. We have to prove that range (T_α) is closed. Let z be a limit point of range (T_α). Then there exists a sequence $T_\alpha x_n$ in $(\ell_M)_\varrho$, for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} T_\alpha x_n = z$. Clearly, the sequence $T_\alpha x_n$ is a Cauchy sequence. Now, since M is nondecreasing, we have

$$\begin{aligned}
\varrho(T_\alpha x_n - T_\alpha x_m) &= \sum_{k=0}^{\infty} M\bigl(\bigl|\alpha_k(x_n)_k - \alpha_k(x_m)_k\bigr|\bigr) \\
&= \sum_{k=0, k\in\ker(\alpha)^c}^{\infty} M\bigl(\bigl|\alpha_k(x_n)_k - \alpha_k(x_m)_k\bigr|\bigr) \\
&\quad + \sum_{k=0, k\notin\ker(\alpha)^c}^{\infty} M\bigl(\bigl|\alpha_k(x_n)_k - \alpha_k(x_m)_k\bigr|\bigr) \\
&\geq \sum_{k=0, k\in\ker(\alpha)^c}^{\infty} M\bigl(|\alpha_k|\bigl|(x_n)_k - (x_m)_k\bigr|\bigr) \\
&= \sum_{k=0}^{\infty} M\bigl(|\alpha_k|\bigl|(y_n)_k - (y_m)_k\bigr|\bigr) \\
&> \sum_{k=0}^{\infty} M\bigl(\bigl|\epsilon(y_n)_k - \epsilon(y_m)_k\bigr|\bigr) = \varrho\bigl(\epsilon(y_n - y_m)\bigr),
\end{aligned}$$

where

$$(y_n)_k = \begin{cases} (x_n)_k, & k \in \ker(\alpha)^c, \\ 0, & k \notin \ker(\alpha)^c. \end{cases}$$

This proves that $\{y_n\}$ is a Cauchy sequence in $(\ell_M)_\varrho$. But $(\ell_M)_\varrho$ is complete. Therefore, there exists $x \in (\ell_M)_\varrho$ such that $\lim_{n\to\infty} y_n = x$. In view of the continuity of T_α, $\lim_{n\to\infty} T_\alpha y_n = T_\alpha x$. But $\lim_{n\to\infty} T_\alpha x_n = \lim_{n\to\infty} T_\alpha y_n = z$. Therefore, $T_\alpha x = z$. Hence $z \in \text{range}(T_\alpha)$. This proves that T_α has closed range. Conversely, suppose that T_α has closed range. Then T_α is bounded away from zero on $((\ell_M)_\varrho)_{\ker(\alpha)^c}$. That is, there exists $\epsilon > 0$ such that $\varrho(T_\alpha x) \geq \varrho(\epsilon x)$ for all $x \in ((\ell_M)_\varrho)_{\ker(\alpha)^c}$. Let $D = \{k \in \ker(\alpha)^c : |\alpha_k| < \epsilon\}$. If $D \neq \phi$, then for $n_0 \in D$, we have

$$\varrho(T_\alpha e_{n_0}) = \varrho\bigl(\bigl(\alpha_k(e_{n_0})_k\bigr)_{k=0}^{\infty}\bigr) = \sum_{k=0}^{\infty} M\bigl(\bigl|\alpha_k(e_{n_0})_k\bigr|\bigr) < \sum_{k=0}^{\infty} M\bigl(\bigl|\epsilon(e_{n_0})_k\bigr|\bigr) = \varrho(\epsilon e_{n_0}),$$

which is a contradiction. Hence, $D = \phi$ so that $|\alpha_k| \geq \epsilon$ for all $k \in \ker(\alpha)^c$. This proves the theorem. □

Theorem 4.8 *Let $\alpha : \mathbb{N} \to \mathbb{C}$ be a mapping and $(\ell_M)_\varrho$ be a pre-quasi Banach (sss), where $\varrho(x) = \sum_{k=0}^{\infty} M(|x_k|)$ for all $x \in \ell_M$. There exist $a > 0$ and $A > 0$ such that $a < \alpha_n < A$; for all $n \in \mathbb{N}$ if and only if $T_\alpha \in L((\ell_M)_\varrho)$ is invertible.*

