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Approximation on a class of Szász–Mirakyan operators via second kind of beta operators

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Abstract

In the present article, we construct a new sequence of positive linear operators via Dunkl analogue of modified Szász–Durrmeyer operators. We study the moments and basic results. Further, we investigate the pointwise approximation and uniform approximation results in various functional spaces for these sequences of positive linear operators. Finally, we prove the global approximation and A-statistical convergence results for these operators.

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1 Introduction

In recent past, the Szász–Mirakjan operators and Szász–Durrmeyer type operators have been intensively investigated in various functional spaces to approximate the continuous functions and Lebesgue measurable functions respectively. Mazhar and Totik [17] gave an integral modification of Szász–Mirakyan [37] operators to study the Lebesgue measurable functions as follows:

$$S_n^*(g;u) = ne^{-nu}\sum_{k=0}^{\infty}\frac{(nu)^k}{k!}\int_0^{\infty}e^{-nt}\frac{(nt)^k}{k!}g(t)\,dt, \quad u\geq 0, n\in\mathbb{N}=\{1,2,3,\dots\}. \tag{1.1}$$

Researchers have obtained several approximations of Szász–Mirakyan type operators via Dunkl generalization; for instance, see [6, 18, 26, 28, 29, 32, 39]. Related to these results, more approximation results have been studied in different functional spaces (see [1, 2, 4, 5, 14, 38] and [3, 16, 27, 31]). Sucu [36] introduced Szász–Mirakyan type operators for $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ as follows:

$$S_n(g;u) := \frac{1}{e_i(nu)}\sum_{k=0}^{\infty}\frac{(nu)^k}{\gamma_i(k)}g\left(\frac{k+2i\theta_k}{n}\right), \tag{1.2}$$

using generalized exponential function [33] given by

$$e_i(t) = \sum_{j=0}^{\infty} \frac{t^j}{\gamma_i(j)}, \quad t \in [0,\infty), \tag{1.3}$$

where the coefficients γ_i are defined as follows:

$$\gamma_i(2j) = \frac{2^{2j} j! \Gamma(j+i+\frac{1}{2})}{\Gamma(i+\frac{1}{2})} \quad \text{and} \quad \gamma_i(2j+1) = 2^{2j+1} j! \frac{\Gamma(j+i+\frac{3}{2})}{\Gamma(i+\frac{1}{2})}.$$

Here, Γ is the gamma function and $i > -\frac{1}{2}$. In [36] the pointwise and uniform approximation results for the operators defined by (1.2) are obtained. Several extensions of the Szász–Mirakyan operators defined by (1.2) and their approximation results are studied by the authors viz İçöz et al. [11, 12]. In [40], we have constructed Szász–Durrmeyer type operators to approximate the Lebesgue measurable functions as follows:

$$D_n(g;u) = \frac{1}{e_i(nu)} \sum_{j=0}^{\infty} \frac{(nu)^j}{\gamma_i(j)} \frac{n^{j+2i\theta_j+\lambda+1}}{\Gamma(j+2i\theta_j+\lambda+1)} \int_0^{\infty} t^{j+2i\theta_j+\lambda} e^{-nt} g(t)\,dt, \tag{1.4}$$

where $\Gamma(t) = \int_0^{\infty} u^t e^{-u}\,du$ is the gamma function, $\lambda \geq 0$, and

$$\theta_j = \begin{cases} 0, & j \in 2\mathbb{N}, \\ 1, & j \in 2\mathbb{N}+1. \end{cases}$$

In order to obtain a better rate of convergence in comparison to (1.4), in this paper we introduce a modification with two nonnegative shifted nodes as follows:

$$M_n^{\alpha,\beta}(g;u) = \sum_{k=0}^{\infty} \nu_n(u;k) \int_0^{\infty} t^{k+2i\theta_k+\lambda} e^{-nt} g\left(\frac{nt+\alpha}{n+\beta}\right) dt, \tag{1.5}$$

where $\nu_n(u;k) = \frac{1}{e_i(nu)} \frac{(nu)^k}{\gamma_i(k)} \frac{n^{k+2i\theta_k+\lambda+1}}{\Gamma(k+2i\theta_k+\lambda+1)}$, $0 \leq \alpha \leq \beta$, and $f \in C[0,\infty)$ whenever the above series converges. One can notice that (i) for $i = \lambda = \alpha = \beta = 0$, the sequence of operators introduced in (1.5) reduces to the operators defined in (1.1) and (ii) for $\alpha = \beta = 0$, the sequence of positive linear operators defined in (1.5) reduces to the operators defined in [40].

In the subsequent sections, we deduce some basic results, direct approximation results. Further, global approximation results are studied in [19–21, 24, 25, 35].

