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Generalizations and refinements of three-tuple Diamond-Alpha integral Hölder's inequality on time scales

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Abstract

In this paper, based on the existing Hölder's inequality, some new three-tuple diamond-alpha integral Hölder's inequalities on time scales are proposed and the related theorems and corollaries are given. At the same time, we also give the relevant conclusions and proof of n-tuple diamond-alpha integral Hölder's inequalities on time scales.

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Keywords: Hölder's inequality; Diamond-Alpha integral; Three-tuple; Time scales; Minkowski's inequality

1 Introduction

Let $f(\delta) > 0, g(\delta) > 0, p > 1, 1/p + 1/q = 1$. If $f(\delta)$ and $g(\delta)$ are continuous real-valued functions on $[\xi, \sigma]$, then

$$\int_{\xi}^{\sigma} f(\delta)g(\delta)\,d\delta \leq \left(\int_{\xi}^{\sigma} f^{p}(x)\,d\delta\right)^{1/p}\left(\int_{\xi}^{\sigma} g^{q}(x)\,d\delta\right)^{1/q}.$$

This famous Hölder inequality is extended in article [1] to the diamond-α integral Hölder inequality on time scales, in the following form:

Let $f, g, h : [\xi, \sigma] \to \mathbb{R}$ be $\Diamond_\alpha$-integrable functions, and $1/p + 1/q = 1$ with $p > 1$, then

$$\int_{\xi}^{\sigma} \left|h(\delta)\right|\left|f(\delta)g(\delta)\right|\Diamond_\alpha\delta \leq \left(\int_{\xi}^{\sigma} \left|h(\delta)\right|\left|f(\delta)\right|^{p}\Diamond_\alpha\delta\right)^{\frac{1}{p}}\left(\int_{\xi}^{\sigma} \left|h(\delta)\right|\left|g(\delta)\right|^{q}\Diamond_\alpha\delta\right)^{\frac{1}{q}}.$$

Since Hilger [2] proposed the time-scale theory in 1998, many researchers [3, 4] have made extensive promotions and applications of his theory. The classical analytic inequality [5–8], especially Hölder's inequality, plays a very important role in modern mathematics. Due to the importance of both, more and more scholars [9–15] have studied the intersections of two inequalities. The purpose of this article is to derive some generalizations and refinements of the three-tuple diamond-α integral Hölder inequality on time scales. The relevant conclusions of the n-tuple diamond-α integral Hölder inequality on time scales are also given.

2 Main lemmas

Before the main results are given in this paper, we need to introduce the following lemmas, which are helpful for the results of this paper.

Lemma 2.1 ([16]) *Let* $\sum_{j=1}^{m}\frac{1}{p_j}=1, \lambda_j\geq 0$ $(j=1,2,\ldots,m)$. *Then*

(1) *for* $p_j>1$, *we have*

$$\prod_{j=1}^{m}\lambda_j\leq\sum_{j=1}^{m}\frac{\lambda_j^{p_j}}{p_j}, \tag{1}$$

(2) *for* $0<p_m<1, p_j<0$ $(j=1,2,\ldots,m-1)$, *we have*

$$\prod_{j=1}^{m}\lambda_j\geq\sum_{j=1}^{m}\frac{\lambda_j^{p_j}}{p_j}. \tag{2}$$

Lemma 2.2 ([10]) *Let* $\mathbb{T}$ *be a time scale*, $a,b\in\mathbb{T}$ *with* $a<b$ *and* $\sum_{j=1}^{m}\frac{1}{p_j}=1$. *If* $f_j(\delta)>0$, *and* f_j $(j=1,2,\ldots,m)$ *is continuous real-valued function on* $[\xi,\sigma]_{\mathbb{T}}$, *then*

(1) *for* $p_j>1$, *we have*

$$\int_{\xi}^{\sigma}\prod_{j=1}^{m}f_j(\delta)\Diamond_\alpha\delta\leq\prod_{j=1}^{m}\left(\int_{\xi}^{\sigma}f_j^{p_j}(\delta)\Diamond_\alpha\delta\right)^{1/p_j}, \tag{3}$$

