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Some logarithmic Minkowski inequalities for nonsymmetric convex bodies and related problems

Lewen Ji[1,2*]

*Correspondence:
jilewen2008@163.com
[1]Department of Mathematics, East China University of Technology, Nanchang, China
[2]Department of Mathematics, Shanghai University, Shanghai, China

Abstract

In this paper, we show the existence of a solution to an even logarithmic Minkowski problem for p-capacity and prove some analogue inequalities of the logarithmic Minkowski inequality for general nonsymmetric convex bodies involving p-capacity.

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1 Introduction

A convex body in an n-dimensional Euclidean space $\mathbb{R}^n$ is a compact convex set that has nonempty interior. The cone-volume measure V_K of a convex body K is a Borel measure on the unit sphere S^{n-1} defined for a Borel $\omega \in S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n} \int_{x \in g_K^{-1}(\omega)} x \cdot g_K \, d\mathcal{H}^{n-1}(x),$$

where $g_K : \partial K \to S^{n-1}$ is the Gauss map of K, defined on ∂K, the set of boundary points of K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is an $(n-1)$-dimensional Hausdorff measure, see, e.g., [5, 20, 24, 28]. The cone-volume measure of a convex body has clear geometric significance. Böröczky et al. in [4] posed the subspace concentration condition and completely solved the even Minkowski problem. The problem asks: *What are the necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} such that μ is the cone-volume measure of a convex body in $\mathbb{R}^n$?* In [30], Zhu solved the case of discrete measures whose supports are in general position. Uniqueness for the logarithmic Minkowski problem was completely settled for even measures in $\mathbb{R}^2$ in [3]. Recently, Stancu [23] proved the logarithmic Minkowski inequality for nonsymmetric convex bodies. Wang, Xu, and Zhou [25] gave the L_p version of Stancu's results. For more results, see, e.g., [2, 4, 21, 25, 26, 30].

In his celebrated paper [17], Jerison solved the Minkowski problem for the capacitary measure, the measure that is the variational functional arising from the electrostatic capacity. Colesanti et al. in [9] extended Jerison's work on electrostatic capacity to p-capacity.

Naturally, the Minkowski problem for p-capacity was posed [9]: *Given a finite Borel measure μ on S^{n-1}, what are the necessary and sufficient conditions on μ so that μ is the p-capacitary measure $\mu_p(K,\cdot)$ of convex body K in $\mathbb{R}^n$?* The authors in [9] proved the uniqueness of the solution when $1 < p < n$ and existence and regularity when $1 < p < 2$, the existence for $2 < p < n$ was solved by Akman et al. [1]. Inspired by the L_p Minkowski problem for volume, Zou and Xiong [31] initiated the research into the L_q Minkowski problem for p-capacitary measure. For more results, see, e.g., [6–8, 15–17, 19, 27, 29].

1.1 Main results

In this paper, we study the even logarithmic Minkowski problem and logarithmic Minkowski inequality for p-capacity. Our first result is to solve the existence part of the even logarithmic Minkowski problem for p-capacity. The problem asks: *What are the necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} such that μ is the L_0 p-capacitary measure of an origin-symmetric convex body in $\mathbb{R}^n$?* Our proof is based on the techniques in [4, 9, 23]. In order to solve the existence of the even logarithmic Minkowski problem, we use the definition of subspace concentration inequality in [4].

Definition 1.1 ([4]) A finite Borel measure μ on S^{n-1} is said to satisfy the subspace concentration inequality if, for every subspace ξ of $\mathbb{R}^n$ such that $0 < \dim\xi < n$,

$$\mu\left(\xi \cap S^{n-1}\right) \leq \frac{1}{n}\mu\left(S^{n-1}\right)\dim\xi. \tag{1.1}$$

The measure is said to satisfy the subspace concentration condition if, in addition to satisfying the subspace concentration inequality (1.1), whenever

$$\mu\left(\xi \cap S^{n-1}\right) = \frac{1}{n}\mu\left(S^{n-1}\right)\dim\xi,$$

for some subspace ξ, then there exists a subspace ξ', which is complementary to ξ in $\mathbb{R}^n$, so that

$$\mu\left(\xi' \cap S^{n-1}\right) = \frac{1}{n}\mu\left(S^{n-1}\right)\dim\xi'.$$

Theorem 1.1 *Let μ be a nonzero finite even Borel measure on S^{n-1} and $1 < p < 2$. Suppose that μ satisfies the strict subspace concentration inequality and $\mu(\{-u\}) = 0$ whenever $\mu(\{u\}) > 0$ for $u \in S^{n-1}$. Then the measure μ is the L_0 p-capacitary measure of an origin-symmetric convex body in $\mathbb{R}^n$.*

Next, we prove the modified logarithmic Minkowski inequality for p-capacity.

Theorem 1.2 *Suppose $K, L \in \mathcal{K}_o^n$ and $1 < p < n$. Then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p^*(L,K,u) \geq \ln\left(\frac{C_p(L,K)}{C_p(L)}\right) \geq \frac{1}{n-p}\ln\left(\frac{C_p(K)}{C_p(L)}\right).$$

Equality holds if and only if K and L are homothetic.