2 Preliminary results

For $i > -\frac{1}{2}$, $u \geq 0$, and $n \in \mathbb{N}$, we denote

$$E_n(u) = \frac{e_i(-nu)}{e_i(nu)}. \tag{2.1}$$

Lemma 2.1 ([40]) *For the operators given by* (1.4), *we have*

$$D_n(1;u) = 1,$$

$$
\begin{aligned}
D_n(t;u) &= u + \frac{\lambda+1}{n},\\
D_n\big(t^2;u\big) &= u^2 + \big(4+2\lambda+2iE_n(u)\big)\frac{u}{n} + \frac{(\lambda+1)(\lambda+2)}{n^2},\\
D_n\big(t^3;u\big) &= u^3 + \big(9+3\lambda-2iE_n(u)\big)\frac{u^2}{n} + \big(18+3\lambda(\lambda+5)+4i^2\\
&\qquad + 2i(8+3\lambda)E_n(u)\big)\frac{u}{n^2} + \frac{\lambda^3+6\lambda^2+11\lambda+6}{n^3},\\
D_n\big(t^4;u\big) &= u^4 + \mathcal{O}\left(\frac{1}{n}\right) \quad \textit{for each compact subset of } [0,\infty).
\end{aligned}
$$

Lemma 2.2 *For operators* (1.5), *we have*

$$
M_n^{\alpha,\beta}\big(t^r;u\big) = \sum_{i=0}^{r}\binom{r}{i}\frac{n^i\alpha^{r-i}}{(n+\beta)^r}D_n\big(t^i;u\big), \quad r\in\mathbb{N}.
$$

Proof From (1.5), we get

$$
\begin{aligned}
M_n^{\alpha,\beta}\big(t^r;u\big) &= \sum_{k=0}^{\infty}\nu_n(u;t)\int_0^{\infty} t^{k+2i\theta_k+\lambda}e^{-nt}\left(\frac{nt+\alpha}{n+\beta}-u\right)^r\\
&= \sum_{j=0}^{r}\binom{r}{j}\frac{n^j\alpha^{r-j}}{(n+\beta)^r}\sum_{k=0}^{\infty}\nu_n(u;t)\int_0^{\infty} t^{k+2i\theta_k+\lambda}e^{-nt}t^j\,dt\\
&= \sum_{j=0}^{r}\binom{r}{j}\frac{n^j\alpha^{r-j}}{(n+\beta)^r}D_n\big(t^j;u\big).
\end{aligned}
$$

□

Lemma 2.3 *For $r\in\mathbb{N}$, we have the following relation*:

$$
M_n^{\alpha,\beta}\big((t-u)^r;u\big) = \sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}M_n^{\alpha,\beta}\big(t^i;u\big),
$$

where $M_n^{\alpha,\beta}$ is defined in (1.5).

Proof From relation (1.5), we get

$$
\begin{aligned}
M_n^{\alpha,\beta}\big((t-u)^r;u\big) &= \sum_{k=0}^{\infty}\nu_n(u;k)\int_0^{\infty} t^{k+2i\theta_k+\lambda}e^{-nt}\left(\frac{nt+\alpha}{n+\beta}-u\right)^r dt\\
&= \sum_{j=0}^{r}\binom{r}{j}(-u)^{r-j}M_n^{\alpha,\beta}\big(t^j;u\big).
\end{aligned}
$$

□

Lemma 2.4 *Let $e_r(t)=t^r$ for $r\in\{0,1,2,3,4\}$ be the test functions and $E_n(u)$ be defined in* (2.1). *Then*

$$
\begin{aligned}
M_n^{\alpha,\beta}(e_0;u) &= 1,\\
M_n^{\alpha,\beta}(e_1;u) &= \frac{n}{n+\beta}u + \frac{\lambda+\alpha+1}{n+\beta},
\end{aligned}
$$

$$
\begin{aligned}
M_n^{\alpha,\beta}(e_2;u) &= \frac{n^2}{(n+\beta)^2}u^2 + \bigl(4+2\lambda+2\alpha+2iE_n(u)\bigr)\frac{n}{(n+\beta)^2}u \\
&\quad + \frac{(\lambda+1)(\lambda+2\alpha+2)+\alpha^2}{(n+\beta)^2}, \\
M_n^{\alpha,\beta}(e_3;u) &= \frac{n^3}{(n+\beta)^3}u^3 + \bigl(9+3\alpha+3\lambda-2iE_n(u)\bigr)\frac{n^2}{(n+\beta)^3}u^2 \\
&\quad + \bigl(18+12\alpha+3\lambda(\lambda+5+2\alpha)+3\alpha^2+4i^2+2i(8+3\lambda+3\alpha)E_n(u)\bigr) \\
&\quad \times \frac{n}{(n+\beta)^3}u + \frac{\lambda^3+\alpha^3+\lambda^2(6+3\alpha)+\lambda(11+9\alpha+3\alpha^2)+6+6\alpha+3\alpha^2}{(n+\beta)^3}, \\
M_n^{\alpha,\beta}(e_4;u) &= \frac{n^4}{(n+\beta)^4}u^4 + \mathcal{O}\left(\frac{1}{n+\beta}\right) \quad \textit{for each compact subset of } [0,\infty).
\end{aligned}
$$

