Alansari et al. *Journal of Inequalities and Applications* (2020) 2020:42
https://doi.org/10.1186/s13660-020-02313-z

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Strong convergence of an inertial iterative algorithm for variational inequality problem, generalized equilibrium problem, and fixed point problem in a Banach space

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Abstract

We propose and analyze an inertial iterative algorithm to approximate a common solution of generalized equilibrium problem, variational inequality problem, and fixed point problem in the framework of a 2-uniformly convex and uniformly smooth real Banach space. Further, we study the convergence analysis of our proposed iterative method. Finally, we give application and a numerical example to illustrate the applicability of the main algorithm.

MSC: 47H09; 47H05; 47J25

Keywords: Generalized equilibrium problem; Variational inequality problem; Relatively nonexpansive mapping; Inertial hybrid iterative method

1 Introduction

Let C be a nonempty closed convex subset of a real Banach space X and X^* be the dual space of X; let the pairing between X and X^* be denoted by $\langle\cdot,\cdot\rangle$. A mapping $J : X \to 2^{X^*}$ such that

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X, \tag{1.1}$$

is called normalized duality mapping.

Let $g, b : C \times C \to \mathbb{R}$ be bifunctions, where $\mathbb{R}$ is the set of real numbers. We study the generalized equilibrium problem (in short, GEP) which was to find $x \in C$ such that

$$g(x,y) + b(x,y) - b(x,x) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solution of (1.2) is denoted by Sol(GEP(1.2)). If we consider $b(x,y) = 0$, $\forall x,y \in C$, (1.2) reduces to the equilibrium problem (in short, EP): Find $x \in C$ such that

$$g(x,y) \geq 0, \quad \forall y \in C, \tag{1.3}$$

which was studied by Blum and Oettli [1]. The solution of (1.3) is denoted by Sol(EP(1.3)).

In the development of various fields of science and engineering, the equilibrium problem has a great importance. It provides various mathematical problems as special cases, like variational inclusion problem, variational inequality problem, mathematical programming problem, saddle point problem, complementary problem, Nash equilibrium problem in noncooperative games, minimization problem, minimax inequality problem, and fixed point problem (see [1–3]). If we consider $g(x,y) = h(y) - h(x)$, where $h : C \to \mathbb{R}$ is a nonlinear function, then (1.3) becomes the optimization problem: Find $x \in C$ such that

$$h(x) \leq h(y), \quad \forall y \in C. \tag{1.4}$$

If we consider $g(x,y) = \langle y - x, Dx\rangle$, $\forall x, y \in C$, where $D : C \to X^*$ is a nonlinear mapping, then (1.3) becomes the variational inequality problem (in short, VIP): Find $x \in C$ such that

$$\langle y - x, Dx\rangle \geq 0, \quad \forall y \in C, \tag{1.5}$$

which was studied by Hartmann and Stampacchia [4]. The set of solutions of (1.5) is denoted by Sol(VIP(1.5)).

In 2006, using the extragradient iterative method for VIP(1.5) given in [5], Nadezkhina and Takahashi [6] introduced and studied the following extragradient method and proved a strong convergence as follows:

$$\left.\begin{array}{l} x_0 \in C \subseteq H, \\ u_n = P_C(x_n - r_n D x_n), \\ y_n = \alpha_n x_n + (1-\alpha_n) T P_C(x_n - r_n D u_n), \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{array}\right\} \tag{1.6}$$

For further generalizations of iterative method (1.6), see [7–10].

One drawback of algorithm (1.6) is the computation of values of mapping D at two different points and the necessity of two projections on the admissible set C to pass to the next iteration. To overcome this drawback partially, recently, by adopting the idea of Popov [11], Malitsky and Semenov [12] showed that with some other choice of C_n it is possible to drop from (1.6) the step of extrapolation, which consists in $u_n = P_C(x_n - r_n D x_n)$, and introduced the following iteration without extrapolating step and proved a strong convergence:

$$\left.\begin{array}{l} x_0, z_0 \in C \subseteq H, \\ z_{n+1} = P_C(x_n - \lambda D z_n), \\ C_n = \{z \in H : \|z_{n+1} - z\|^2 \leq \|x_n - z\|^2 + k\|x_n - x_{n-1}\|^2 \\ \qquad - (1 - \frac{1}{k} - \lambda L)\|z_{n+1} - z_n\|^2 + \lambda L \|x_n - x_{n-1}\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{array}\right\} \tag{1.7}$$

where L is a Lipschitz constant and $\lambda > 0$, $k > 0$ are parameters. We note that algorithm (1.7) on every iteration needs only one computation of projection and one value of D.

The iterative method given in [12] extended the methods given in [5, 6]. Further, Dong and Lu [13] extended (1.7) and showed that the algorithm given by them could be faster than algorithm (1.6) by a numerical example. Very recently, Kazmi et al. [14] extended (1.7) for the mixed equilibrium problem.

In 2009, Takahashi et al. [15] introduced and studied the following iterative method and studied strong convergence for a relatively nonexpansive mapping to approximate the common solution of a fixed point problem and an equilibrium problem in Banach space:

$$\left.\begin{array}{l} x_0 \in C, \\ u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ z_n \in C \text{ such that } g(z_n,y) + \frac{1}{r_n}\langle y - z_n, Jz_n - Ju_n\rangle \geq 0, \quad \forall y \in C, \\ C_n = \{z \in C : \phi(z,z_n) \leq \phi(z,x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{array}\right\} \tag{1.8}$$

where $\Pi_C : X \to C$ is the generalized projection. For further extension of [13, 15], see[16–18].

On the other hand, Mainge [19] extended and unified the Krasnosel'skiĭ–Mann algorithm as follows:

$$\left.\begin{array}{l} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\alpha_n)w_n + \alpha_n Tw_n, \end{array}\right\} \tag{1.9}$$

for each $n \geq 1$ and proved a weak convergence for a nonexpansive mapping T under some conditions.

The term $\theta_n(x_n - x_{n-1})$ given in (1.9) is called the inertial term. It plays a crucial role in speeding up the convergence properties of iterative method (1.9); for details see [19–27]. It is worth to mention that, if we consider $\theta_n = 0$, then iterative method (1.9) becomes Krasnosel'skiĭ–Mann type iterative methods; for details, see [28–30]. Due to this importance, a number of researchers have been working on inertial type methods; see for details the following: inertial Douglas–Rachford splitting methods [31], inertial forward-backward splitting methods [32, 33], inertial forward-backward-forward method [34], and inertial proximal ADMM [35]. Further it is worth to mention that the study of convergence analysis of inertial type iterative methods is still unexplored in the setting of Banach space.

Therefore, inspired and motivated by the work given in [12, 15, 19], we introduce and study a hybrid iterative algorithm for approximating a common solution of GEP(1.2), VIP(1.5), and a fixed point problem for a relatively nonexpansive mapping. Further, we prove a strong convergence theorem in a uniformly smooth and 2-uniformly convex Banach spaces. Finally, we give a numerical example to justify the main theorem and demonstrate that our proposed inertial iterative algorithm is faster than the algorithms due to [15, 16].