Write $(\frac{h_K}{h_L})_{\mathrm{average}} := \frac{\int_{S^{n-1}} \frac{h_K(u)}{h_L(u)}\, dC_p(L,u)}{\int_{S^{n-1}} dC_p(L,u)} = \frac{C_p(L,K)}{C_p(L)}$, $(\frac{h_K}{h_L})_{\max} := \max_{u\in \operatorname{supp} C_p(L,\cdot)} \frac{h_K(u)}{h_L(u)}$, and $(\frac{h_K}{h_L})_{\min} := \min_{u\in \operatorname{supp} C_p(K,\cdot)} \frac{h_K(u)}{h_L(u)}$. We obtain the third result.

Theorem 1.3 *Suppose $K, L \in \mathcal{K}_o^n$ with $L \subseteq K$ and $1 < p < n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p^*(L,u) \geq \frac{(\frac{h_K}{h_L})_{\mathrm{average}}}{(\frac{h_K(u)}{h_L(u)})_{\max}} \frac{1}{n-p} \ln\left(\frac{C_p(K)}{C_p(L)}\right).$$

Equality holds if and only if $K = L$.

In general, we have

$$\int_{S^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p^*(L,u) \geq \frac{(\frac{h_K}{h_L})_{\mathrm{average}}}{(\frac{h_K(u)}{h_L(u)})_{\max}} \frac{1}{n-p} \ln\left(\frac{C_p(K)}{C_p(L)}\right) + \ln\left(\left(\frac{h_K}{h_L}\right)_{\min}\right)\left(1 - \frac{(\frac{h_K}{h_L})_{\mathrm{average}}}{(\frac{h_K}{h_L})_{\max}}\right).$$

This paper is organized as follows. In Sect. 2, we give a minimization problem for p-capacity. In Sect. 3, we give the proof of Theorem 1.1. In Sect. 4, we prove Theorem 1.2 and its application.

1.2 Preliminaries

For quick reference, we collect some basic facts on the theory of convex bodies. Good references are the books by Schneider [22].

Denote by $\mathcal{K}^n$ the set of convex bodes in $\mathbb{R}^n$ and by $\mathcal{K}_o^n$ the set of convex bodies with the origin o in its interiors. Let h_K and h_L be the support functions ($h_K(u) = h(K,u) := \max_{u\in S^{n-1}}\{x \cdot u : x \in K\}$, where $x \cdot u$ denotes the inner product of u and x) of K. Let $I \subset \mathbb{R}$ be an interval containing 0, $C^+(S^{n-1})$ be a class of continuous and positive functions on S^{n-1}, $C_e^+(S^{n-1})$ means the subsets of $C^+(S^{n-1})$ are the even functions, and assume that $h_t(u) = h(t,u) : I \times S^{n-1} \to (0, +\infty)$ is continuous. Put

$$\Omega_t = \bigcap_{u\in S^{n-1}} \left\{x \in \mathbb{R}^n : x \cdot u \leq h_t(u)\right\}.$$

The convex body Ω_t is called the Aleksandrov body associated with h_t. Via the support function, Böröczky et al. in [3] defined the log Minkowski combination $(1-\lambda)\cdot K +_0 \lambda \cdot L$, that is,

$$(1-\lambda)\cdot K +_0 \lambda \cdot L = \bigcap_{u\in S^{n-1}} \left\{x \in \mathbb{R}^n : x \cdot u \leq h_K(u)^{1-\lambda} h_L(u)^{\lambda}\right\}. \tag{1.2}$$

The surface area measure S_K of a convex body K is a Borel measure on the unit sphere S^{n-1} defined for a Borel $\omega \in S^{n-1}$ by

$$S_K(\omega) = \int_{x\in g_K^{-1}(\omega)} d\mathcal{H}^{n-1}(x). \tag{1.3}$$

Let Ω be a bounded convex domain in $\mathbb{R}^n, n \geq 3$, $\overline{\Omega}$ be its closure. The equilibrium potential $u = u_\Omega$ is the unique solution to the boundary value problem

$$\begin{cases} \Delta_p u = 0 \quad \text{in } \mathbb{R}^n \backslash \overline{\Omega}, \\ u = 1 \quad \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.4}$$

where Δ_p is the p-Laplace operator and $1 < p < n$. Due to Dahlbeg [12], we can see that ∇u has non-tangential limits almost everywhere on $\partial\Omega$ and $|\nabla u| \in L^p(\partial\Omega, \mathcal{H}^{n-1})$. The p-capacity was defined by

$$C_p(\Omega) = \inf\left\{\int_{\mathbb{R}^n} |\nabla u_\Omega|^p \, dx : u \in C_c^\infty\big(\mathbb{R}^n\big), u \geq 1 \text{ on } \Omega\right\},$$

where $C_c^\infty(\mathbb{R}^n)$ is the set of C^∞ functions in $\mathbb{R}^n$ with compact support. In [9], the authors proved the following Hadamard variational formula for p-capacity. Let $K, L \in \mathcal{K}_o^n$ and $1 < p < n$,