Proof The proof follows immediately from Lemma 2.2. □

Lemma 2.5 *Let $\psi_u^r(t) = (t-u)^r$, $r \in \mathbb{N}$, be the central moments of $M_n^{\alpha,\beta}$. Then*

$$
\begin{aligned}
M_n^{\alpha,\beta}\bigl(\psi_u^0;u\bigr) &= 1, \\
M_n^{\alpha,\beta}\bigl(\psi_u^1;u\bigr) &= \frac{1}{(n+\beta)}(1+\lambda+\alpha-\beta u), \\
M_n^{\alpha,\beta}\bigl(\psi_u^2;u\bigr) &= \frac{\beta^2}{(n+\beta)^2}u^2 + 2\left(1+iE_n(u)-2\beta\frac{1+\lambda+\alpha}{n}\right)\frac{n}{(n+\beta)^2}u \\
&\quad + \frac{\alpha^2+(\lambda+1)(\lambda+2\alpha+2)}{(n+\beta)^2}, \\
M_n^{\alpha,\beta}\bigl(\psi_u^4;u\bigr) &= \mathcal{O}\bigl((n+\beta)^{-1}\bigr) \quad \textit{for each compact subset of } [0,\infty).
\end{aligned}
$$

Proof In the light of Lemmas 2.2 and 2.3, we can easily prove Lemma 2.5. □

Definition 2.6 Let $g \in C[0,\infty)$. Then the modulus of continuity for a uniformly continuous function g is defined as follows:

$$
\omega(g;\delta) = \sup_{|t_1-t_2|\le\delta} \bigl|g(t_1)-g(t_2)\bigr|, \quad t_1,t_2 \in [0,\infty).
$$

For a uniformly continuous function g in $C[0,\infty)$ and $\delta > 0$, we get

$$
\bigl|g(t_1)-g(t_2)\bigr| \le \left(1+\frac{(t_1-t_2)^2}{\delta^2}\right)\omega(g;\delta). \tag{2.2}
$$

Theorem 2.7 *Let $M_n^{\alpha,\beta}$ be the operators defined by (1.5) and $g \in C[0,\infty) \cap \{g : u \ge 0, \frac{g(u)}{1+u^2}$ is convergent as $u \to \infty\}$. Then the operators are defined by (1.5), $M_n^{\alpha,\beta} \rightrightarrows g$ on each compact subset of $[0,\infty)$, where $\rightrightarrows$ stands for uniform convergence.*

Proof For the proof of uniformity for the operators $M_n^{\alpha,\beta}$, we use the well-known Korovkin theorem. So, it is sufficient to show that

$$
M_n^{\alpha,\beta}(e_\nu;u) \to e_\nu(u) \quad \text{for } \nu = 0,1,2.
$$

Using Lemma 2.4, it is obvious that $M_n^{\alpha,\beta}(e_\nu;u) \to e_\nu(u)$ as $n \to \infty$, $\nu = 0,1,2$, which proves Theorem 2.7. □

Theorem 2.8 (See [34]) *Let $L : C([a,b]) \to B([a,b])$ be a linear and positive operator, and let φ_x be the function defined by*

$$\varphi_x(t) = |t-x|, \quad (x,t) \in [a,b] \times [a,b].$$

If $f \in C_B([a,b])$ for any $x \in [a,b]$ and any $\delta > 0$, the operator L verifies

$$\begin{aligned} \bigl|(Lf)(x) - f(x)\bigr| &\le \bigl|f(x)\bigr|\bigl|(Le_0)(x) - 1\bigr| \\ &\quad + \Bigl\{(Le_0)(x) + \delta^{-1}\sqrt{(Le_0)(x)\bigl(L\varphi_x^2\bigr)(x)}\Bigr\}\omega_f(\delta), \end{aligned}$$

where $C[a,b]$ is the space of all continuous functions defined on $[a,b]$, and $C_B[a,b]$ is the space of all continuous and bounded functions defined on $[a,b]$.