2 Preliminaries

Suppose that weak and strong convergence are denoted by the symbols $\rightharpoonup$ and $\to$, respectively. Suppose that the unit sphere N is defined as $N = \{x \in X : \|x\| = 1\}$ on a Banach space X. If $\frac{\|x+y\|}{2} < 1$, $\forall x, y \in N$ with $x \neq y$, then X is said to be strictly convex. If for any $\varepsilon \in (0, 2]$

there exists $\delta > 0$ such that

$$\|x-y\| \geq \varepsilon \quad \text{implies} \quad \frac{\|x+y\|}{2} \leq 1-\delta \quad \text{for any } x, y \in N, \tag{2.1}$$

then X is said to be uniformly convex. Notice that X is reflexive and strictly convex if it is a uniformly convex Banach space and smooth if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in N$. If the limit exists uniformly, then X is uniformly smooth and X is said to enjoy the Kadec–Klee property if for any $\{x_n\} \in X$ and $x \in X$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. Notice that X enjoys the Kadec–Klee property if X is a uniformly convex Banach space. Also J is single-valued if X is smooth, J is uniformly norm-to-norm continuous on bounded subsets of X if X is uniformly smooth, and X is strictly convex if J is strictly monotone.

The function $\phi : X \times X \to \mathbb{R}$ is said to be *Lyapunov function* and is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in X. \tag{2.2}$$

It is obvious that

$$\bigl(\|x\| - \|y\|\bigr)^2 \leq \phi(x,y) \leq \bigl(\|x\| + \|y\|\bigr)^2, \quad \forall x, y \in X, \tag{2.3}$$

$$\phi\bigl(x, J^{-1}\bigl(\lambda Jy + (1-\lambda)Jz\bigr)\bigr) \leq \lambda\phi(x,y) + (1-\lambda)\phi(x,z), \quad \forall x, y \in X, \lambda \in [0,1], \tag{2.4}$$

and

$$\phi(x,y) = \|x\|\|Jx - Jy\| + \|y\|\|x-y\|, \quad \forall x, y \in X. \tag{2.5}$$

Remark 2.1 If X is a reflexive, strictly convex, and smooth Banach space, then $\forall x, y \in X$, $\phi(x,y) = 0 \Leftrightarrow x = y$.

Lemma 2.2 ([36]) *Let X be a 2-uniformly convex Banach space, then for all $x, y \in X$ the following inequality holds*:

$$\|x-y\| \leq \frac{2}{c}\|Jx - Jy\|,$$

where c is a 2-uniformly convex constant and $c \in (0,1]$.

Lemma 2.3 ([37]) *Let X be a smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Remark 2.4 If $\{x_n\}$ and $\{y_n\}$ are bounded, then by using (2.5) it is obvious that the converse of Lemma 2.3 is also true.

Definition 2.1 Let $T : C \to C$ be a mapping. Then:

(i) $\operatorname{Fix}(T) = \{x \in C : Tx = x\}$ is the collection of all fixed points of T;

(ii) A point $x_0 \in C$ is defined as an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup x_0$ and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. $\widehat{\operatorname{Fix}}(T)$ denotes the collection of all asymptotic fixed points of T;

(iii) T is said to be relatively nonexpansive if

$$\widehat{\mathrm{Fix}}(T) = \mathrm{Fix}(T) \neq \emptyset \quad \text{and} \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, p \in \mathrm{Fix}(T).$$

Lemma 2.5 ([38]) *Let X be a reflexive, strictly convex, and smooth Banach space. Let C be a nonempty closed convex subset of X. Let $T : C \to C$ be a relatively nonexpansive mapping. Then $\mathrm{Fix}(T)$ is a closed convex subset of C.*

Lemma 2.6 ([39]) *Let C be a nonempty closed convex subset of X and D be a monotone and hemicontinuous mapping of C into X^*. Then $\mathrm{VIP}(C, D)$ is closed and convex.*

Lemma 2.7 ([37]) *Let C be a nonempty closed convex subset of a real reflexive, strictly convex, and smooth Banach space X, and let $x \in X$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \inf_{y\in C} \phi(y, x)$.*

Definition 2.2 ([40]) A mapping $\Pi_C : X \to C$ is said to be a generalized projection if, for any point $x \in X$, $\Pi_C x = \bar{x}$, where $\bar{x}$ is a solution of the minimization problem $\phi(\bar{x}, x) = \inf_{y\in C} \phi(y, x)$.

Lemma 2.8 ([40]) *Let X be a reflexive, strictly convex, and smooth Banach space, and let C be a nonempty closed convex subset of X. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C \text{ and } y \in X.$$

Lemma 2.9 ([40]) *Let X be reflexive, strictly convex, and let C be a nonempty closed convex subset of a smooth Banach space X, let $x \in X$ and $z \in C$. Then*

$$z = \Pi_C x \iff \langle z - y, Jx - Jz\rangle \geq 0, \quad \forall y \in C.$$

Assumption 2.1 Let $g : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the following:
(i) $g(x, x) = 0$, $\forall x \in C$;
(ii) g is monotone, that is, $g(x, y) + g(y, x) \leq 0$, $\forall x \in C$;
(iii) $\limsup_{t\to 0} g(tz + (1 - t)x, y) \leq g(x, y)$, $\forall x, y, z \in C$;
(iv) For each $x \in C$, $y \to g(x, y)$ is convex and lower semicontinuous.

Assumption 2.2 Let $b : C \times C \to \mathbb{R}$ be a bifunction satisfying the following:
(i) b is skew-symmetric, i.e., $b(x, x) - b(x, y) - b(y, x) + b(y, y) \geq 0$, $\forall x, y \in C$;
(ii) b is convex in the second argument;
(iii) b is continuous.

Lemma 2.10 ([41]) *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space X, and let $g : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 2.1 and $b : C \times C \to \mathbb{R}$ satisfying Assumption 2.2. For all $r > 0$ and $x \in X$, define a mapping $T_r : X \to C$ as follows:*

$$T_r x = \left\{ z \in C : g(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle + b(z, y) - b(z, z) \geq 0, \forall y \in C \right\}, \quad \forall x \in X. \quad (2.6)$$

Then the following hold:

(a) $T_r x$ *is single-valued*;

(b) $T_r x$ *is a firmly nonexpansive type mapping, i.e., for all* $x, y \in X$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \le \langle T_r x - T_r y, J_x - Jy\rangle;$$

(c) $\mathrm{Fix}(T_r) = \mathrm{Sol}(\mathrm{GEP}(1.2))$ *is closed and convex*;

(d) $T_r x$ *is quasi-ϕ-nonexpansive*;

(e) $\phi(q, T_r x) + \phi(T_r x, x) \le \phi(q, x)$, $\forall q \in F(T_r)$.

In the sequel, we make use of the function $\Phi : X \times X^* \to \mathbb{R}$, defined by

$$\Phi(x, x^*) = \|x\|^2 - \langle x, x^*\rangle + \|x^*\|^2.$$

Observe that $\Phi(x, x^*) = \phi(x, J^{-1}x^*)$.

Lemma 2.11 ([40]) *Let X be a smooth, strictly convex, and reflexive Banach space with X^* as its dual. Then*

$$\Phi(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le \Phi(x, x^* + y^*), \quad \forall x \in X \text{ and all } x^*, y^* \in X^*.$$

3 Main result

In this section, we prove a strong convergence theorem for the inertial hybrid iterative algorithm to approximate a common solution of GEP(1.2), VIP(1.5), and fixed point problem for a relatively-nonexpansive mapping in uniformly smooth and 2-uniformly convex real Banach spaces.

Iterative Algorithm 3.1 Let the sequences $\{x_n\}$ and $\{z_n\}$ be generated by the iterative algorithm:

$$\left.\begin{aligned}
&x_0 = x_{-1}, \qquad z_0 \in C, \qquad C_0 := C,\\
&w_n = x_n + \theta_n(x_n - x_{n-1}),\\
&y_n = \Pi_C J^{-1}(Jw_n - \mu_n D w_n),\\
&u_n = J^{-1}(\alpha_n J z_n + (1-\alpha_n) JTy_n),\\
&z_{n+1} = T_{r_n} u_n,\\
&C_n = \{z \in C : \phi(z, z_{n+1}) \le \alpha_n \phi(z, z_n) + (1-\alpha_n)\phi(z, w_n)\},\\
&Q_n = \{z \in C : \langle x_n - z, Jx_n - Jx_0\rangle \le 0\},\\
&x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \qquad \forall n \ge 0,
\end{aligned}\right\} \tag{3.1}$$

where $\{\alpha_n\} \in [0,1]$, $r_n \in [a, \infty)$ for some $a > 0$, $\{\theta_n\} \in (0,1)$ and $\{\mu_n\} \in (0, \infty)$.