$$\frac{d}{dt} C_p(K + tL)|_{t=0^+} = (p-1) \int_{S^{n-1}} h(L,u) \, d\mu_p(K,u),$$

where the p-capacitary measure $\mu_p(K,E)$ is defined by

$$\mu_p(K,E) = \int_{g_K^{-1}(E)} |\nabla u_K|^p \, d\mathcal{H}^{n-1},$$

and the Poincaré p-capacity formula

$$C_p(K) = \frac{p-1}{n-p} \int_{S^{n-1}} h(K,u) \, d\mu_p(K,u) = \frac{p-1}{n-p} \int_{\partial K} |\nabla u|^p (x \cdot g_K) \, d\mathcal{H}^{n-1}(x). \tag{1.5}$$

2 A minimization problem

In this section, we study a minimization problem, its solution also solves the logarithmic Minkowski problem for p-capacity. The following lemma will be needed.

Lemma 2.1 ([9]) *Let $h(t,u) : I \times S^{n-1} \to (0, +\infty)$ be continuous and $\{K_t\}_{t\in I}$ be the family of Aleksandrov domain associated with h_t. If*

$$h'_+(0,u) = \lim_{t\to 0^+} \frac{h(t,u) - h(0,u)}{t}$$

is uniform on S^{n-1}, then

$$\lim_{t\to 0^+} \frac{C_p(K_t) - C_p(K_0)}{t} = (p-1) \int_{S^{n-1}} h'_+(0,u) \, d\mu_p(K_0,u).$$

With Lemma 2.1 in hand, we use the definition of log Minkowski combination to prove the following result.

Lemma 2.2 *Let $K \in \mathcal{K}_o^n$ and $f \in C^+(S^{n-1})$ be nonnegative. If $1 < p < n$, then*

$$\lim_{t\to 0^+} \frac{C_p(K +_0 t\cdot f) - C_p(K)}{t} = (p-1)\int_{S^{n-1}} h(K,u)\ln f \, d\mu_p(K,u).$$

Proof Let $0 < t_0 < +\infty$, consider the interval $[0, t_0]$. We have

$$h'_+(0,u) = \lim_{t\to 0^+} \frac{h(t,u) - h(0,u)}{t} = h(K,u)\ln f \quad \text{for all } u \in S^{n-1}.$$

Then, by Lemma 2.1, we obtain

$$\lim_{t\to 0^+} \frac{C_p(K +_0 t\cdot f) - C_p(K)}{t} = (p-1)\int_{S^{n-1}} h(K,u)\ln f \, d\mu_p(K,u). \qquad \square$$

Corollary 2.1 *Suppose $K, L \in \mathcal{K}_o^n$ and $1 < p < n$, then*

$$\lim_{t\to 0^+} \frac{C_p(K +_0 t\cdot L) - C_p(K)}{t} = (p-1)\int_{S^{n-1}} h(K,u)\ln h(L,u)\, d\mu_p(K,u).$$

In view of the above result, we give the following definitions.

Definition 2.1 Let $1 < p < n$ and $K \in \mathcal{K}_o^n$, then the Borel measure $\mu_{p,0}(K,\cdot)$ on S^{n-1}, defined by

$$\mu_{p,0}(K,\omega) = \int_\omega h(K,u)\, d\mu_p(K,u)$$

for $\omega \subset S^{n-1}$, is called the L_0 p-capacitary measure of K.

Definition 2.2 Let $1 < p < n$ and $K, L \in \mathcal{K}_o^n$, then L_0 mixed p-capacity of K and L is defined by

$$C_{p,0}(K,L) = \frac{p-1}{n-p}\int_{S^{n-1}} h(K,u)\ln h(L,u)\, d\mu_p(K,u).$$

The next lemma shows that $C_{p,0}(K,L)$ is continuous in (K,L).

Lemma 2.3 *Let $K_i, L_i, K, L \in \mathcal{K}_o^n$ and $1 < p < n$. Assume that $(K_i, L_i) \to (K,L)$ as $i \to \infty$, then $C_{p,0}(K_i, L_i) \to C_{p,0}(K,L)$.*

Proof By the assumption $(K_i, L_i) \to (K,L)$, we have $h_{K_i} \to h_K$, $h_{L_i} \to h_L$ uniformly on S^{n-1}, and $\mu_p(K_i,\cdot) \to \mu_p(K,\cdot)$ weakly. It follows that $h_{K_i}\ln h_{L_i} \to h_K \ln h_L$ uniformly on S^{n-1}. By Definition 2.2, the desired limit is obtained. $\square$

The weak convergence of p-capacitary measure implies the weak convergence of $\mu_{p,0}$ as follows.