Proof Suppose that the condition is true. Define $\beta : \mathbb{N} \to \mathbb{C}$ by $\beta_n = \frac{1}{\alpha_n}$. Then T_α and T_β are bounded linear operators in view of Theorem 4.2. Also $T_\alpha.T_\beta = T_\beta.T_\alpha = I$. Hence, T_β is the inverse of T_α. Conversely, suppose that T_α is invertible. Then range$(T_\alpha) = ((\ell_M)_\varrho)_{\mathbb{N}}$. Therefore, range$(T_\alpha)$ is closed. Hence, by Theorem 4.7, there exists $a > 0$ such that $|\alpha_n| \geq a$ for all $n \in \ker(\alpha)^c$. Now $\ker(\alpha) = \phi$; otherwise $\alpha_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, in which case $e_{n_0} \in \ker(T_\alpha)$, which is a contradiction, since $\ker(T_\alpha)$ is trivial. Hence, $|\alpha_n| \geq a$ for all $n \in \mathbb{N}$. Since T_α is bounded, so by Theorem 4.2, there exists $A > 0$ such that $|\alpha_n| \leq A$ for all $n \in \mathbb{N}$. Thus, we have proved that $a \leq |\alpha_n| \leq A$ for all $n \in \mathbb{N}$. □

Theorem 4.9 *Let $(\ell_M)_\varrho$ be a pre-quasi Banach (sss), where $\varrho(x) = \sum_{k=0}^{\infty} M(|x_k|)$ for all $x \in \ell_M$ and $T_\alpha \in L((\ell_M)_\varrho)$. Then T_α is a Fredholm operator if and only if*

(i) $\ker(\alpha)$ *is a finite subset of* $\mathbb{N}$.
(ii) $|\alpha_n| \geq \epsilon$ *for all* $n \in \ker(\alpha)^c$.

Proof Suppose that T_α is Fredholm. If $\ker(\alpha)$ is an infinite subset of $\mathbb{N}$, then $e_n \in \ker(T_\alpha)$ for all $n \in \ker(\alpha)$. But e_ns are linearly independent, which shows that $\ker(T_\alpha)$ is infinite dimensional, which is a contradiction. Hence, $\ker(\alpha)$ must be a finite subset of $\mathbb{N}$. Condition (ii) follows from Theorem 4.7. Conversely, if conditions (i) and (ii) are true, then we prove that T_α is Fredholm. In view of Theorem 4.7, condition (ii) implies that T_α has closed range. Condition (i) implies that $\ker(T_\alpha)$ and $(\text{range}(T_\alpha))^c$ are finite dimensional. This proves that T_α is Fredholm. □

5 Pre-quasi closed ideal components

For which Orlicz sequence space ℓ_M, are the components of pre-quasi operator ideal S_{ℓ_M} closed?

Theorem 5.1 *$(S_{(\ell_M)_\rho}, g)$ is a pre-quasi closed operator ideal, where $\rho(x) = \sum_{n=0}^{\infty} M(|x_n|)$ and $g(T) = \rho((s_n(T))_{n=0}^{\infty})$, if X, Y are normed spaces and M is an Orlicz function satisfying Δ_2-condition.*

Proof Since ℓ_M is a pre-modular (sss) by Theorem 2.13, then from Theorem 2.16, the function $g(T) = \rho((s_n(T))_{n=0}^{\infty})$ is a pre-quasi norm on $S_{(\ell_M)_\rho}$. Let $T_m \in S_{(\ell_M)_\rho}(X, Y)$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} g(T_m - T) = 0$, then by using Definition 2.12(vii) there exists a constant $\xi > 0$, and since $L(X, Y) \supseteq S_{(\ell_M)_\rho}(X, Y)$, we get

$$g(T - T_m) = \sum_{n=0}^{\infty} M\big(s_n(T - T_m)\big) \geq \xi \|T - T_m\| M(1),$$

then $(T_m)_{m\in\mathbb{N}}$ is a convergent sequence in $L(X, Y)$. While $(s_n(T_m))_{n=0}^{\infty} \in (\ell_M)_\rho$ for each $m \in \mathbb{N}$, hence using Definition 2.12 conditions (iii), (iv), and M is continuous from right at 0, we obtain

$$g(T) = \sum_{n=0}^{\infty} M\big(s_n(T)\big) = \sum_{n=0}^{\infty} M\big(s_n(T - T_m + T_m)\big)$$

$$\leq K\left(\sum_{n=0}^{\infty} M\big(s_{[\frac{n}{2}]}(T - T_m)\big) + \sum_{n=0}^{\infty} M\big(s_{[\frac{n}{2}]}(T_m)\big)\right)$$
$$\leq K\left(\sum_{n=0}^{\infty} M\big(\|T_m - T\|\big) + K_0 \sum_{n=0}^{\infty} M\big(s_n(T_m)\big)\right) < \varepsilon,$$

we have $(s_n(T))_{n=0}^{\infty} \in (\ell_M)_\rho$, then $T \in S_{(\ell_M)_\rho}(X, Y)$. □

6 Pre-quasi simple Banach operator ideal

We give here the sufficient conditions on Orlicz sequence space such that the pre-quasi operator ideal formed by the sequence of s-numbers and this sequence space are strictly contained for different Orlicz functions.