Theorem 2.9 *For the operators $M_n^{\alpha,\beta}$ given by (1.5) and $g \in C_B[0,\infty)$, we have*

$$\bigl|M_n^{\alpha,\beta}(g;u) - g(u)\bigr| \le 2\omega(g;\delta),$$

where $\delta = \sqrt{M_n^{\alpha,\beta}(\psi_u^2;u)}$ and $C_B[0,\infty)$ is the space of continuous and bounded functions defined on $[0,\infty)$.

Proof Using Lemma 2.4, Lemma 2.5, and Theorem 2.8, we get

$$\bigl|M_n^{\alpha,\beta}(g;u) - g(u)\bigr| \le \Bigl\{1 + \delta^{-1}\sqrt{M_n^{\alpha,\beta}\bigl(\psi_u^2;u\bigr)}\Bigr\}\omega(g;\delta).$$

On taking $\delta = \sqrt{M_n^{\alpha,\beta}(\psi_u^2;u)}$, we arrive at the required result. □

3 Direct results

Let us equip the norm on g such as $\|g\| = \sup_{0\le u<\infty}|g(u)|$. For any $g \in C_B[0,\infty)$ and $\delta > 0$, Peetre's K-functional is defined as follows:

$$K_2(g,\delta) = \inf\bigl\{\|g-h\| + \delta\bigl\|h''\bigr\| : h \in C_B^2[0,\infty)\bigr\}, \tag{3.1}$$

where $C_B^2[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$. From DeVore and Lorentz ([8], p.177, Theorem 2.4), there exists an absolute constant $C > 0$ in such a way that

$$K_2(g;\delta) \le C\omega_2(g;\sqrt{\delta}).$$

Lemma 3.1 *Let the auxiliary operators be*

$$\widehat{M}_n^{\alpha,\beta}(g;u) = M_n^{\alpha,\beta}(g;u) + g(u) - g\biggl(\frac{n}{n+\beta}u + \frac{\lambda+\alpha+1}{n+\beta}\biggr). \tag{3.2}$$

For $h,g \in C_B^2[0,\infty)$, $u \geq 0$, and $\mu,\lambda \geq 0$, we get

$$\left|\widehat{M}_n^{\alpha,\beta}(g;u) - g(u)\right| \leq \xi_n^u \|h''\|,$$

where

$$\xi_n^u = M_n^{\alpha,\beta}\left(\psi_u^2;u\right) + \left(M_n^{\alpha,\beta}\left(\psi_u^1;u\right)\right)^2.$$

Proof From (3.2), we have

$$\widehat{M}_n^{\alpha,\beta}(1;u) = 1, \qquad \widehat{M}_n^{\alpha,\beta}(\psi_u;u) = 0 \quad \text{and} \quad \left|\widehat{M}_n^{\alpha,\beta}(g;u)\right| \leq 3\|g\|. \tag{3.3}$$

From Taylor's series for $h \in C_B^2[0,\infty)$, we have

$$h(t) = h(u) + (t-u)h'(u) + \int_u^t (t-v)h''(v)\,dv. \tag{3.4}$$

Using $\widehat{M}_n^{\alpha,\beta}(g;u)$ in (3.4), we get

$$\widehat{M}_n^{\alpha,\beta}(h;u) - h(u) = h'(u)\widehat{M}_n^{\alpha,\beta}(t-u;u) + \widehat{M}_n^{\alpha,\beta}\left(\int_u^t (t-v)h''(v)\,dv;u\right).$$

From (3.2) and (3.3), we have

$$\begin{aligned}
\widehat{M}_n^{\alpha,\beta}(h;u) - h(u) &= \widehat{M}_n^{\alpha,\beta}\left(\int_u^t (t-v)h''(v)\,dv;u\right) \\
&= M_n^{\alpha,\beta}\left(\int_u^t (t-v)h''(v)\,dv;u\right) \\
&\quad - \int_u^{\frac{n}{n+\beta}u+\frac{\lambda+\alpha+1}{n+\beta}} \left(\frac{n}{n+\beta}u + \frac{\lambda+\alpha+1}{n+\beta} - v\right)g''(v)\,dv,
\end{aligned}$$

$$\begin{aligned}
&\left|\widehat{M}_n^{\alpha,\beta}(h;u) - h(u)\right| \\
&\quad \leq \left|M_n^{\alpha,\beta}\left(\int_u^t (t-v)h''(v)\,dv;u\right)\right| \\
&\qquad + \left|\int_u^{\frac{n}{n+\beta}u+\frac{\lambda+\alpha+1}{n+\beta}} \left(\frac{n}{n+\beta}u + \frac{\lambda+\alpha+1}{n+\beta} - v\right)h''(v)\,dv\right|.
\end{aligned} \tag{3.5}$$