Lemma 2.4 *Let $K_i, K \in \mathcal{K}_o^n$ and $1 < p < n$. If $K_i \to K$ as $i \to \infty$, then $\mu_{p,0}(K_i) \to \mu_{p,0}(K)$ weakly.*

We now consider the minimization problem

$$\inf\{\Phi_\mu(K) : C_p(K) = |\mu| \text{ and } K \in \mathcal{K}_e^n\},$$

where $\mathcal{K}_e^n$ is an origin-symmetric convex body with nonempty interior, μ is a finite even Borel measure on S^{n-1} with total mass $|\mu| > 0$, and the logarithmic functional $\Phi_\mu : \mathcal{K}_e^n \to \mathbb{R}$ is defined by

$$\Phi_\mu(K) = \int_{S^{n-1}} \ln h_K \, d\mu.$$

The following theorem shows a solution to the logarithmic Minkowski type problem for the measure μ is a solution to minimization problem for the function Φ_μ.

Theorem 2.1 *Let μ be a finite even Borel measure on S^{n-1} with $|\mu| > 0$ and $1 < p < n$. If K is an origin-symmetric convex body such that $C_p(K) = |\mu|$ and*

$$\Phi_\mu(K) = \inf\{\Phi_\mu(Q) : C_p(Q) = |\mu|, Q \in \mathcal{K}_e^n\},$$

then the measure μ is the L_0 p-capacitary measure of K.

Proof Clearly, we may assume that μ is a probability measure. Next, we consider the minimization problem

$$\inf_{f \in C_e^+(S^{n-1})} F(f),$$

where the continuous functional $F : C_e^+(S^{n-1}) \to (0, +\infty)$ by

$$F(f) = \frac{\exp \int_{S^{n-1}} \ln f \, d\mu}{C_p(K_f)^{\frac{1}{n-p}}},$$

here K_f denotes the Wulff shape of $f \in C_e^+(S^{n-1})$. Notice that F is homogeneous of degree 0, i.e., $F(sf) = F(f)$ for $s > 0$. By the properties of Aleksandrov body, we have $h_{K_f} \le f$. According to Lemma 2.1 in [31], we have $C_p(f) = C_p(K_f)$; in addition, $C_p(K_f) = C_p(h_{K_f})$, which yields $F(K_f) \le F(f)$. Therefore, we shall search for the infimum of F among the support functions of origin-symmetric convex bodies. It follows that the infimum of F

$$\inf_{f \in C_e^+(S^{n-1})} F(f) = \inf\{\exp(\Phi_\mu(Q)) : C_p(Q) = 1, Q \in \mathcal{K}_e^n\}.$$

Obviously, the right infimum is attained at $K \in \mathcal{K}_e^n$. Thus, the support function $h_K > 0$ is a solution of minimization problem, i.e.,

$$\inf_{f \in C_e^+(S^{n-1})} F(f) = F(h_K).$$

Given the function $h_t = h(\cdot, t) : S^{n-1} \times \mathbb{R} \to (0, \infty)$ is defined by

$$h_t = h(\cdot, t) = h_K \exp(tf),$$

by the function $F(h_t)$ has a minimum at $t = 0$, this implies that

$$\frac{d}{dt}F(h_t) = 0.$$

On the other hand,

$$\lim_{t\to 0}\frac{h_t - h_0}{t} = fh_K, \quad \text{uniformly on } S^{n-1},$$

it follows from $C_p(K) = 1$ that

$$\left[-\frac{p-1}{n-p}\int_{S^{n-1}} f(u)h_K(u)\,d\mu_p(K,u) + \int_{S^{n-1}} f(u)\,d\mu(u)\right]\exp\left(\int_{S^{n-1}} \ln h_K(u)\,d\mu(u)\right) = 0,$$

which yields

$$\frac{p-1}{n-p}\int_{S^{n-1}} f(u)\,d\mu_{p,0}(K,u) = \int_{S^{n-1}} f(u)\,d\mu(u).$$

Since $f \in C_e^+(S^{n-1})$ is arbitrary, we conclude that

$$d\mu(\cdot) = \frac{p-1}{n-p}\,d\mu_{p,0}(K,\cdot),$$

this completes the proof of the theorem. □

3 Logarithmic Minkowski problem

In the previous section, we have proved the existence of a solution to the logarithmic Minkowski problem by using the variational argument. In this section, we show the proof of the main result.

Lemma 3.1 *Let μ be an even finite Borel measure on S^{n-1} satisfying the strict subspace concentration inequality. If $1 < p < n$ and $\mu(\{-u\}) = 0$ whenever $\mu(\{u\}) > 0$, then there exists an origin-symmetric convex body K so that*

$$\inf\{\Phi_\mu(Q) : C_p(Q) = |\mu|, Q \in \mathcal{K}_e^n\} = \int_{S^{n-1}} \ln h_K\,d\mu.$$

Proof Without loss of generality, we assume that $|\mu| = 1$. Let Q_l be the minimizing sequence of origin-symmetric convex bodies, that is, Q_l satisfies the $C_p(Q_l) = 1$ and

$$\lim_{l\to\infty}\Phi_\mu(Q_l) = \inf_{Q\in\mathcal{K}_e^n}\{\Phi_\mu(Q) : C_p(Q) = 1, Q \in \mathcal{K}_e^n\}.$$