Theorem 3.2 *Let C be a nonempty, closed, and convex subset of a 2-uniformly convex and uniformly smooth real Banach space X, and let X^* be the dual of X. Let $D : X \to X^*$ be a γ-inverse strongly monotone mapping with constant $\gamma \in (0,1)$; $g : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1, and $b : C \times C \to \mathbb{R}$ satisfy Assumption 2.2. Let $T : C \to C$ be a relatively nonexpansive mapping such that $\Gamma := \mathrm{Sol}(\mathrm{GEP}(1.2)) \cap \mathrm{Sol}(\mathrm{VIP}(1.5)) \cap \mathrm{Fix}(T) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{z_n\}$ be generated by iterative algorithm (3.1) and the control sequences $\{\alpha_n\} \in [0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$, $r_n \in [a, \infty)$ for some $a > 0$, $\{\theta_n\} \in (0,1)$,* *and* $\{\mu_n\} \in (0,\infty)$ *satisfying the condition* $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{c^2\gamma}{2}$, *where* c *is the* 2*-uniformly convex constant of* X. *Then* $\{x_n\}$ *converges strongly to* $\hat{x} \in \varGamma$, *where* $\hat{x} = \varPi_\varGamma x_0$ *and* $\varPi_\varGamma x_0$ *is the generalized projection of* X *onto* $\varGamma$.

Now, we give some lemmas for the main result in this paper as follows.

Lemma 3.3 *For each* $n \geq 0$, $\varGamma$ *and* $C_n \cap Q_n$ *are closed and convex.*

Proof It follows from Lemmas 2.5–2.6 and Lemma 2.10 that $\varGamma$ is a nonempty closed and convex set, and hence $\varPi_\varGamma x_0$ is well defined. Evidently, $C_0 = C$ is closed and convex. Further, the closedness of C_n is also obvious. We only prove the convexity of C_n. For $q_1, q_2 \in C_n$, we have $q_1, q_2 \in C$, $tq_1 + (1-t)q_2 \in C$, where $t \in (0,1)$, and

$$\phi(q_1, z_{n+1}) \leq \alpha_n \phi(q_1, z_n) + (1-\alpha_n)\phi(q_1, w_n) \tag{3.2}$$

and

$$\phi(q_2, z_{n+1}) \leq \alpha_n \phi(q_2, z_n) + (1-\alpha_n)\phi(q_2, w_n). \tag{3.3}$$

The above two inequalities are equivalent to

$$\begin{aligned} &2\alpha_n\langle q_1, Jz_n\rangle + 2(1-\alpha_n)\langle q_1, Jw_n\rangle - 2\langle q_1, Jz_{n+1}\rangle \\ &\quad\leq \alpha_n\|z_n\|^2 + (1-\alpha_n)\|w_n\|^2 - \|z_{n+1}\|^2 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &2\alpha_n\langle q_2, Jz_n\rangle + 2(1-\alpha_n)\langle q_2, Jw_n\rangle - 2\langle q_2, Jz_{n+1}\rangle \\ &\quad\leq \alpha_n\|z_n\|^2 + (1-\alpha_n)\|w_n\|^2 - \|z_{n+1}\|^2. \end{aligned} \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} &2\alpha_n\bigl\langle tq_1 + (1-t)q_2, Jz_n\bigr\rangle + 2(1-\alpha_n)\bigl\langle tq_1 + (1-t)q_2, Jw_n\bigr\rangle - 2\bigl\langle tq_1 + (1-t)q_2, Jz_{n+1}\bigr\rangle \\ &\quad\leq \alpha_n\|z_n\|^2 + (1-\alpha_n)\|w_n\|^2 - \|z_{n+1}\|^2. \end{aligned} \tag{3.6}$$

Hence, we have

$$\phi\bigl(tq_1 + (1-t)q_2, z_{n+1}\bigr) \leq \alpha_n\phi\bigl(tq_1 + (1-t)q_2, z_n\bigr) + (1-\alpha_n)\phi\bigl(tq_1 + (1-t)q_2, w_n\bigr), \tag{3.7}$$

which implies that $tq_1 + (1-t)q_2 \in C_n$, hence C_n is closed and convex for all $n \geq 0$. By using the definition of Q_n, it is obvious that Q_n is closed and convex. This implies that $C_n \cap Q_n$, $\forall n \geq 0$ is closed and convex. □

Lemma 3.4 *For each* $n \geq 0$, $\varGamma \subset C_n \cap Q_n$, *and the sequence* $\{x_n\}$ *is well defined.*

Proof Let $p \in \Gamma$, we have

$$\begin{aligned}
\phi(p, z_{n+1}) &= \phi(p, T_{r_n} u_n) \\
&\leq \phi(p, u_n) \\
&\leq \phi\bigl(p, J^{-1}\bigl(\alpha_n J z_n + (1-\alpha_n) J T y_n\bigr)\bigr) \\
&\leq \alpha_n \phi(p, z_n) + (1-\alpha_n)\phi(p, T y_n) \\
&\leq \alpha_n \phi(p, z_n) + (1-\alpha_n)\phi(p, y_n). \qquad (3.8)
\end{aligned}$$

Additionally, by Lemmas 2.2 and 2.11, we obtain

$$\begin{aligned}
\phi(p, y_n) &= \phi\bigl(p, \Pi_C J^{-1}(J w_n - \mu_n D w_n)\bigr) \\
&\leq \phi\bigl(p, J^{-1}(J w_n - \mu_n D w_n)\bigr) \\
&= \Phi(p, J w_n - \mu_n D w_n) \\
&\leq \Phi\bigl(p, (J w_n - \mu_n D w_n) + \mu_n D w_n\bigr) - 2\bigl\langle J^{-1}(J w_n - \mu_n D w_n) - p, \mu_n D w_n\bigr\rangle \\
&= \Phi(p, J w_n) - 2\mu_n \bigl\langle J^{-1}(J w_n - \mu_n D w_n) - p, D w_n\bigr\rangle \\
&= \phi(p, w_n) - 2\langle w_n - p, D w_n\rangle - 2\mu_n \bigl\langle J^{-1}(J w_n - \mu_n D w_n) - w_n, D w_n\bigr\rangle \\
&= \phi(p, w_n) - 2\langle w_n - p, D w_n - D p\rangle - 2\mu_n \bigl\langle J^{-1}(J w_n - \mu_n D w_n) - w_n, D w_n\bigr\rangle \\
&\leq \phi(p, w_n) - 2\mu_n \gamma \|D w_n\|^2 + 2\mu_n \bigl\|J^{-1}(J w_n - D w_n) - J^{-1} J w_n\bigr\| \|D w_n\|^2 \\
&\leq \phi(p, w_n) - 2\mu_n \gamma \|D w_n\|^2 + \frac{4\mu_n^2}{c^2}\|D w_n\|^2 \\
&= \phi(p, w_n) - 2\mu_n\left(\gamma - \frac{2\mu_n}{c^2}\right)\|D w_n\|^2, \qquad (3.9)
\end{aligned}$$