(2) *for* $0<p_m<1, p_j<0$ $(j=1,2,\ldots,m-1)$, *we have*

$$\int_{\xi}^{\sigma}\prod_{j=1}^{m}f_j(\delta)\Diamond_\alpha\delta\geq\prod_{j=1}^{m}\left(\int_{\xi}^{\sigma}f_j^{p_j}(\delta)\Diamond_\alpha\delta\right)^{1/p_j}. \tag{4}$$

Lemma 2.3 ([17]) *Let* $f,g,h:\mathbb{T}\to\mathbb{R}$ *be* $\Diamond$*-integrable on* $[\xi,\sigma]_{\mathbb{T}}, p>1$ *with* $q=p/(p-1)$. *Then we have*

$$\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|f(\delta)g(\delta)\right|\Diamond\delta\leq\left(\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|f(\delta)\right|^p\Diamond\delta\right)^{1/p}\left(\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|g(\delta)\right|^q\Diamond\delta\right)^{1/q}. \tag{5}$$

Lemma 2.4 ([17]) *Let* $f,g,h:\mathbb{T}\to\mathbb{R}$ *be* $\Diamond$*-integrable on* $[\xi,\sigma]_{\mathbb{T}}, p>1$ *with* $q=p/(p-1)$. *Then we have*

$$\begin{aligned}&\left(\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|f(\delta)+g(\delta)\right|^p\Diamond\delta\right)^{1/p}\\&\quad\leq\left(\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|f(\delta)\right|^p\Diamond\delta\right)^{1/p}+\left(\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|g(\delta)\right|^p\Diamond\delta\right)^{1/p}.\end{aligned} \tag{6}$$

Lemma 2.5 ([17]) *Let* $f,g,h:\mathbb{T}\to\mathbb{R}$ *be* $\Diamond$*-integrable on* $[\xi,\sigma]_{\mathbb{T}}, 0<p<1$ *with* $q=p/(p-1)$. *If* g^q *is* $\Diamond$*-integrable on* $[\xi,\sigma]_{\mathbb{T}}$, *then*

$$\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|f(\delta)g(\delta)\right|\Diamond\delta\geq\left(\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|f(\delta)\right|^p\Diamond\delta\right)^{1/p}\left(\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|g(\delta)\right|^q\Diamond\delta\right)^{1/q}. \tag{7}$$

Lemma 2.6 ([17]) *Let $f,g,h:\mathbb{T}\to\mathbb{R}$ be $\Diamond$-integrable on $[\xi,\sigma]_{\mathbb{T}}$, $0<p<1$ with $q=p/(p-1)$. Then we have*

$$\begin{aligned}&\left(\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|f(\delta)+g(\delta)\right|^{p}\Diamond\delta\right)^{1/p}\\&\quad\geq\left(\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|f(\delta)\right|^{p}\Diamond\delta\right)^{1/p}+\left(\int_{\xi}^{\sigma}\left|h(\delta)\right|\left|g(\delta)\right|^{p}\Diamond\delta\right)^{1/p}.\end{aligned}\tag{8}$$

3 Main results about diamond-α integral Hölder's inequality

Now, based on Tian's [18, 19] research results, we will give the following generalizations and refinements of the three-tuple diamond-α integral and n-tuple diamond-α integral Hölder inequality on time scales.

Theorem 3.1 *Let $\mathbb{T}$ be a time scale $a,b\in\mathbb{T}$ with $a<b$ and $\alpha_{kj}\in\mathbb{R}$ ($j=1,2,\ldots,m$, $k=1,2,\ldots,s$), $\sum_{k=1}^{s}\frac{1}{p_k}=1$, $\sum_{k=1}^{s}\alpha_{kj}=0$. If $f_j(\delta)>0$, and f_j ($j=1,2,\ldots,m$) is a continuous real-valued function on $[\xi,\sigma]_{\mathbb{T}}$, then*

(1) *for $p_k>1$, we have*

$$\begin{aligned}&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3\\&\quad\leq\prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\prod_{j=1}^{m}f_j^{1+p_k\alpha_{kj}}(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3\right)^{\frac{1}{p_k}},\end{aligned}\tag{9}$$

(2) *for $0<p_s<1$, $p_k<0$ ($k=1,2,\ldots,s-1$), we have*

$$\begin{aligned}&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3\\&\quad\geq\prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\prod_{j=1}^{m}f_j^{1+p_k\alpha_{kj}}(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3\right)^{\frac{1}{p_k}}.\end{aligned}\tag{10}$$