Taking $L = \gamma^{-\frac{1}{n-p}}B_n$, $\gamma = (\frac{p-1}{n-p})^{1-p}\omega_n$, here B_n is a unit ball. Then $C_p(L) = 1$, it follows that

$$\lim_{l\to\infty}\Phi_\mu(Q_l) \le -\frac{1}{n-p}\ln\gamma. \tag{3.1}$$

By John's theorem [18] associated with Q_l, there exists an origin-symmetric ellipsoid E_l so that

$$E_l \subseteq Q_l \subseteq \sqrt{n} E_l.$$

Note that, for every origin-symmetric ellipsoid E_l, there is a cross-polytope P_l denoted by $P_l = [\pm a_{1l} u_{1l}, \dots, \pm a_{nl} u_{nl}]$ such that $P_l \subseteq E_l \subseteq \sqrt{n} P_l$. Hence, $P_l \subseteq Q_l \subseteq n P_l$. It follows from $C_p(Q_l) = 1$ that $C_p(P_l) \geq n^{-\frac{1}{n-p}}$. We next claim that suppose K is a convex body containing the origin and satisfying $V(K) = 0$, then $C_p(K) = 0$. In fact, since

$$\begin{aligned} 0 = V(K) &= \frac{1}{n} \int_{S^{n-1}} h_K(u)\, dS(K,u) \\ &= \frac{1}{n} \int_{\{h_K > 0\}} h_K(u)\, dS(K,u) + \frac{1}{n} \int_{\{h_K = 0\}} h_K(u)\, dS(K,u) \\ &= \frac{1}{n} \int_{\{h_K > 0\}} h_K(u)\, dS(K,u). \end{aligned}$$

It follows that $S(K, \{h_K > 0\}) = 0$, which implies $\mathcal{H}^{n-1}(g_K^{-1}(\{h_K > 0\})) = 0$. Together with the fact that $|\nabla u|^p$ is integrable on ∂K, we have

$$\mu_p\big(K, \{h_K > 0\}\big) = \int_{g_K^{-1}(\{h_K > 0\})} |\nabla u|^p \mathcal{H}^{n-1}(x) = 0.$$

By (1.5), we have $C_p(K) = 0$. Thus, there exists a constant $c > 0$ such that $V(K) \geq c C_p(K)$. This yields

$$\prod_{i=1}^{n} a_{il} = \frac{n! V(P_l)}{2^n} \geq \frac{c n! C_p(P_l)}{2^n} := \gamma_1.$$

Assume that Q_l is not bounded. Then P_l is not bounded, thus there exists a_{nl} such that $\lim_{l \to \infty} a_{nl} = \infty$. Applying Lemma 6.2 in [3] to $P_l' = \gamma_1^{-\frac{1}{n}} P_l$ implies that $\Phi_\mu(P_l')$ is not bounded, which gives that $\Phi_\mu(Q_l)$ is not bounded. This contradicts (3.1). Therefore, the sequence Q_l is bounded. By Blaschke's selection theorem, Q_l has a subsequence that converges to an origin-symmetric convex body K. In the following, we prove $K \in \mathcal{K}_o^n$ is n-dimensional. Let $\dim K \leq n-2$. Since $1 < p < 2$, we have $\dim K \leq n-2 < n-p$, so $C_p(K) = 0$ by [13], p. 179, which contradicts that $C_p(K) = 1$. Let $\dim K = n-1$, there exists a unit vector $u \in S^{n-1}$ such that $K \subset u^\perp$. Then $u, -u \in \operatorname{supp} \mu$. But μ satisfies $\mu(\{-u\}) = 0$ whenever $\mu(\{u\}) > 0$ for any $u \in S^{n-1}$, which is a contradiction. This gives the desired result. □

We get the proof of Theorem 1.1 directly from Theorem 2.1 and Lemma 3.1.

4 Logarithmic Minkowski type inequality

In this section, we set $dC_p(L,K,\cdot) = \frac{1}{n-p} h_K \, d\mu_p(L,\cdot)$ and $dC_p(L,\cdot) = \frac{1}{n-p} h_L \, d\mu_p(L,\cdot)$. Then $C_p(L,K) = \int_{S^{n-1}} dC_p(L,K,u)$ and $C_p(L) = \int_{S^{n-1}} dC_p(L,u)$. Clearly,

$$dC_p^*(L,K,\cdot) = \frac{1}{C_p(L,K)} dC_p(L,K,\cdot)$$

and $dC_p^*(L,\cdot) = \frac{1}{C_p(L)}\,dC_p(L,\cdot)$ are their normalization, respectively. Now, we give the proof of Theorem 1.2.