which, combined with $\mu_n < \frac{c^2\gamma}{2}$, leads to

$$\phi(p, y_n) \leq \phi(p, w_n). \qquad (3.10)$$

By (3.8) and (3.10), we have

$$\phi(p, z_{n+1}) \leq \alpha_n \phi(p, z_n) + (1-\alpha_n)\phi(p, w_n), \qquad (3.11)$$

which implies that $p \in C_n$. Thus, $\Gamma \subset C_n$, $\forall n \geq 0$. Next, we show by induction that $\Gamma \subset C_n \cap Q_n$, $\forall n \geq 0$. Since $Q_0 = C$, we have $\Gamma \subset C_0 \cap Q_0$. Let $\Gamma \subset C_k \cap Q_k$ for some $k > 0$. Then there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = \Pi_{C_k \cap Q_k} x_0$. From the definition of x_{k+1}, we have, for all $z \in C_k \cap Q_k$, that $\langle x_{k+1} - z, J x_0 - J x_{k+1}\rangle \geq 0$. Since $\Gamma \subset C_k \cap Q_k$, we have

$$\langle x_{k+1} - p, J x_0 - J x_{k+1}\rangle \geq 0, \quad \forall p \in \Gamma, \qquad (3.12)$$

and hence $p \in Q_{k+1}$. Thus, we obtain $\Gamma \subset C_{k+1} \cap Q_{k+1}$ as $\Gamma \subset C_n$ for all n. Therefore, $\Gamma \subset C_n \cap Q_n$, $\forall n \geq 0$, and hence $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ is well defined $\forall n \geq 0$. Thus, $\{x_n\}$ is well defined. □

Lemma 3.5 *The sequences* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, *and* $\{u_n\}$ *generated by iterative algorithm* (3.1) *are bounded.*

Proof By the definition of Q_n, $x_n = \Pi_{Q_n}x_0$. Using $x_n = \Pi_{Q_n}x_0$ and Lemma 2.8, we obtain

$$\begin{aligned}\phi(x_n,x_0) &= \phi(\Pi_{Q_n}x_0,x_0)\\ &\leq \phi(u,x_0) - \phi(u,\Pi_{Q_n}x_0) \leq \phi(u,x_0), \quad \forall u \in \Gamma \subset Q_n.\end{aligned}$$

This shows that $\{\phi(x_n,x_0)\}$ is bounded and hence from (2.3) that $\{x_n\}$ is bounded. Further,

$$\begin{aligned}\phi(p,x_n) &= \phi(p,\Pi_{C_{n-1}\cap Q_{n-1}}x_0)\\ &= \phi(p,x_0) - \phi(x_n,x_0)\end{aligned}$$

implies that $\{\phi(p,x_n)\}$ is bounded and by the fact $\phi(p,Tx_n) \leq \phi(p,x_n)$, $\forall p \in \Gamma$ that $\{Tx_n\}$ is also bounded. Therefore, $\{w_n\}$ and $\{y_n\}$ are also bounded. Now, setting $M = \max\{\phi(p,z_0),\sup_n \phi(p,w_n)\}$. Then obviously $\phi(p,z_0) \leq M$. Let $\phi(p,z_n) \leq M$ for some n, then from (3.11)

$$\phi(p,z_{n+1}) \leq \alpha_n M + (1-\alpha_n)M \leq M.$$

Thus, $\{\phi(p,z_{n+1})\}$ is bounded and hence $\{z_n\}$ is also bounded. □

Lemma 3.6 *The sequences* $x_n \to \hat{x}$, $u_n \to \hat{x}$, *and* $z_{n+1} \to \hat{x}$ *as* $n \to \infty$, *where* $\hat{x}$ *is some point in* C.

Proof Since $x_{n+1} = \Pi_{C_n\cap Q_n}x_0 \in Q_n$ and $x_n \in \Pi_{Q_n}x_0$, we get

$$\phi(x_n,x_0) \leq \phi(x_{n+1},x_0), \quad \forall n \geq 0.$$

This shows that $\{\phi(x_n,x_0)\}$ is nondecreasing and hence from boundedness of $\{\phi(x_n,x_0)\}$, $\lim_{n\to\infty}\phi(x_n,x_0)$ exists. Further,

$$\begin{aligned}\phi(x_{n+1},x_n) &= \phi(x_{n+1},\Pi_{Q_n}x_0)\\ &\leq \phi(x_{n+1},x_0) - \phi(\Pi_{Q_n}x_0,x_0)\\ &= \phi(x_{n+1},x_0) - \phi(x_n,x_0), \quad \forall n \geq 0,\end{aligned}$$

and hence

$$\lim_{n\to\infty}\phi(x_{n+1},x_n) = 0. \tag{3.13}$$

Since X is uniformly convex and smooth, by Lemma 2.3, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.14}$$

Since X is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. Since $C_n \cap Q_n$ is closed and convex, $\hat{x} \in C_n \cap Q_n$. Using weak lower semicontinuity of $\|\cdot\|^2$, we obtain

$$\begin{aligned}\phi(\hat{x},x_0) &= \|\hat{x}\|^2 - 2\langle\hat{x},Jx_0\rangle + \|x_0\|^2 \\ &\le \liminf_{k\to\infty}\bigl(\|x_{n_k}\|^2 - 2\langle x_{n_k},Jx_0\rangle + \|x_0\|^2\bigr) \\ &= \liminf_{k\to\infty}\phi(x_{n_k},x_0) \\ &\le \limsup_{k\to\infty}\phi(x_{n_k},x_0) \\ &\le \phi(\hat{x},x_0),\end{aligned}$$

which implies that $\lim_{k\to\infty}\phi(x_{n_k},x_0) = \phi(\hat{x},x_0)$, and hence we have $\lim_{k\to\infty}\|x_{n_k}\| = \|\hat{x}\|$. Further, from the Kadec–Klee property of X, $x_{n_k} \to \hat{x}$ as $k \to \infty$. Since $\lim_{n\to\infty}\phi(x_n,x_0)$ exists, $\lim_{n\to\infty}\phi(x_n,x_0) = \phi(\hat{x},x_0)$. If there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to \tilde{x}$ as $j \to \infty$, then

$$\begin{aligned}\phi(\hat{x},\tilde{x}) &= \lim_{k,j\to\infty}\phi(x_{n_k},x_{n_j}) \\ &= \lim_{k,j\to\infty}\phi(x_{n_k},\Pi_{Q_{n_j}}x_0) \\ &\le \lim_{k,j\to\infty}\bigl\{\phi(x_{n_k},x_0) - \phi(x_{n_j},x_0)\bigr\} = 0,\end{aligned}$$

which shows $\hat{x} = \tilde{x}$ and thus $x_n \to \hat{x}$ as $n \to \infty$.