Since

$$\left|\int_u^t (t-v)h''(v)\,dv\right| \leq (t-v)^2\|h''\|, \tag{3.6}$$

we get

$$\left|\int_u^{M_n^{\alpha,\beta}(e_1;u)} \left(M_n^{\alpha,\beta}(e_1;u) - v\right)h''(v)\,dv\right| \leq \left(M_n^{\alpha,\beta}(t-v;u)\right)^2\|h''\|. \tag{3.7}$$

Combining (3.5), (3.6), and (3.7), we have

$$\widehat{M}_n^{\alpha,\beta}(h;u) - h(u)| \leq \xi_n^u\|h''\|. \qquad \square$$

Theorem 3.2 *Let $f \in C_B^2[0,\infty)$. Then there exists a constant $C > 0$ such that*

$$\left|M_n^{\alpha,\beta}(g;u) - g(u)\right| \leq C\omega_2\left(g; \frac{1}{2}\sqrt{\xi_n^u}\right) + \omega(g; M_n^{\alpha,\beta}(\psi_u;u),$$

where ξ_n^u is defined in Lemma 3.1.

Proof For $h \in C_B^2[0,\infty)$ and $g \in C_B[0,\infty)$ and by the definition of $\widehat{M}_n^{\alpha,\beta}$, we have

$$\begin{aligned}\left|M_n^{\alpha,\beta}(g;u) - g(u)\right| &\leq \left|\widehat{M}_n^{\alpha,\beta}(g-h;u)\right| + \left|(g-h)(u)\right| + \left|\widehat{M}_n^{\alpha,\beta}(h;u) - h(u))\right| \\ &\quad + \left|g\left(\frac{n}{n+\beta}u + \frac{\lambda+\alpha+1}{n+\beta}\right) - g(u)\right|.\end{aligned}$$

From Lemma 3.1 and relations in (3.3), one gets

$$\begin{aligned}\left|M_n^{\alpha,\beta}(g;u) - g(u)\right| &\leq 4\|g-h\| + \left|\widehat{M}_n^{\alpha,\beta}(h;u) - h(u)\right| \\ &\quad + \left|g\left(\frac{n}{n+\beta}u + \frac{\lambda+\alpha+1}{n+\beta}\right) - g(u)\right| \\ &\leq 4\|g-h\| + \xi_n^u \left\|h''\right\| + \omega\big(g; M_n^{\alpha,\beta}(\psi_u;u)\big).\end{aligned}$$

Using the definition of Peetre's K-functional, we have

$$\left|M_n^{\alpha,\beta}(g;u) - g(u)\right| \leq C\omega_2\left(g; \frac{1}{2}\sqrt{\xi_n^u}\right) + \omega(g; M_n^{\alpha,\beta}(\psi_u;u).$$

This completes the proof of Theorem 3.2. □

We consider the Lipschitz type space [30]:

$$\mathrm{Lip}_M^{k_1,k_2}(\rho) := \left\{g \in C_B[0,\infty) : \left|g(t) - g(u)\right| \leq M\frac{|t-u|^\rho}{(t+k_1u+k_2u^2)^{\frac{\rho}{2}}} : u,t \in (0,\infty)\right\},$$

where $M \geq 0$ is a constant, $k_1, k_2 > 0$, $\rho > 0$, and $\rho \in (0,1]$.

Theorem 3.3 *For $g \in \mathrm{Lip}_M^{k_1,k_2}(\rho)$, we have*

$$\left|M_n^{\alpha,\beta}(g;u) - g(u)\right| \leq M\left(\frac{\eta_n^*(u)}{k_1u+k_2u^2}\right)^{\frac{\rho}{2}}, \tag{3.8}$$

where $u > 0$ and $\eta_n^*(u) = M_n^{\alpha,\beta}(\psi_u^2;u)$.

Proof For $\rho = 1$, we have

$$\begin{aligned}\left|M_n^{\alpha,\beta}(g;u) - g(u)\right| &\leq M_n^{\alpha,\beta}\big(\left|g(t) - g(u)\right|\big)(u) \\ &\leq MM_n^{\alpha,\beta}\left(\frac{|t-u|}{(t+k_1u+k_2u^2)^{\frac{1}{2}}}; u\right).\end{aligned}$$