Theorem 6.1 *Let φ_1 and φ_2 be two Orlicz functions satisfying Δ_2-condition. For any infinite dimensional Banach spaces X, Y and $\varphi_2(t) < \varphi_1(t)$ for all $t \in (0, \infty)$, then*

$$S_{\ell_{\varphi_1}}(X, Y) \subsetneqq S_{\ell_{\varphi_2}}(X, Y) \subsetneqq L(X, Y).$$

Proof Let X and Y be infinite dimensional Banach spaces, if $T \in S_{\ell_{\varphi_1}}(X, Y)$, then $(s_n(T)) \in \ell_{\varphi_1}$. We have

$$\sum_{n=0}^{\infty} \varphi_2\big(s_n(T)\big) < \sum_{n=0}^{\infty} \varphi_1\big(s_n(T)\big) < \infty,$$

hence $T \in S_{\ell_{\varphi_2}}(X, Y)$. Next, if we take $(s_n(T))_{n=0}^{\infty}$ such that $\varphi_1(s_n(T)) = \frac{1}{\sqrt[q]{n+1}}$ and $\varphi_2(s_n(T)) = \frac{1}{(n+1)^{\frac{q}{p}}}$ for any $q > p > 0$, we can find $T \in L(X, Y)$ with T does not belong to $S_{\ell_{\varphi_1}}(X, Y)$ and $T \in S_{\ell_{\varphi_2}}(X, Y)$.

It is easy to verify that $S_{\ell_{\varphi_2}}(X, Y) \subset L(X, Y)$. Next, if we take $(s_n(T))_{n=0}^{\infty}$ such that $\varphi_2(s_n(T)) = \frac{1}{n+1}$, one can find $T \in L(X, Y)$ such that T does not belong to $S_{\ell_{\varphi_2}}(X, Y)$. This completes the proof. □

Corollary 6.2 *For any infinite dimensional Banach spaces X, Y and $0 < p < q < \infty$, then*

$$S_{\ell_p}(X, Y) \subsetneqq S_{\ell_q}(X, Y) \subsetneqq L(X, Y).$$

The following question arises naturally: For which Orlicz sequence space, is the pre-quasi Banach ideal simple?

Theorem 6.3 *Let φ_1 and φ_2 be two Orlicz functions satisfying Δ_2-condition. For any infinite dimensional Banach spaces X, Y and $\varphi_2(t) < \varphi_1(t)$ for all $t \in (0, \infty)$, then*

$$L(S_{\ell_{\varphi_2}}, S_{\ell_{\varphi_1}}) = \Psi(S_{\ell_{\varphi_2}}, S_{\ell_{\varphi_1}}).$$

Proof Suppose that there exists $T \in L(S_{\ell_{\varphi_2}}, S_{\ell_{\varphi_1}})$ which is not approximable. According to Lemma 2.3, we can find $X \in L(S_{\ell_{\varphi_2}}, S_{\ell_{\varphi_2}})$ and $B \in L(S_{\ell_{\varphi_1}}, S_{\ell_{\varphi_1}})$ with $BTXI_k = I_k$. Then it follows for all $k \in \mathbb{N}$ that

$$\|I_k\|_{S_{\ell_{\varphi_1}}} = \sum_{n=0}^{\infty} \varphi_1\big(s_n(I_k)\big) \leq \|BTX\| \|I_k\|_{S_{\ell_{\varphi_2}}} \leq \sum_{n=0}^{\infty} \varphi_2\big(s_n(I_k)\big).$$

This contradicts Theorem 6.1, which is a contradiction. Hence $T \in \Psi(S_{\ell_{\varphi_2}}, S_{\ell_{\varphi_1}})$, which finishes the proof. □

Corollary 6.4 *Let φ_1 and φ_2 be two Orlicz functions satisfying Δ_2-condition. For any infinite dimensional Banach spaces X, Y and $\varphi_2(t) < \varphi_1(t)$ for all $t \in (0,\infty)$, then*

$$L(S_{\ell_{\varphi_2}}, S_{\ell_{\varphi_1}}) = L_C(S_{\ell_{\varphi_2}}, S_{\ell_{\varphi_1}}).$$

Proof Since every approximable operator is compact. □

7 Eigenvalues of s-type operators

We give here the sufficient conditions on Orlicz sequence space such that the pre-quasi operator ideal is formed by the sequence of s-numbers and this sequence space is strictly contained in the class of all bounded linear operators whose sequence of eigenvalues belongs to this sequence space.