Proof of Theorem 1.2 Consider the function $f : [1,+\infty] \to \mathbb{R}$ defined by

$$f(q) = \frac{1}{C_p(L,K)} \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{\frac{q}{q+n}} dC_p(L,u). \tag{4.1}$$

According to Lebesgue's dominated convergence theorem, we obtain: as $q \to \infty$,

$$\int_{S^{n-1}} \left(\frac{h_K(u)}{h_L(u)}\right)^{\frac{q}{q+n}} dC_p(L,u) \to C_p(L,K)$$

and

$$\int_{S^{n-1}} \left(\frac{h_K(u)}{h_L(u)}\right)^{\frac{q}{q+n}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p(L,u) \to \int_{S^{n-1}} \frac{h_K(u)}{h_L(u)} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p(L,u).$$

Using L'Hôpital's rule, we have

$$\begin{aligned}
\lim_{q\to\infty} \ln\bigl(f(q)\bigr)^{q+n} &= \lim_{q\to\infty} \frac{\frac{n}{(n+q)^2 C_p(L,K)} \int_{S^{n-1}} (\frac{h_K(u)}{h_L(u)})^{\frac{q}{q+n}} \ln(\frac{h_K(u)}{h_L(u)})\, dC_p(L,u)}{-\frac{f(q)}{(q+n)^2}} \\
&= -\frac{n}{C_p(L,K)} \int_{S^{n-1}} \frac{h_K(u)}{h_L(u)} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p(L,u) \\
&= -\frac{n}{C_p(L,K)} \int_{S^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p(L,K,u).
\end{aligned} \tag{4.2}$$

By (4.1) and (4.2), we have

$$\begin{aligned}
&\exp\left[-n \int_{S^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p^*(L,K,u)\right] \\
&\quad = \lim_{q\to\infty}\left[\frac{1}{C_p(L,K)} \int_{S^{n-1}} \left(\frac{h_K(u)}{h_L(u)}\right)^{\frac{q}{q+n}} dC_p(L,u)\right]^{n+q}.
\end{aligned}$$

The reverse Hölder inequality gives

$$\left(\int_{S^{n-1}} \left(\frac{h_K(u)}{h_L(u)}\right)^{\frac{q}{q+n}} dC_p(L,u)\right)^{\frac{n+q}{q}} \left(\int_{S^{n-1}} dC_p(L,u)\right)^{-\frac{n}{q}} \le C_p(L,K).$$

Equality holds if and only if K is homothetic to L. This gives the first inequality of Theorem 1.2.

Now, we prove the second inequality of Theorem 1.2. Using Minkowski's inequality for p-capacity [10], we have

$$C_p(L,K) \ge C_p(K)^{\frac{1}{n-p}} C_p(L)^{1-\frac{1}{n-p}}.$$

Equality holds if and only if K is homothetic to L. □

Similarly, we obtain the reverse form of Theorem 1.2.

Theorem 4.1 *Let $K, L \in \mathcal{K}_o^n$ and $1 < p < n$. Then*

$$\int_{S^{n-1}} \ln \frac{h_K(u)}{h_L(u)}\, dC_p^*(L,u) \le \ln \frac{C_p(L,K)}{C_p(L)} \le \int_{S^{n-1}} \ln\left(\frac{h_L(u)}{h_K(u)}\right) dC_p^*(L,K,u),$$

with equality if and only if K is homothetic to L.

First proof Via the same idea in Theorem 1.2, we have

$$\begin{aligned}
&\exp\left(-\frac{1}{C_p(L)} \int_{S^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p(L,u)\right) \\
&\quad = \lim_{q\to\infty}\left[\frac{1}{C_p(L)} \int_{S^{n-1}} \left(\frac{h_K(u)}{h_L(u)}\right)^{\frac{1}{q+n}} dC_p(L,u)\right]^{n+q}.
\end{aligned}$$

From the reverse Hölder inequality, we have

$$\left(\int_{S^{n-1}} \left(\frac{h_K(u)}{h_L(u)}\right)^{\frac{1}{q+n}} dC_p(L,u)\right)^{q+n} \left(\int_{S^{n-1}} dC_p(L,u)\right)^{1-q-n} \le C_p(L,K).$$

Thus,

$$\frac{1}{C_p(L)} \int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) dC_p(L) \le \ln\left(\frac{C_p(L,K)}{C_p(L)}\right). \qquad \square$$

Second proof We will use Gibbs' inequality [11], i.e., let f and g be the probability density functions on a measure space (X,ν), then $\int f \ln f\, d\nu \ge \int f \ln g\, d\nu$, equality holds if and only if $f = g$. On the one hand, let $f\, d\nu(\cdot) = \frac{h_L}{h_K C_p(L)}\, dC_p(L,K,\cdot)$ and $g\, d\nu(\cdot) = \frac{1}{C_p(L,K)}\, dC_p(L,K,\cdot)$. By Gibbs' inequality in [11], we have

$$\begin{aligned}
&\int_{S^{n-1}} \frac{h_L(u)}{h_K(u)C_p(L)} \ln\left(\frac{h_L(u)}{h_K(u)C_p(L)}\right) dC_p(L,K,u) \\
&\quad \ge \int_{S^{n-1}} \frac{1}{C_p(L,K)} \ln\left(\frac{1}{C_p(L,K)}\right) dC_p(L,K,u),
\end{aligned}$$

which implies that

$$\int_{S^{n-1}} \ln\left(\frac{h_L(u)}{h_K(u)}\right) dC_p^*(L,u) \le \ln\left(\frac{C_p(L,K)}{C_p(L)}\right).$$

According to the equality condition, we know that K is homothetic to L.