Proof (1) Set

$$g_k(\delta_1,\delta_2,\delta_3)=\left(\prod_{j=1}^{m}f_j^{1+p_k\alpha_{kj}}(\delta_1,\delta_2,\delta_3)\right)^{1/p_k}.$$

Applying the assumptions $\sum_{k=1}^{s}\frac{1}{p_k}=1$ and $\sum_{k=1}^{s}\alpha_{kj}=0$, by computing, we can observe that

$$\begin{aligned}&\prod_{k=1}^{s}g_k(\delta_1,\delta_2,\delta_3)\\&\quad=g_1g_2\cdots g_s\\&\quad=\left(\prod_{j=1}^{m}f_j^{1+a_1\alpha_{1j}}(\delta_1,\delta_2,\delta_3)\right)^{1/a_1}\left(\prod_{j=1}^{m}f_j^{1+a_2\alpha_{2j}}(\delta_1,\delta_2,\delta_3)\right)^{1/a_2}\cdots\end{aligned}$$

$$
\begin{aligned}
&\times \left(\prod_{j=1}^{m} f_j^{1+a_s\alpha_{s_j}}(\delta_1,\delta_2,\delta_3)\right)^{1/a_s} \\
&= \prod_{j=1}^{m} f_j^{1/a_1+\alpha_{1j}}(\delta_1,\delta_2,\delta_3) \prod_{j=1}^{m} f_j^{1/a_2+\alpha_{2j}}(\delta_1,\delta_2,\delta_3)\cdots \\
&\quad \times \prod_{j=1}^{m} f_j^{1/a_s+\alpha_{sj}}(\delta_1,\delta_2,\delta_3) \\
&= \prod_{j=1}^{m} f_j^{1/a_1+1/a_2+\cdots+1/a_s+\alpha_{1j}+\alpha_{2j}+\cdots+\alpha_{sj}}(\delta_1,\delta_2,\delta_3) = \prod_{j=1}^{m} f_j(\delta_1,\delta_2,\delta_3).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \prod_{j=1}^{m} f_j(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3 \\
&= \int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \prod_{k=1}^{s} g_k(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3. \qquad (11)
\end{aligned}
$$

By the Hölder inequality (3), we find

$$
\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \prod_{k=1}^{s} g_k(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3 \\
&\le \prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} g_k^{p_k}(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3\right)^{1/p_k}. \qquad (12)
\end{aligned}
$$

Substituting $g_k(\delta_1,\delta_2,\delta_3)$ into the inequality (12) can be obtained

$$
\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \prod_{j=1}^{m} f_j(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3 \\
&\le \prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \prod_{j=1}^{m} f_j^{1+p_k\alpha_{kj}}(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3\right)^{\frac{1}{p_k}}.
\end{aligned}
$$

(2) After the same proof as inequality (9), we get

$$
\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \prod_{k=1}^{s} g_k(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3 \\
&\ge \prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} g_k^{p_k}(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3\right)^{1/p_k}. \qquad (13)
\end{aligned}
$$

Substituting $g_k(\delta_1,\delta_2.\delta_3)$ into the (12) can be obtained

$$
\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \prod_{j=1}^{m} f_j(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3
$$

$$\geq \prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\prod_{j=1}^{m}f_j^{1+p_k\alpha_{kj}}(\delta_1,\delta_2,\delta_3)\diamondsuit_\alpha\delta_1\diamondsuit_\alpha\delta_2\diamondsuit_\alpha\delta_3\right)^{\frac{1}{p_k}}.$$

The proof of Theorem 3.1 is accomplished. □

Corollary 3.2 *Under the conditions of Theorem* 3.1, *let* $s = m, \alpha_{kj} = -t/p_k$ *for* $k \neq j$ *and* $\alpha_{jj} = t(1 - 1/p_j)$ *with* $t \in \mathbb{R}$, *then*

(1) *for* $p_k > 1$, *we have the following inequality*:

$$\begin{aligned}&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\delta_3)\diamondsuit_\alpha\delta_1\diamondsuit_\alpha\delta_2\diamondsuit_\alpha\delta_3\\&\quad\leq\prod_{k=1}^{m}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left(\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\delta_3)\right)^{1-t}\right.\\&\qquad\left.\times\left(f_k^{p_k}(\delta_1,\delta_2,\delta_3)\right)^t\diamondsuit_\alpha\delta_1\diamondsuit_\alpha\delta_2\diamondsuit_\alpha\delta_3\right)^{1/p_k},\end{aligned}\tag{14}$$