Since $\frac{1}{t+k_1u+k_2u^2} < \frac{1}{k_1u+k_2u^2}$ for all $t,u \in (0,\infty)$, we get

$$\begin{aligned}\left|M_n^{\alpha,\beta}(g;u)-g(u)\right| &\leq \frac{M}{(k_1u+k_2u^2)^{\frac{1}{2}}}\left(M_n^{\alpha,\beta}\left((t-u)^2;x\right)\right)^{\frac{1}{2}}\\ &\leq M\left(\frac{\eta_n^*(u)}{k_1u+k_2u^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Thus Theorem 3.3 holds for $\rho = 1$. Next, for $\rho \in (0,1)$ and from Hölder's inequality with $p = \frac{2}{\rho}$ and $q = \frac{2}{2-\rho}$, we get

$$\begin{aligned}\left|M_n^{\alpha,\beta}(g;u)-g(u)\right| &\leq \left(M_n^{\alpha,\beta}\left(\left(\left|g(t)-g(u)\right|\right)^{\frac{2}{\rho}};u\right)\right)^{\frac{\rho}{2}}\\ &\leq M\left(M_n^{\alpha,\beta}\left(\frac{|t-u|^2}{(t+k_1u+k_2u^2)};u\right)\right)^{\frac{\rho}{2}}.\end{aligned}$$

Since $\frac{1}{t+k_1u+k_2u^2} < \frac{1}{k_1u+k_2u^2}$ for all $t,u \in (0,\infty)$, we obtain

$$\begin{aligned}\left|M_n^{\alpha,\beta}(g;u)-g(u)\right| &\leq M\left(\frac{M_n^{\alpha,\beta}(|t-u|^2;u)}{k_1u+k_2u^2}\right)^{\frac{\rho}{2}}\\ &\leq M\left(\frac{\eta_n^*(u)}{k_1u+k_2u^2}\right)^{\frac{\rho}{2}}.\end{aligned}$$

Hence, we prove Theorem 3.3. □

4 Global approximation

From [10], we recall some notation to prove the global approximation results.

For the weight function $1+u^2$ and $0 \leq u < \infty$, we have

$$B_{1+u^2}[0,\infty) = \left\{f(u) : \left|f(u)\right| \leq M_f\left(1+u^2\right)\right\}, \tag{4.1}$$

where M_f is a constant depending on f and $C_{1+u^2}[0,\infty) \subset B_{1+u^2}[0,\infty)$ is a space of continuous functions equipped with the norm $\|f\|_{1+u^2} = \sup_{u\in[0,\infty)} \frac{|f|}{1+u^2}$.

Denote

$$C_{1+u^2}^k[0,\infty) = \left\{f \in C_{1+u^2} : \lim_{u\to\infty}\frac{f(u)}{1+u^2} = k,\ \text{where } k \text{ is a constant}\right\}. \tag{4.2}$$

Theorem 4.1 *Let the operators be defined by (1.5) acting from $C_{1+u^2}^k[0,\infty)$ to $B_{1+u^2}[0,\infty)$. Then, for every $f \in C_{1+u^2}^k[0,\infty)$, we have*

$$\lim_{n\to\infty}\left\|M_n^{\alpha,\beta}(g;u)-g\right\|_{1+u^2} = 0,$$

where B_{1+u^2} and $C_{1+u^2}^k$ are defined in (4.1) and (4.2).

Proof To prove this result, it is sufficient to show that

$$\lim_{n\to\infty}\left\|M_n^{\alpha,\beta}(e_i;u)-u^i\right\|_{1+u^2} = 0, \quad i = 0,1,2.$$

Using Lemma 2.4, we obtain

$$\left\|M_n^{\alpha,\beta}(e_0;u)-u^0\right\|_{1+u^2}=\sup_{u\in[0,\infty)}\frac{|M_n^{\alpha,\beta}(1;u)-1|}{1+u^2}=0\quad\text{for } i=0.$$

For $i=1$,

$$\begin{aligned}\left\|M_n^{\alpha,\beta}(e_1;u)-u\right\|_{1+u^2}&=\sup_{u\in[0,\infty)}\frac{(\frac{n}{n+\beta}-1)u+\frac{\lambda+\alpha+1}{n+\beta}}{1+u^2}\\&=\left(\frac{n}{n+\beta}-1\right)\sup_{u\in[0,\infty)}\frac{u}{1+u^2}+\frac{\lambda+\alpha+1}{n+\beta}\sup_{u\in[0,\infty)}\frac{1}{1+u^2}.\end{aligned}$$