From the definition of w_n, we have $\|w_n - x_n\| = \|\theta_n(x_n - x_{n-1})\| \le \|x_n - x_{n-1}\|$, which implies by (3.14) that

$$\lim_{n\to\infty}\|w_n - x_n\| = 0. \tag{3.15}$$

Since $\{w_n\}$ is bounded and by Remark 2.4, we get

$$\lim_{n\to\infty}\phi(x_n,w_n) = 0. \tag{3.16}$$

By (3.14) and (3.15), we have

$$\lim_{n\to\infty}\|x_{n+1} - w_n\| = 0, \tag{3.17}$$

using Remark 2.4

$$\lim_{n\to\infty}\phi(x_{n+1},w_n) = 0. \tag{3.18}$$

As $x_{n+1} = \Pi_{C_n\cap Q_n}x_0 \in C_n$, therefore

$$\phi(x_{n+1},z_{n+1}) \le \alpha_n\phi(x_{n+1},z_n) + (1-\alpha_n)\phi(x_{n+1},w_n). \tag{3.19}$$

By (3.16), (3.19), and assumption $\lim_{n\to\infty}\alpha_n = 0$,

$$\phi(x_{n+1},z_{n+1}) = 0.$$

Using (2.3), we get

$$\lim_{n\to\infty}\left(\|x_{n+1}\|-\|z_{n+1}\|\right)=0,$$

and using $\lim_{n\to\infty}\|x_n\|=\|\hat{x}\|$, we have

$$\lim_{n\to\infty}\|z_{n+1}\|=\|\hat{x}\|. \tag{3.20}$$

Hence, we have

$$\lim_{n\to\infty}\|Jz_{n+1}\|=\lim_{n\to\infty}\|z_{n+1}\|=\|\hat{x}\|=\|J\hat{x}\|. \tag{3.21}$$

This shows that $\{\|Jz_{n+1}\|\}$ is bounded. Since X and X^* are reflexive, we may assume that $Jz_{n+1}\rightharpoonup x^*\in X^*$. By reflexivity of X, we see that $J(X)=X^*$, that is, there exists $x\in X$ such that $Jx=x^*$. Since

$$\begin{aligned}\phi(x_{n+1},z_{n+1})&=\|x_{n+1}\|^2-2\langle x_{n+1},Jz_{n+1}\rangle+\|z_{n+1}\|^2,\\ \phi(x_{n+1},z_{n+1})&=\|x_{n+1}\|^2-2\langle x_{n+1},Jz_{n+1}\rangle+\|Jz_{n+1}\|^2.\end{aligned}$$

By using $\liminf_{n\to\infty}$ in the above equality, we have

$$\begin{aligned}0&\geq\|\hat{x}\|^2-2\langle\hat{x},x^*\rangle+\|x^*\|^2\\ &=\|\hat{x}\|^2-2\langle\hat{x},Jx\rangle+\|Jx\|^2\\ &=\|\hat{x}\|^2-2\langle\hat{x},Jx\rangle+\|x\|^2\\ &=\phi(\hat{x},x),\end{aligned}$$

i.e., $\hat{x}=x$, and hence $x^*=J\hat{x}$. This implies that $Jz_{n+1}\rightharpoonup J\hat{x}\in X^*$. Since X^* and (3.21) satisfy the Kadec–Klee property, then

$$\lim_{n\to\infty}\|Jz_{n+1}-J\hat{x}\|=0.$$

As $J^{-1}:X^*\to X$ is demicontinuous, therefore $z_{n+1}\rightharpoonup\hat{x}$. Using (3.20) and the Kadec–Klee property of X

$$\lim_{n\to\infty}z_{n+1}=\hat{x}. \tag{3.22}$$

Next, by using the weak lower semicontinuity of $\|\cdot\|^2$, we arrive at

$$\begin{aligned}\phi(p,\hat{x})&=\|p\|^2-2\langle p,J\hat{x}\rangle+\|\hat{x}\|^2\\ &\leq\liminf_{n\to\infty}\left(\|p\|^2-2\langle p,Jz_{n+1}\rangle+\|z_{n+1}\|^2\right)\\ &=\liminf_{n\to\infty}\phi(p,z_{n+1})\\ &\leq\limsup_{n\to\infty}\phi(p,z_{n+1})\end{aligned}$$

$$= \limsup_{n\to\infty}\big(\|p\|^2 - 2\langle p, Jz_{n+1}\rangle + \|z_{n+1}\|^2\big)$$
$$\le \phi(p,\hat{x}),$$

and hence

$$\lim_{n\to\infty}\phi(p,z_{n+1}) = \phi(p,\hat{x}). \tag{3.23}$$

Since $x_n \to \hat{x}$, $n\to\infty$, and (3.22), we have

$$\lim_{n\to\infty}\|x_n - z_{n+1}\| = 0. \tag{3.24}$$

Since J is uniformly continuous, we get

$$\lim_{n\to\infty}\|Jx_n - Jz_{n+1}\| = 0. \tag{3.25}$$

By using the definition of Lyapunov function, we have, $\forall p \in \Gamma$,

$$\begin{aligned}\phi(p,x_n) - \phi(p,z_{n+1}) &= \|x_n\|^2 - \|z_{n+1}\|^2 - 2\langle p, Jx_n - Jz_{n+1}\rangle \\ &\le \|x_n - z_{n+1}\|\big(\|x_n\| + \|z_{n+1}\|\big) + 2\|p\|\|Jx_n - Jz_{n+1}\|.\end{aligned}$$

From (3.24) and (3.25)

$$\lim_{n\to\infty}\big\{\phi(p,x_n) - \phi(p,z_{n+1})\big\} = 0. \tag{3.26}$$

From (3.23) and (3.26)

$$\lim_{n\to\infty}\phi(p,x_n) = \phi(p,\hat{x}). \tag{3.27}$$

Again, by using weak lower semicontinuity of $\|\cdot\|^2$, we obtain

$$\begin{aligned}\phi(p,\hat{x}) &= \|p\|^2 - 2\langle p, J\hat{x}\rangle + \|\hat{x}\|^2 \\ &\le \liminf_{n\to\infty}\big(\|p\|^2 - 2\langle p, Jw_n\rangle + \|w_n\|^2\big) \\ &= \liminf_{n\to\infty}\phi(p,w_n) \\ &\le \limsup_{n\to\infty}\phi(p,w_n) \\ &= \limsup_{n\to\infty}\big(\|p\|^2 - 2\langle p, Jw_n\rangle + \|w_n\|^2\big) \\ &\le \phi(p,\hat{x}),\end{aligned}$$

which implies

$$\lim_{n\to\infty}\phi(p,w_n) = \phi(p,\hat{x}). \tag{3.28}$$

Thus, for any $p \in \Gamma \subset C_n$ and by (3.8) and (3.10),

$$\begin{aligned} \phi(p,u_n) &\le \phi\bigl(p,J^{-1}\bigl(\alpha_n z_n + (1-\alpha_n)JTy_n\bigr)\bigr) \\ &\le \alpha_n\phi(p,z_n) + (1-\alpha_n)\phi(p,y_n) \\ &\le \alpha_n\phi(p,z_n) + (1-\alpha_n)\phi(p,w_n). \end{aligned} \tag{3.29}$$

From (3.23), (3.28), (3.29), and $\lim_{n\to\infty}\alpha_n = 0$,

$$\lim_{n\to\infty}\phi(p,u_n) = \phi(p,\hat{x}). \tag{3.30}$$

From Lemma 2.10(e), we obtain, for any $p \in \Gamma$ and $z_{n+1} = T_{r_n}u_n$,

$$\begin{aligned} \phi(z_{n+1},u_n) &= \phi(T_{r_n}u_n,u_n) \\ &\le \phi(p,u_n) - \phi(p,T_{r_n}u_n) \\ &= \phi(p,u_n) - \phi(p,z_{n+1}). \end{aligned} \tag{3.31}$$

It follows from (3.23), (3.30), and (3.31) that

$$\lim_{n\to\infty}\phi(z_{n+1},u_n) = 0,$$

and hence from (2.3) we have

$$\lim_{n\to\infty}\bigl(\|z_{n+1}\| - \|u_n\|\bigr) = 0.$$

From (3.20)

$$\lim_{n\to\infty}\|u_n\| = \|\hat{x}\|, \tag{3.32}$$

and hence

$$\lim_{n\to\infty}\|Ju_n\| = \|J\hat{x}\|, \tag{3.33}$$

i.e., $\{\|Ju_n\|\}$ is bounded in X^*. Since X^* is reflexive, we can assume that $Ju_n \rightharpoonup u^* \in X^*$ as $n \to \infty$. Since $J(X) = X^*$, there exists $u \in X$ such that $Ju = u^*$. Since

$$\begin{aligned} \phi(z_{n+1},u_n) &= \|z_{n+1}\|^2 - 2\langle z_{n+1},Ju_n\rangle + \|u_n\|^2 \\ &= \|z_{n+1}\|^2 - 2\langle z_{n+1},Ju_n\rangle + \|Ju_n\|^2. \end{aligned}$$