Notations 7.1

$$\begin{aligned} S_E^\lambda &:= \bigl\{S_E^\lambda(X,Y); X \text{ and } Y \text{ are Banach spaces}\bigr\}, \quad \text{where} \\ S_E^\lambda(X,Y) &:= \bigl\{T \in L(X,Y) : \bigl(\bigl(\lambda_i(T)\bigr)_{n=0}^\infty \in E \text{ and } \bigl\|T - \lambda_n(T)\bigr\| \\ &\qquad \text{is not invertible for all } n \in \mathbb{N}\bigr\}. \end{aligned}$$

Theorem 7.2 *For any infinite dimensional Banach spaces X and Y. If M is an Orlicz function satisfying Δ_2-condition, then*

$$S_{\ell_M}(X,Y) \subsetneqq S_{\ell_M}^\lambda(X,Y).$$

Proof Let $T \in S_{\ell_M}(X,Y)$, then $(s_n(T))_{n=0}^\infty \in \ell_M$, we have $\sum_{n=0}^\infty M(s_n(T)) < \infty$, and since M is continuous, so $\lim_{n\to\infty} s_n(T) = 0$. Suppose $\|T - s_n(T)\|$ is invertible for all $n \in \mathbb{N}$, then $\|T - s_n(T)\|^{-1}$ exists and is bounded for all $n \in \mathbb{N}$. This gives $\lim_{n\to\infty} \|T - s_n(T)\|^{-1} = \|T\|^{-1}$ exists and is bounded. Since (S_{ℓ_M}, g) is a pre-quasi operator ideal, we have

$$I = TT^{-1} \in S_{\ell_M}(X,Y) \quad \Rightarrow \quad \bigl(s_n(I)\bigr)_{n=0}^\infty \in \ell_M \quad \Rightarrow \quad \lim_{n\to\infty} s_n(I) = 0.$$

But $\lim_{n\to\infty} s_n(I) = 1$. This is a contradiction, then $\|T - s_n(T)\|$ is not invertible for all $n \in \mathbb{N}$. Therefore the sequence $(s_n(T))_{n=0}^\infty$ is the eigenvalues of T. Next, on considering $(s_n(T))_{n=0}^\infty$ such that $M(s_n(T)) = \frac{1}{n+1}$, we find $T \in L(X,Y)$ with T does not belong to $S_{\ell_M}(X,Y)$, and if we take $(\lambda_n(T))_{n=0}^\infty$ such that $M(\lambda_n(T)) = \frac{1}{(n+1)^2}$, hence $T \in S_{\ell_M}^\lambda(X,Y)$. This finishes the proof. □

Acknowledgements
The authors thank the anonymous referees for their constructive suggestions and helpful comments which led to significant improvement of the original manuscript of this paper.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 20 November 2019 Accepted: 14 February 2020 Published online: 28 February 2020