This shows that $\|M_n^{\alpha,\beta}(e_1;u)-u\|_{1+u^2}\to 0$, $n\to\infty$. For $i=2$,

$$\begin{aligned}\left\|M_n^{\alpha,\beta}(e_2;u)-u^2\right\|_{1+u^2}&=\sup_{u\in[0,\infty)}\frac{|M_n^{\alpha,\beta}(e_2;u)-u^2|}{1+u^2}\\&\le\left(\frac{n^2}{(n+\beta)^2}-1\right)\sup_{u\in[0,\infty)}\frac{u^2}{1+u^2}\\&\quad+\left(4+2\lambda+2\alpha+2iE_n(u)\right)\frac{n}{(n+\beta)^2}\sup_{u\in[0,\infty)}\frac{u}{1+u^2}\\&\quad+\frac{(\lambda+1)(\lambda+2\alpha+2)+\alpha^2}{(n+\beta)^2}\sup_{u\in[0,\infty)}\frac{1}{1+u^2},\end{aligned}$$

which shows that $\|M_n^{\alpha,\beta}(e_2;u)-u^2\|_{1+u^2}\to 0$, $n\to\infty$. □

Let $f\in C_\rho^k[0,\infty)$, Yüksel and Ispir [41] introduced the weighted modulus of continuity as follows:

$$\Omega(g;\delta)=\sup_{u\in[0,\infty),0<h\le\delta}\frac{|g(u+h)-g(u)|}{1+(u+h)^2}.$$

Lemma 4.2 ([41]) *Let $f\in C_{1+x^2}^k[0,\infty)$. Then*
(i) *$\Omega(f;\delta)$ is a monotone increasing function of δ;*
(ii) $\lim_{\delta\to 0^+}\Omega(f;\delta)=0$;
(iii) *for each $\lambda\in[0,\infty)$, $\Omega(f;\lambda\delta)\le(1+\lambda)(1+\delta^2)\Omega(f;\delta)$.*
For $t,x\in[0,\infty)$, one gets

$$\left|f(t)-f(x)\right|\le 2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\left(1+x^2\right)\left(1+(t-x)^2\right)\Omega(f;\delta).\tag{4.3}$$

Theorem 4.3 *Let $g\in C_{1+u^2}^k[0,\infty)$. Then we have*

$$\sup_{u\in[0,\infty)}\frac{|M_n^{\alpha,\beta}(g;u)-g(x)|}{(1+u^2)^3}\le C(n)\left(1+\frac{1}{n+\beta}\right)\Omega\left(g;\frac{1}{\sqrt{n+\beta}}\right),$$

where $C(n)=1+\frac{C_1}{n+\beta}\cdot\frac{3\sqrt{(3)}}{16}+\frac{\sqrt{C_1}}{2}+\frac{1}{4}\sqrt{\frac{C_1C_2}{n+\beta}}$ and $C_1,C_2\in(0,\infty)$.

Proof Using (4.3) and $x, t \in (0,\infty)$, we have

$$\begin{aligned}\left|M_n^{\alpha,\beta}(g;u)-g(u)\right| \leq 2\left(1+\frac{M_n^{\alpha,\beta}(|t-u|;u)}{\delta}\right)\left(1+\delta^2\right)\left(1+u^2\right) \\ \times \left(1+M_n^{\alpha,\beta}\left((t-u)^2;u\right)\right)\Omega(g;\delta).\end{aligned} \tag{4.4}$$

Applying the Cauchy–Schwarz inequality for (4.4), we get

$$\begin{aligned}&\left|M_n^{\alpha,\beta}(g;u)-g(u)\right| \\ &\quad\leq 2\left(1+\delta^2\right)\left(1+u^2\right)\Omega(g;\delta)\Bigg(1+M_n^{\alpha,\beta}\left((t-u)^2;u\right) \\ &\qquad+\frac{\sqrt{M_n^{\alpha,\beta}((t-u)^2;u)}}{\delta}+\frac{\sqrt{M_n^{\alpha,\beta}((t-u)^2;u)M_n^{\alpha,\beta}((t-u)^4;u)}}{\delta}\Bigg).\end{aligned} \tag{4.5}$$

By elementary calculations, we get

$$\lim_{n\to\infty}(n+\beta)M_n^{\alpha,\beta}\left(\psi_u^2;u\right)=2(1+i)u;$$
$$\lim_{n\to\infty}(n+\beta)M_n^{\alpha,\beta}\left(\psi_u^4;u\right)=3(8i+1)u^3.$$

It follows that, for each fixed point $u>0$, there exists $N_u \in \mathbb{N}$ such that, for all $n>N_u$,

$$M_n^{\alpha,\beta}\left(\psi^2;u\right) \leq C_1\frac{u}{n+\beta}, \tag{4.6}$$

$$M_n^{\alpha,\beta}\left(\psi^4;u\right) \leq C_2\frac{u^3}{n+\beta}, \tag{4.7}$$

where $C_1, C_2 \in (0,\infty)$. From (4.5)–(4.7) and choosing $\delta=\frac{1}{\sqrt{n+\beta}}$, we get the required result. □