On the other hand, we set $f\, d\nu(\cdot) = \frac{1}{C_p(L,K)}\, dC_p(L,K,\cdot)$ and $g\, d\nu(\cdot) = \frac{h_L}{h_K C_p(L)}\, dC_p(L,K,\cdot)$, then

$$\begin{aligned}
&\int_{S^{n-1}} \frac{1}{C_p(L,K)} \ln\left(\frac{1}{C_p(L,K)}\right) dC_p(L,K,u) \\
&\quad \ge \int_{S^{n-1}} \frac{h_L(u)}{h_K(u)C_p(L)} \ln\left(\frac{h_L(u)}{h_K(u)C_p(L)}\right) dC_p(L,K,u),
\end{aligned}$$

which is equivalent to the following inequality:

$$\int_{S^{n-1}} \ln\left(\frac{h_L(u)}{h_K(u)}\right) dC_p^*(L,K,u) \geq \ln\left(\frac{C_p(L,K)}{C_p(L)}\right),$$

with equality if and only if K is homothetic to L. This completes the proof of the theorem. □

Moreover, we show the logarithmic Minkowski type inequality for p-capacity.

Proof of Theorem 1.3 Let $q \in \mathbb{R}$ and $1 < p < n$, consider the function

$$G(q) = \int_{S^{n-1}} \left(\frac{h_K(u)}{h_L(u)}\right)^q \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p(L,u),$$

since $L \subseteq K$, the function $G(q)$ is nonnegative. If $\ln(\frac{h_K(u)}{h_L(u)}) \equiv 0$, $u \in \operatorname{supp} C_p(L,\cdot)$, then $G \equiv 0$, which implies that $G(1) \geq G(0) > 0$. If $G(1) = G(0)$, then $K = L$. So we assume $G(1) > G(0)$.

We now claim that $G(q)$ is a log-convex function. In fact, let $t \in (0,1)$, by Hölder's inequality, we get that

$$\begin{aligned} G\bigl((1-t)q_1 + tq_2\bigr) &= \int_{S^{n-1}} \left(\frac{h_K(u)}{h_L(u)}\right)^{(1-t)q_1+tq_2} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p(L,u) \\ &\leq \left(\int_{S^{n-1}} \left(\frac{h_K(u)}{h_L(u)}\right)^{q_1} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p(L,u)\right)^{1-t} \\ &\quad \times \left(\int_{S^{n-1}} \left(\frac{h_K(u)}{h_L(u)}\right)^{q_2} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p(L,u)\right)^{t} \\ &= G(q_1)^{1-t} G(q_2)^t. \end{aligned}$$

Applying the Hadamard type inequality in [14], we have

$$\frac{G(1)-G(0)}{\ln G(1) - \ln G(0)} \geq \int_0^1 \int_{S^{n-1}} \left(\frac{h_K(u)}{h_L(u)}\right)^q \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p(L,u)\, dq. \tag{4.3}$$

From Fubini–Tonelli's theorem, we obtain

$$\begin{aligned} G(0) &\geq G(1) \exp\left(-\frac{G(1)-G(0)}{\int_0^1 \int_{S^{n-1}} (\frac{h_K(u)}{h_L(u)})^q \ln(\frac{h_K(u)}{h_L(u)})\, dC_p(L,u)\, dq}\right) \\ &= G(1) \exp\left(-\frac{G(1)-G(0)}{\int_{S^{n-1}} (\frac{h_K(u)}{h_L(u)} - 1)\, dC_p(L,u)}\right). \end{aligned}$$

Notice that

$$\frac{G(1)-G(0)}{\int_{S^{n-1}} (\frac{h_K(u)}{h_L(u)} - 1)\, dC_p(L,u)} \leq \ln\left(\frac{h_K}{h_L}\right)_{\max}.$$

Thus, by Theorem 1.2, we have

$$\begin{aligned}
&\int_{S^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p^*(L,u) \\
&\quad \geq \exp\left(-\ln\left(\frac{h_K}{h_L}\right)_{\max}\right)\frac{C_p(L,K)}{C_p(L)}\int_{S^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p^*(L,K,u) \\
&\quad \geq \frac{(\frac{h_K}{h_L})_{\text{average}}}{(\frac{h_K}{h_L})_{\max}}\int_{S^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p^*(L,K,u) \\
&\quad \geq \frac{(\frac{h_K}{h_L})_{\text{average}}}{(\frac{h_K}{h_L})_{\max}}\frac{1}{n-p}\ln\left(\frac{C_p(K)}{C_p(L)}\right).
\end{aligned} \tag{4.4}$$

Suppose $G(q) \equiv 0$, then $h_K(u) = h_L(u)$ for almost all u in L_0 p-capacitary measure of L, or equivalently with respect to the p-capacitary measure of L. This implies that $C_p(L,K) = C_p(L)$. According to the equality condition of Minkowski inequality for p-capacity and $L \subseteq K$, we obtain $K = L$.