By using $\liminf_{n\to\infty}$ in the above equality, we have

$$\begin{aligned} 0 &\ge \|\hat{x}\|^2 - 2\bigl\langle \hat{x},u^*\bigr\rangle + \bigl\|u^*\bigr\|^2 \\ &= \|\hat{x}\|^2 - 2\langle \hat{x},Ju\rangle + \|Ju\|^2 \\ &= \|\hat{x}\|^2 - 2\langle \hat{x},Ju\rangle + \|u\|^2 \\ &= \phi(\hat{x},u). \end{aligned}$$

Using Remark 2.1, we have $\hat{x} = u$, i.e., $u^* = J\hat{x}$. Therefore $Ju_n \rightharpoonup J\hat{x} \in X^*$. From the Kadec–Klee property of X^* and (3.33), we obtain

$$\lim_{n\to\infty} \|Ju_n - J\hat{x}\| = 0. \tag{3.34}$$

As J^{-1} is demicontinuous and (3.34), therefore $u_n \rightharpoonup \hat{x}$. From the Kadec–Klee property of X and (3.32), we obtain

$$\lim_{n\to\infty} u_n = \hat{x}.$$

□

Proof of Theorem 3.2 By (3.8) and (3.9), we have

$$\begin{aligned}\phi(p,u_n) &\le \alpha_n\phi(p,z_n) + (1-\alpha_n)\phi(p,y_n)\\ &\le \alpha_n\phi(p,z_n) + (1-\alpha_n)\phi(p,w_n) + 2(1-\alpha_n)\mu_n\left(\frac{2\mu_n}{c^2} - \gamma\right)\|Dw_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}2(1-\alpha_n)\mu_n\left(\gamma - \frac{2\mu_n}{c^2}\right)\|Dw_n\|^2 &\le \phi(p,w_n) - \phi(p,u_n)\\ &\quad + \alpha_n\big[\phi(p,w_n) - \phi(p,u_n)\big],\end{aligned} \tag{3.35}$$

and hence from (3.28),(3.30), (3.35), and $\lim_{n\to\infty}\alpha_n = 0$, $\mu_n(\gamma - \frac{2\mu_n}{c^2}) > 0$,

$$\lim_{n\to\infty} \|Dw_n\| = 0. \tag{3.36}$$

Since D is γ-inverse strongly monotone, it is $\frac{1}{\gamma}$-Lipschitz continuous. It follows from $\lim_{n\to\infty} w_n = \hat{x}$ and (3.36) that $\hat{x} \in D^{-1}(0)$. Hence $\hat{x} \in \mathrm{Sol}(\mathrm{VIP}(1.5))$.

Furthermore, (3.1) combined with (3.36) yields that

$$\begin{aligned}\lim_{n\to\infty}\|y_n - \hat{x}\| &= \lim_{n\to\infty}\big\|\Pi_C J^{-1}(Jw_n - \mu_n Dw_n) - \Pi_C\hat{x}\big\|\\ &\le \lim_{n\to\infty}\big\|J^{-1}(Jw_n - \mu_n Dw_n) - \hat{x}\big\|\\ &= 0.\end{aligned} \tag{3.37}$$

Using Lemmas 2.2 and 2.11, we estimate

$$\begin{aligned}\phi(w_n,y_n) &= \phi\big(w_n, \Pi_C J^{-1}(Jw_n - \mu_n Dw_n)\big)\\ &\le \phi\big(w_n, J^{-1}(Jw_n - \mu_n Dw_n)\big)\\ &\le \Phi\big(w_n, (Jw_n - \mu_n Dw_n)\big)\\ &\le \Phi\big(w_n, (Jw_n - \mu_n Dw_n) + \mu_n Dw_n\big) - 2\big\langle J^{-1}(Jw_n - \mu_n Dw_n) - w_n, \mu_n Dw_n\big\rangle\\ &= \phi(w_n,w_n) + 2\big\langle J^{-1}(Jw_n - \mu_n Dw_n) - w_n, -\mu_n Dw_n\big\rangle\\ &= 2\mu_n\big\langle J^{-1}(Jw_n - \mu_n Dw_n) - w_n, -Dw_n\big\rangle\end{aligned}$$

$$\begin{aligned} &\leq \left\| J^{-1}(Jw_n - \mu_n Dw_n) - J^{-1}Jw_n \right\| \\ &\leq \frac{4}{c^2}\mu_n^2 \|Dw_n\|^2, \end{aligned} \tag{3.38}$$

then using (3.36) we have

$$\lim_{n\to\infty} \phi(w_n, y_n) = 0, \tag{3.39}$$

which implies by Lemma 2.3 that

$$\lim_{n\to\infty} \|w_n - y_n\| = 0. \tag{3.40}$$

As $\lim_{n\to\infty} \phi(z_{n+1}, u_n) = 0$, hence from Lemma 2.3 we have

$$\lim_{n\to\infty} \|z_{n+1} - u_n\| = 0. \tag{3.41}$$

By the uniform continuity of J,

$$\lim_{n\to\infty} \|Jz_{n+1} - Ju_n\| = 0. \tag{3.42}$$

Since $r_n \geq a$ and using (3.42), we get

$$\lim_{n\to\infty} \frac{\|Jz_{n+1} - Ju_n\|}{r_n} = 0. \tag{3.43}$$

By $z_{n+1} = T_{r_n}u_n$, we have

$$g(z_{n+1}, y) + \frac{1}{r_n}\langle y - z_{n+1}, Jz_{n+1} - Ju_n\rangle + b(y, z_{n+1}) - b(z_{n+1}, z_{n+1}) \geq 0, \quad \forall y \in C.$$

It follows from Assumption 2.1(ii) that

$$\begin{aligned} \frac{1}{r_n}\langle y - z_{n+1}, Jz_{n+1} - Ju_n\rangle &\geq -g(z_{n+1}, y) - b(y, z_{n+1}) + b(z_{n+1}, z_{n+1}) \\ &\geq g(y, z_{n+1}) - b(y, z_{n+1}) + b(z_{n+1}, z_{n+1}). \end{aligned}$$

Setting $n \to \infty$, by (3.43) and the lower semicontinuity of $y \to f(y, \cdot)$, we have

$$g(y, \hat{x}) - b(y, \hat{x}) + b(\hat{x}, \hat{x}) \leq 0, \quad \forall y \in C.$$

Setting $y_t := ty + (1-t)\hat{x}$, $\forall t \in (0, 1]$, and $y \in C$, then $y_t \in C$, and thus

$$g(y_t, \hat{x}) - b(y_t, \hat{x}) + b(\hat{x}, \hat{x}) \leq 0.$$

It follows from Assumption 2.1(i)–(iv) that

$$\begin{aligned} 0 &= g(y_t, y_t) \\ &\leq tg(y_t, y) + (1-t)g(y_t, \hat{x}) \\ &\leq tg(y_t, y) + (1-t)\bigl[b(y_t, \hat{x}) - b(\hat{x}, \hat{x})\bigr] \\ &\leq tg(y_t, y) + (1-t)\bigl[b(y, \hat{x}) - b(\hat{x}, \hat{x})\bigr]. \end{aligned}$$

Letting $t > 0$, we have from Assumption 2.1(iii)

$$g(\hat{x}, y) + b(y, \hat{x}) - b(\hat{x}, \hat{x}) \geq 0, \quad \forall y \in C.$$

Therefore $\hat{x} \in \mathrm{Sol}(\mathrm{GEP}(1.2))$.