5 A-statistical convergence

From [7, 15] we first introduce convergence approximation theorems in operators theory. Here, we recall same notation from [7, 15]. Let $A=(a_{nk})$ be a nonnegative infinite summability matrix. For a given sequence $x:=(x_k)$, the A-transform of x denoted by $Ax:((Ax)_n)$ is defined as follows:

$$(Ax)_n=\sum_{k=1}^{\infty}a_{nk}x_k,$$

provided the series converges for each n. A is said to be regular if $\lim(Ax)_n=L$ whenever $\lim x=L$. Then $x=(x_n)$ is said to be an A-statistical convergence to L i.e. $st_A-\lim x=L$ if, for every $\epsilon>0$, $\lim_n\sum_{k:|x_k-L|\geq\epsilon}a_{nk}=0$. In addition, A-statistical convergence results are studied in [13, 22, 23].

Theorem 5.1 *Let $A=(a_{nk})$ be a nonnegative regular summability matrix and $x \geq 0$. Then we have*

$$st_A-\lim_n\left\|M_n^{\alpha,\beta}(g;u)-g\right\|_{1+u^2}=0 \quad \textit{for all } g \in C_{1+u^2}^k[0,\infty).$$

Proof From ([9], p. 191, Th. 3), it is sufficient to show that, for $\lambda_1 = 0$,

$$st_A - \lim_n \left\| M_n^{\alpha,\beta}(e_i;u) - e_i \right\|_{1+u^2} = 0 \quad \text{for } i \in \{0,1,2\}. \tag{5.1}$$

Using Lemma 2.4, we have

$$\begin{aligned}\left\| M_n^{\alpha,\beta}(e_1;u) - u \right\|_{1+u^2} &= \sup_{u\in[0,\infty)} \frac{1}{1+u^2}\left|\left(\frac{n}{n+\beta} - 1\right)u + \frac{\lambda+\alpha+1}{n+\beta}\right| \\ &\leq \left|\frac{n}{n+\beta} - 1\right| \sup_{u\in[0,\infty)} \frac{u}{1+u^2} + \left|\frac{\lambda+\alpha+1}{n+\beta}\right| \sup_{u\in[0,\infty)} \frac{1}{1+u^2} \\ &\leq \frac{1}{2}\left|\frac{n}{n+\beta} - 1\right| + \left|\frac{\lambda+\alpha+1}{n+\beta}\right|.\end{aligned}$$

We have

$$st_A - \lim_n \frac{1}{2}\left|\frac{n}{n+\beta} - 1\right| = st_A - \lim_n \left|\frac{\lambda+\alpha+1}{n+\beta}\right| = 0. \tag{5.2}$$

Now, for given $\epsilon > 0$, we define the following sets:

$$\begin{aligned} J_1 &:= \left\{n : \left\| M_n^{\alpha,\beta}(e_1;u) - u \right\| \geq \epsilon\right\}, \\ J_2 &:= \left\{n : \frac{1}{2}\left|\frac{n}{n+\beta} - 1\right| \geq \frac{\epsilon}{2}\right\}, \\ J_3 &:= \left\{n : \left|\frac{\lambda+\alpha+1}{n+\beta}\right| \geq \frac{\epsilon}{2}\right\}. \end{aligned}$$

This implies that $J_1 \subseteq J_2 \cup J_3$, which shows that $\sum_{k_1\in J_1} a_{nk_1} \leq \sum_{k_1\in J_2} a_{nk} + \sum_{k_1\in J_3} a_{nk}$. Hence, from (5.2) we get

$$st_A - \lim_n \left\| M_n^{\alpha,\beta}(e_1;u) - u \right\|_{1+u^2} = 0. \tag{5.3}$$

In a similar way the following can be proved:

$$st_A - \lim_n \left\| M_n^{\alpha,\beta}(e_2;u) - u^2 \right\|_{1+u^2} = 0. \tag{5.4}$$

This completes the proof of Theorem 5.1. □

6 Conclusion

The motive of the present paper is to give a generalized error estimation of convergence by modification of Szász–Mirakyan integral operators via Dunkl analogue. We have defined the Szász–Mirakyan integral operators based on Dunkl analogue with the aid of two non-negative parameters $0 \leq \alpha \leq \beta$. This type of modification enables us to give a generalied error estimation for a certain function in comparison to the Szász–Mirakyan integral operators based on Dunkl analogue. We investigated some approximation results by means of the well-known Korovkin type theorem. We have also calculated the rate of convergence of operators by means of Peetre's K-functional and second order modulus of continuity. Lastly, we studied the global approximation results.

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