Assume that $K, L \in \mathcal{K}_o^n$ are arbitrary and L is not included in K, then there exists $0 < t < 1$ such that $tL \subseteq K$. By (4.4), we have

$$\int_{S^{n-1}} \ln\left(\frac{h_K(u)}{th_L(u)}\right) dC_p^*(tL,u) \geq \frac{(\frac{h_K}{th_L})_{\text{average}}}{(\frac{h_K}{th_L})_{\max}}\frac{1}{n-p}\ln\left(\frac{C_p(K)}{C_p(tL)}\right),$$

which is equivalent to the following inequality:

$$\begin{aligned}
&\int_{S^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p^*(L,u) \\
&\quad \geq \frac{(\frac{h_K}{h_L})_{\text{average}}}{(\frac{h_K}{h_L})_{\max}}\frac{1}{n-p}\ln\left(\frac{C_p(K)}{C_p(L)}\right) + \ln t\left(1 - \frac{(\frac{h_K}{h_L})_{\text{average}}}{(\frac{h_K}{h_L})_{\max}}\right).
\end{aligned}$$

Taking $t = \min_{u \in \operatorname{supp} C_p(K,\cdot)} \frac{h_K(u)}{h_L(u)}$, we obtain the second inequality, with equality if and only if K and L are homothetic. □

Obviously, Theorem 1.3 implies the following corollary.

Corollary 4.1 *Let $K, L \in \mathcal{K}_o^n$ and $1 < p < n$. If there exists a constant $t > 0$ such that $tL \subseteq K$ with $h_K(u) = h_{tL}(u)$ for u in L_0 p-capacitary measure of L, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) dC_p^*(L,u) \geq \frac{1}{n-p}\ln\left(\frac{C_p(K)}{C_p(L)}\right).$$

Equality holds if and only if $K = tL$.

Finally, we shall give an application of Theorem 1.2. Let $\mathcal{F}_n$ be the set of all convex bodies with positive continuous curvature functions and L be a convex body in $\mathcal{F}_n$ with a curvature function f_L. To simplify the notation, we write $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}$ to be the polar body of K and $\Phi_p(L) = \int_{S^{n-1}} f_L^{\frac{n}{n+1}} |\nabla u_L|^{\frac{np}{n+1}} \, dS(u)$.

Theorem 4.2 *If $K \in \mathcal{K}_o^n$, $1 < p < n$, and $L \in \mathcal{K}_o^n \cap \mathcal{F}_n$, then*

$$V(K)V\big(K^*\big) \geq \frac{1}{n(n-p)^n} \frac{\Phi_p(L)^{n+1}}{C_p(L)^n} \frac{V(K)}{c(p,K,L)},$$

where $c(p,K,L) := (\exp(\int_{S^{n-1}} \ln(\frac{h_K(u)}{h_L(u)}))\, dC_p^*(L,K,u))^n$.

Proof Let K and L be distinct, by Theorem 1.2, we have

$$\begin{aligned}\exp\biggl(\int_{S^{n-1}} \ln\biggl(\frac{h_K(u)}{h_L(u)}\biggr)\biggr)\, dC_p^*(L,K,u) &\geq \frac{C_p(L,K)}{C_p(L)} \\ &= \frac{1}{(n-p)C_p(L)} \int_{S^{n-1}} h_K(u) f_L |\nabla u_L|^p \, dS(u).\end{aligned}$$

Using the reverse Hölder inequality, we obtain

$$\begin{aligned}\int_{S^{n-1}} h_K f_L |\nabla u_L|^p \, dS(u) &\geq \biggl(\int_{S^{n-1}} h_K^{-n}\, dS(u)\biggr)^{-\frac{1}{n}} \biggl(\int_{S^{n-1}} f_L^{\frac{n}{n+1}} |\nabla u_L|^{\frac{np}{n+1}} \, dS(u)\biggr)^{\frac{n+1}{n}} \\ &= \big(nV\big(K^*\big)\big)^{-\frac{1}{n}} \Phi_p(L)^{\frac{n+1}{n}}.\end{aligned}$$

Thus

$$\exp\biggl(\int_{S^{n-1}} \ln\biggl(\frac{h_K(u)}{h_L(u)}\biggr)\biggr)\, dC_p^*(L,K,u) \geq \frac{1}{(n-p)C_p(L)} \big(nV\big(K^*\big)\big)^{-\frac{1}{n}} \Phi_p(L)^{\frac{n+1}{n}},$$

which implies that

$$\biggl(\exp\biggl(\int_{S^{n-1}} \ln\biggl(\frac{h_K(u)}{h_L(u)}\biggr)\biggr)\, dC_p^*(L,K,u)\biggr)^n \geq \frac{1}{n(n-p)^n C_p(L)^n V(K^*)} \Phi_p(L)^{n+1}.$$

This gives the desired inequality.

For the case $K = L \in \mathcal{K}_o^n \cap \mathcal{F}_n$, according to the above proof, we also obtain the desired inequality. □

Taking $K = L$, we obtain the following result.

Corollary 4.2 *If $K \in \mathcal{K}_o^n \cap \mathcal{F}_n$ and $1 < p < n$, then*

$$V(K)V\big(K^*\big) \geq \frac{1}{n(n-p)^n} \frac{V(K)\Phi_p(K)^{n+1}}{C_p(K)^n}.$$

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