Next, we show that $\hat{x} \in \mathrm{Fix}(T)$. In view of $u_n = J^{-1}(\alpha_n Jz_n + (1-\alpha_n)JTy_n)$, we have

$$Jz_{n+1} - Ju_n = \alpha_n(Jz_{n+1} - Jz_n) + (1-\alpha_n)(Jz_{n+1} - JTy_n).$$

Hence, we have

$$(1-\alpha_n)\|Jz_{n+1} - JTy_n\| \leq \|Jz_{n+1} - Ju_n\| + \alpha_n\|Jz_{n+1} - Jz_n\|. \tag{3.44}$$

From assumption $\lim_{n\to\infty} \alpha_n = 0$, (3.42), and (3.44), we obtain

$$\lim_{n\to\infty} \|Jz_{n+1} - JTy_n\| = 0, \tag{3.45}$$

$$\lim_{n\to\infty} \|z_{n+1} - Ty_n\| = 0. \tag{3.46}$$

Further, using (3.40) and (3.46), the inequality

$$\|Ty_n - y_n\| \leq \|Ty_n - z_{n+1}\| + \|z_{n+1} - w_n\| + \|w_n - y_n\|$$

implies

$$\lim_{n\to\infty} \|Ty_n - y_n\| = 0. \tag{3.47}$$

From (3.17), (3.40), and (3.41) it follows that $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{w_n\}$, and $\{z_n\}$ all have the same asymptotic behavior, hence from (3.47) we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0, \tag{3.48}$$

which implies that $\hat{x} = T\hat{x}$, i.e., $\hat{x} \in \mathrm{Fix}(T)$. Then $\hat{x} \in \Gamma$.

Finally, we show $\hat{x} = \Pi_\Gamma x_0$. Taking $k \to \infty$ in (3.12), we have

$$\langle \hat{x} - p, Jx_0 - J\hat{x}\rangle \geq 0, \quad \forall p \in \Gamma.$$

Now, by Lemma 2.9, $\hat{x} = \Pi_\Gamma x_0$. This completes the proof. □

If X is a Hilbert space, then $J = I$ and $\phi(x,y) = \|x-y\|^2$, $\forall x,y \in C$. Then from Theorem 3.2 we get the following corollaries.

Corollary 3.1 *Let C be a nonempty, closed, and convex subset of a Hilbert space X. Let $D : X \to X^*$ be a γ-inverse strongly monotone mapping with constant $\gamma \in (0,1)$ and $g : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1, and let $b : C \times C \to \mathbb{R}$ satisfy Assumption 2.2. Let $T : C \to C$ be a nonexpansive mapping such that $\Gamma := \mathrm{Sol}(\mathrm{GEP}(1.2)) \cap \mathrm{Sol}(\mathrm{VIP}(1.5)) \cap \mathrm{Fix}(T) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{z_n\}$ be generated by the iterative algorithm*:

$$\left.\begin{aligned}
&x_0 = x_{-1}, \qquad z_0 \in C, \qquad C_0 := C,\\
&w_n = x_n + \theta_n(x_n - x_{n-1}),\\
&y_n = P_C(w_n - \mu_n D w_n),\\
&u_n = \alpha_n z_n + (1-\alpha_n) T y_n,\\
&z_{n+1} = T_{r_n} u_n,\\
&C_n = \{z \in C : \|z_{n+1} - z\|^2 \leq \alpha_n \|z_n - z\|^2 + (1-\alpha_n)\|w_n - z\|^2\},\\
&Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \leq 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned}\right\} \tag{3.49}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$, $r_n \in [a,\infty)$ for some $a > 0$, $\{\theta_n\} \in (0,1)$, and $\{\mu_n\} \in (0,\infty)$ satisfying the condition $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{c^2\gamma}{2}$, where c is the 2-uniformly convex constant of X. Then $\{x_n\}$ converges strongly to $\hat{x} \in \Gamma$, where $\hat{x} = \Pi_\Gamma x_0$ and $\Pi_\Gamma x_0$ is the generalized projection of X onto Γ.

4 Numerical example

If $X = \mathbb{R}$ is a Hilbert space with the norm $\|x\| = |x|$, $\forall x \in X$. Now, we give a numerical example which justifies Theorem 3.2.

Example 4.1 Let $X = \mathbb{R}$, $C = X$, where X is a Hilbert space, and let $g : C \times C \to \mathbb{R}$ be defined by $g(x,y) = x(y-x)$, $\forall x,y \in C$, and $b : C \times C \to \mathbb{R}$ be defined by $b(x,y) = xy$, $\forall x,y \in C$. Let the mapping $D : \mathbb{R} \to \mathbb{R}$ be defined by $Dx = \frac{x}{2}$; let $T : C \to C$ be defined by $Tx = \frac{1}{3}x$. Setting $\{\mu_n\} = \{\frac{0.9}{n}\}$, $r_n = \frac{1}{4}$, $\theta_n = 0.9$, and $\{\alpha_n\} = \{\frac{1}{n^3}\}$, $\forall n \geq 0$, let the sequences $\{x_n\}$, $\{u_n\}$, and $\{z_n\}$ be generated by hybrid iterative algorithm (3.1) converges to $\hat{x} = \{0\} \in \Gamma$.

Proof Note that for the case $C = X$, where X is the Hilbert space, the sets C_n and Q_n in iterative algorithms (3.1) are half spaces. Therefore, the projection onto the intersection of sets C_n and Q_n can be computed using a similar formula as given in [42]. It is easy to observe that g and b satisfy Assumption 2.1 and Assumption 2.2, respectively, and $\mathrm{Sol}(\mathrm{GEP}(1.2)) = \{0\} \neq \emptyset$. Further, it easy to observe that D is a $\frac{1}{2}$-inverse strongly monotone mapping and $\mathrm{Sol}(\mathrm{VIP}(1.5)) = \{0\} \neq \emptyset$. Further, it is easy to observe that T is a relatively nonexpansive mapping with $\mathrm{Fix}(T) = \{0\}$. Therefore, $\Gamma := \mathrm{Sol}(\mathrm{GEP}(1.2)) \cap \mathrm{Sol}(\mathrm{VIP}(1.5)) \cap \mathrm{Fix}(T) = \{0\} \neq \emptyset$. After simplification, hybrid iterative scheme (3.1) is reduced to the following scheme:

Given initial values x_0, x_1, z_0,

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = P_C(w_n - \mu_n D w_n) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 1, \\ w_n - \mu_n \frac{w_n}{2}, & \text{otherwise,} \end{cases} \\ u_n = \alpha_n z_n + \frac{(1-\alpha_n)}{3} y_n; \qquad z_{n+1} = \frac{2u_n}{3}; \\ C_n = [e_n, \infty), \quad \text{where } e_n := \frac{z_{n+1}^2 - w_n^2 + \alpha_n(w_n^2 - z_n^2)}{2z_{n+1} - 2w_n + 2\alpha_n(w_n - z_n)}; \\ Q_n = [x_n, \infty); \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0. \end{cases} \tag{4.1}$$

Finally, using software Matlab 7.8.0, we have Figures 1 and 2 and Table 1 which show that $\{x_n\}$, $\{u_n\}$, and $\{z_n\}$ converge to $\hat{x} = \{0\}$ as $n \to +\infty$. □

For $D = 0$, $b(x, y) = 0$, we now demonstrate that iterative algorithm (3.1) with conditions given in Theorem 3.1 approximates a common element of the solution set of EP(1.3) and the fixed point set of T. Further, we observe that it is faster than iterative algorithm (3.1) due to [16] and iterative algorithm (1.8) due to [15] for a nonexpansive mapping.