References

1. Abrahmse, M.B.: Multiplication operators. In: Hilbert Space Operators, Lecture Notes in Mathematics, vol. 693, pp. 17–36. Springer, Berlin (1978)
2. Alotaibi, A., Mursaleen, M., Raj, K.: Double sequence spaces by means of Orlicz functions. Abstr. Appl. Anal. **2014**, Article ID 260326 (2014)
3. Alotaibi, A., Mursaleen, M., Sharma, S.K.: Double sequence spaces over n-normed spaces defined by a sequence of Orlicz functions. J. Inequal. Appl. **2014**, 216 (2014)
4. Alotaibi, A., Mursaleen, M., Sharma, S.K., Mohiuddine, S.A.: Sequence spaces of fuzzy numbers defined by a Musielak–Orlicz function. Filomat **29**(7), 1461–1468 (2015)
5. Altay, B., Başar, F.: Generalization of the sequence space $\ell(p)$ derived by weighted means. J. Math. Anal. Appl. **330**(1), 147–185 (2007)
6. Bakery, A.A., Mohammed, M.M.: Small pre-quasi Banach operator ideals of type Orlicz–Cesàro mean sequence spaces. J. Funct. Spaces **2019**, Article ID 7265010 (2019)
7. Et, M., Mursaleen, M., Işik, M.: On a class of fuzzy sets defined by Orlicz functions. Filomat **27**(5), 789–796 (2013)
8. Faried, N., Bakery, A.A.: Mappings of type Orlicz and generalized Cesáro sequence space. J. Inequal. Appl. (2013). https://doi.org/10.1186/1029-242X-2013-186
9. Faried, N., Bakery, A.A.: Small operator ideals formed by s numbers on generalized Cesáro and Orlicz sequence spaces. J. Inequal. Appl. (2018). https://doi.org/10.1186/s13660-018-1945-y
10. Komal, B.S., Gupta, S.: Multiplication operators between Orlicz spaces. Integral Equ. Oper. Theory **41**(3), 324–330 (2001)
11. Komal, B.S., Pandoh, S., Raj, K.: Multiplication operators on Cesáro sequence spaces. Demonstr. Math. **49**(4), 430–436 (2016)
12. Krasnoselskii, M.A., Rutickii, Y.B.: Convex Functions and Orlicz Spaces. Gorningen, Netherlands (1961)
13. Lindenstrauss, J., Tzafriri, L.: On Orlicz sequence spaces. Isr. J. Math. **10**, 379–390 (1971)
14. Makarov, B.M., Faried, N.: Some properties of operator ideals constructed by s numbers (in Russian). In: Theory of Operators in Functional Spaces. Academy of Science. Siberian Section, pp. 206–211. Novosibirsk, Russia (1977)
15. Mohiuddine, S.A., Raj, K.: Vector valued Orlicz–Lorentz sequence spaces and their operator ideals. J. Nonlinear Sci. Appl. **10**, 338–353 (2017)
16. Mrowka, T.: A Brief Introduction to Linear Analysis: Fredholm Operators. Geometry of Manifolds. Fall 2004 (Massachusetts Institute of Technology: MIT OpenCourseWare) (2004)
17. Mursaleen, M., Noman, A.K.: Compactness by the Hausdorff measure of noncompactness. Nonlinear Anal. **73**, 2541–2557 (2010)
18. Mursaleen, M., Noman, A.K.: Compactness of matrix operators on some new difference sequence spaces. Linear Algebra Appl. **436**, 41–52 (2012)
19. Mursaleen, M., Raj, K., Sharma, S.K.: Some spaces of difference sequences and lacunary statistical convergence in n-normed space defined by sequence of Orlicz functions. Miskolc Math. Notes **16**(1), 283–304 (2015)
20. Mursaleen, M., Sharma, S., Kiliçman, A.: Sequence spaces defined by Musielak–Orlicz function over n-normed spaces. Abstr. Appl. Anal. **2013**, Article ID 364743 (2013)
21. Mursaleen, M., Sharma, S.K., Mohiuddine, S.A., Kiliçman, A.: New difference sequence spaces defined by Musielak–Orlicz function. Abstr. Appl. Anal. **2014**, Article ID 691632 (2014)
22. Orlicz, W.: Über Räume (L^M). Bull. Int. Acad Polon. Sci. A 93–107 (1936)
23. Pietsch, A.: Small ideals of operators. Stud. Math. **51**, 265–267 (1974)
24. Pietsch, A.: Operator Ideals. North-Holland, Amsterdam (1980)
25. Pietsch, A.: Eigenvalues and s-Numbers. Cambridge University Press, New York (1986)
26. Raj, K., Sharma, S.K., Kumar, A.: Multiplication operator on Musielak–Orlicz spaces of Bochner type. J. Adv. Stud. Topol. **3**, 1–7 (2012)
27. Sharma, A., Raj, K., Sharma, S.K.: Products of multiplication composition and differentiation operators from H^∞ to weighted Bloch spaces. Indian J. Math. **54**, 159–179 (2012)
28. Singh, R.K., Kumar, A.: Multiplication operators with closed range. Bull. Aust. Math. Soc. **16**, 247–252 (1977)
29. Singh, R.K., Manhas, J.S.: Multiplication operators and composition operators with closed ranges. Bull. Aust. Math. Soc. **16**, 247–252 (1977)
30. Singh, R.K., Manhas, J.S.: Composition Operators on Function Spaces. North Holland, Amsterdam (1993)
31. Takagi, H., Yokouchi, K.: Multiplication and composition operators between two L_p-spaces. Contemp. Math. **232**, 321–338 (1999)