Set $D = 0$, $b(x, y) = 0$, in Example 4.1, we have that iterative algorithm (4.1), iterative algorithm (3.1) due to [16], and iterative algorithm (1.8) due to [15] reduce to the following iterative algorithms:

Figure 1 Convergence of $\{x_n\}$, $\{z_n\}$ and $\{u_n\}$ when $x_0 = 1, x_1 = 1, z_0 = 3$

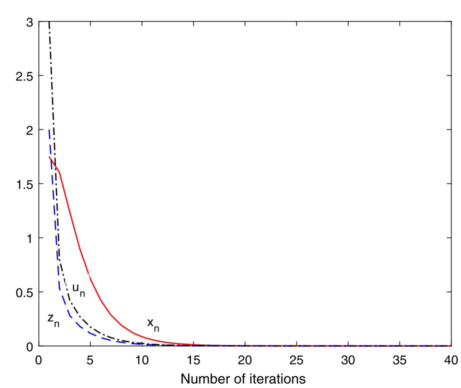


Figure 2 Convergence of $\{x_n\}$, $\{z_n\}$ and $\{u_n\}$ when $x_0 = -1, x_1 = -2, z_0 = -3$

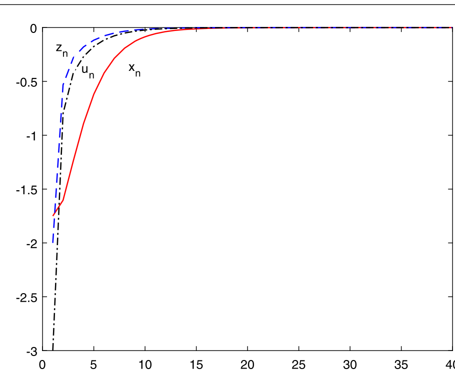


Table 1 Values of x_n, z_n and u_n

No. of iterations	x_n	z_n	u_n
	$x_0 = 1, x_1 = 2$	$z_0 = 3$	
1	1.750000	2.000000	3.000000
2	1.603678	0.532101	0.798151
3	1.238166	0.280883	0.421324
4	0.890899	0.179440	0.269159
5	0.619175	0.116978	0.175467
6	0.422455	0.077010	0.115516
7	0.285321	0.051030	0.076545
8	0.191617	0.033954	0.050931
9	0.128288	0.022649	0.033973
10	0.085748	0.015130	0.022695
15	0.011379	0.002027	0.003041
20	0.001514	0.000272	0.000408
25	0.000202	0.000037	0.000055
30	0.000027	0.000005	0.000007
35	0.000004	0.000001	0.000001
38	0.000001	0.000000	0.000000
39	0.000001	0.000000	0.000000
40	0.000000	0.000000	0.000000

Iterative Algorithm 4.2 Given initial values x_0, x_1, z_0,

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = w_n, \\ u_n = \alpha_n z_n + \frac{(1-\alpha_n)}{3} y_n; \qquad z_{n+1} = \frac{4u_n}{5}; \\ C_n = [e_n, \infty), \quad \text{where } e_n := \frac{z_{n+1}^2 - w_n^2 + \alpha_n(w_n^2 - z_n^2)}{2z_{n+1} - 2w_n + 2\alpha_n(w_n - z_n)}; \\ Q_n = [x_n, \infty); \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0; \end{cases} \tag{4.2}$$

and

Iterative Algorithm 4.3 Given initial values x_0, z_0,

$$\begin{cases} u_n = \alpha_n z_n + \frac{(1-\alpha_n)}{3} y_n; \qquad z_{n+1} = \frac{4u_n}{5}; \\ C_n = [e_n, \infty), \quad \text{where } e_n := \frac{z_{n+1}^2 - x_n^2 + \alpha_n(x_n^2 - z_n^2)}{2z_{n+1} - 2x_n + 2\alpha_n(x_n - z_n)}; \\ Q_n = [x_n, \infty); \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0; \end{cases} \tag{4.3}$$

and

Iterative Algorithm 4.4 Given initial values x_0,

$$\begin{cases} u_n = \alpha_n x_n + \frac{(1-\alpha_n)}{3} x_n; \qquad z_n = \frac{4u_n}{5}; \\ C_n = [e_n, \infty), \quad \text{where } e_n := \frac{z_n + x_n}{2}; \\ Q_n = [x_n, \infty); \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{cases} \tag{4.4}$$

respectively.

Figure 3 Convergence of $\{x_n\}$ when $x_0 = 1, x_1 = 2$, $z_0 = 3$

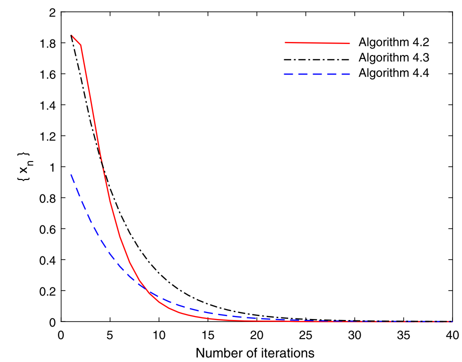
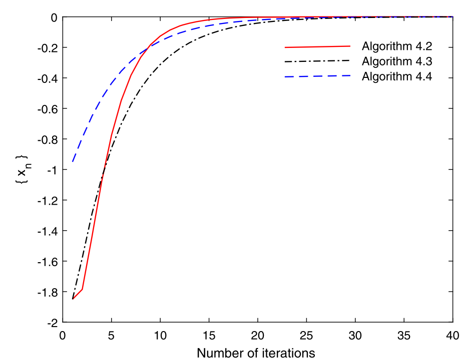


Figure 4 Convergence of $\{x_n\}$ when $x_0 = -1, x_1 = -2$, $z_0 = -3$



Hence, the sequence $\{x_n\}$ defined by Iterative Algorithm 4.2, Iterative Algorithm 4.3 as well as by Iterative Algorithm 4.4 converges strongly to $\hat{x} = 0$.

Finally, using software Matlab 7.8, we have the following figures which show that the sequence $\{x_n\}$ converges to $\hat{x} = 0 \in \Omega$. Figure 3 shows the convergence of $\{x_n\}$ when $x_0 = 1$, $x_1 = 2$, $z_0 = 3$ for Algorithms 4.2–4.3 and 4.4, while Fig. 4 shows the convergence of $\{x_n\}$ when $x_0 = -1$, $x_1 = -2$, $z_0 = -3$ for Algorithms 4.2–4.3. It is evident from figures that the sequence $\{x_n\}$ obtained by Iterative Algorithm 4.2 converges faster than the sequence $\{x_n\}$ obtained by Iterative Algorithm 4.3 and Iterative Algorithm 4.4.

Concluding remark 4.1 We observe that

(i) Iterative algorithm (3.1) is quite different from algorithm (1.8) given by Takahashi [15] and (3.1) given by [16].

(ii) Corollary 3.1 is new and different from that of Theorem 3.2 due to Takahashi [15] and (1.9) given by Mainge [19].

(iii) A numerical example was given to prove the efficiency of the proposed hybrid inertial iterative algorithm, that is, the proposed algorithms in Theorem 3.2 and Corollary 3.1 for $D = 0$ and $b(x, y) = 0$ converge faster than the algorithm presented in [16] and [15].

Acknowledgements

This work was supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (D-022-363-1440). The authors, therefore, gratefully acknowledge the DSR technical and financial support.

Funding
This work was supported by the King Abdulaziz University represented by the Deanship of Scientific Research on the inertial support for this research under grant number D-022-363-1440.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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Received: 29 June 2019 Accepted: 12 February 2020 Published online: 19 February 2020

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