(2) $0 < p_m < 1, p_k < 0$ $(k = 1,2,\ldots,m-1)$, *we have the following reverse inequality*:

$$\begin{aligned}&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\delta_3)\diamondsuit_\alpha\delta_1\diamondsuit_\alpha\delta_2\diamondsuit_\alpha\delta_3\\&\quad\geq\prod_{k=1}^{m}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left(\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\delta_3)\right)^{1-t}\right.\\&\qquad\left.\times\left(f_k^{p_k}(\delta_1,\delta_2,\delta_3)\right)^t\diamondsuit_\alpha\delta_1\diamondsuit_\alpha\delta_2\diamondsuit_\alpha\delta_3\right)^{1/p_k}.\end{aligned}\tag{15}$$

On the basis of Theorem 3.1, we give the n-tuple diamond-α integral Hölder's inequality on time scales.

Theorem 3.3 *Let* $\mathbb{T}$ *be a time scale* $\xi, \sigma \in \mathbb{T}$ *with* $\xi < \sigma$ *and* $\alpha_{kj} \in \mathbb{R}$ $(j = 1,2,\ldots,m, k = 1,2,\ldots,s)$, $\sum_{k=1}^{s}\frac{1}{p_k} = 1, \sum_{k=1}^{s}\alpha_{kj} = 0$. *If* $f_j(\delta) > 0$, *and* f_j $(j = 1,2,\ldots,m)$ *is a continuous real-valued function on* $[\xi,\sigma]_{\mathbb{T}}$, *then*

(1) *for* $p_k > 1$, *we have the following inequality*:

$$\begin{aligned}&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\ldots,\delta_n)\diamondsuit_\alpha\delta_1\diamondsuit_\alpha\delta_2\cdots\diamondsuit_\alpha\delta_n\\&\quad\leq\prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\prod_{j=1}^{m}f_j^{1+p_k\alpha_{kj}}(\delta_1,\delta_2,\ldots,\delta_n)\diamondsuit_\alpha\delta_1\diamondsuit_\alpha\delta_2\cdots\diamondsuit_\alpha\delta_n\right)^{\frac{1}{p_k}},\end{aligned}\tag{16}$$

(2) *for* $0 < p_s < 1, p_k < 0$ $(k = 1,2,\ldots,s-1)$, *we have the following reverse inequality*:

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\ldots,\delta_n)\diamondsuit_\alpha\delta_1\diamondsuit_\alpha\delta_2\cdots\diamondsuit_\alpha\delta_n$$

$$\geq \prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\prod_{j=1}^{m} f_j^{1+p_k\alpha_{kj}}(\delta_1,\delta_2,\ldots,\delta_n)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\cdots\Diamond_\alpha\delta_n\right)^{\frac{1}{p_k}}. \tag{17}$$

Proof Similar to the proof of Theorem 3.1, we get the result of Theorem 3.3. □

Remark 3.4 The three-tuple diamond-*alpha* inequalities in Theorem 3.1 and the n-tuple diamond-α inequalities in Theorem 3.3 are generalizations to Theorem 3.3 in Ref. [10].

Theorem 3.5 *Let* $\mathbb{T}$ *be a time scale,* $\xi,\sigma\in\mathbb{T}$ *with* $\xi<\sigma$ *and* $r\in\mathbb{R},\alpha_{kj}\in\mathbb{R}$ $(j=1,2,\ldots,m,k=1,2,\ldots,s)$, $\sum_{k=1}^{s}\frac{1}{p_k}=r,\sum_{k=1}^{s}\alpha_{kj}=0$. *If* $f_j(\delta)>0$, *and* f_j $(j=1,2,\ldots,m)$ *is a continuous real-valued function on* $[\xi,\sigma]_{\mathbb{T}}$, *then*

(1) *for* $rp_k>1$, *we have the following inequality*:

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\prod_{j=1}^{m} f_j(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3$$
$$\leq \prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\prod_{j=1}^{m} f_j^{1+rp_k\alpha_{kj}}(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3\right)^{\frac{1}{rp_k}}, \tag{18}$$

(2) *for* $0<rp_k<1, rp_k<0$ $(k=1,2,\ldots,s-1)$, *we have the following reverse inequality*:

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\prod_{j=1}^{m} f_j(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3$$
$$\geq \prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\prod_{j=1}^{m} f_j^{1+rp_k\alpha_{kj}}(\delta_1,\delta_2,\delta_3)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\Diamond_\alpha\delta_3\right)^{\frac{1}{rp_k}}. \tag{19}$$

Proof (1) According to $rp_k>1$ and $\sum_{k=1}^{s}\frac{1}{p_k}=r$, we get $\sum_{k=1}^{s}\frac{1}{rp_k}=1$. Then, by inequality (9), we immediately obtain the inequality (18).

(2) According to $0<rp_k<1, rp_k<0$ $(k=1,2,\ldots,s-1)$ and $\sum_{k=1}^{s}\frac{1}{p_k}=r$, we have $\sum_{k=1}^{s}\frac{1}{rp_k}=1$, by inequality (10), we immediately have the inequality (19). This completes the proof. □

Similarly, on the basis of Theorem 3.5, we give the n-tuple diamond-α integral Hölder's inequality on time scales.

Theorem 3.6 *Let* $\mathbb{T}$ *be a time scale,* $\xi,\sigma\in\mathbb{T}$ *with* $\xi<\sigma$ *and* $r\in\mathbb{R},\alpha_{kj}\in\mathbb{R}$ $(j=1,2,\ldots,m,k=1,2,\ldots,s)$, $\sum_{k=1}^{s}\frac{1}{p_k}=r,\sum_{k=1}^{s}\alpha_{kj}=0$. *If* $f_j(\delta)>0$, *and* f_j $(j=1,2,\ldots,m)$ *is a continuous real-valued function on* $[\xi,\sigma]_{\mathbb{T}}$, *then*

(1) *for* $rp_k>1$, *we have the following inequality*:

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\prod_{j=1}^{m} f_j(\delta_1,\delta_2,\ldots,\delta_n)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\cdots\Diamond_\alpha\delta_n$$
$$\leq \prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\prod_{j=1}^{m} f_j^{1+rp_k\alpha_{kj}}(\delta_1,\delta_2,\ldots,\delta_n)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\cdots\Diamond_\alpha\delta_n\right)^{\frac{1}{rp_k}}, \tag{20}$$

(2) *for* $0 < rp_k < 1, rp_k < 0$ $(k = 1, 2, \dots, s-1)$, *we have the following reverse inequality*:

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\prod_{j=1}^{m} f_j(\delta_1,\delta_2,\dots,\delta_n)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\cdots\Diamond_\alpha\delta_n$$
$$\geq \prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\prod_{j=1}^{m} f_j^{1+rp_k\alpha_{kj}}(\delta_1,\delta_2,\dots,\delta_n)\Diamond_\alpha\delta_1\Diamond_\alpha\delta_2\cdots\Diamond_\alpha\delta_n\right)^{\frac{1}{rp_k}}. \tag{21}$$

Proof Similar to the proof of Theorem 3.5, we get the result of Theorem 3.6. □

Remark 3.7 For the inequality of Theorem 3.4 in the Reference [10], we put forward Theorem 3.5 and Theorem 3.6 as the generalization results.

Theorem 3.8 *Assume that* $\mathbb{T}$ *is a time scale,* $\xi, \sigma \in \mathbb{T}$ *with* $\xi < \sigma$ *and* $p_k > 0, \alpha_{kj} \in \mathbb{R}$ $(j = 1, 2, \dots, m, k = 1, 2, \dots, s)$, $\sum_{k=1}^{s}\frac{1}{p_k} = 1, \sum_{k=1}^{s}\alpha_{kj} = 0, f_j, h : \mathbb{T} \to \mathbb{R}$. *If h and* f_j *are* $\Diamond$*-integrable on* $[\xi, \sigma]_{\mathbb{T}}$, *then the following assertions hold true.*

(1) *For* $p_k > 1$, *one has*

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|\prod_{j=1}^{m} f_j(\delta_1,\delta_2,\delta_3)\right|\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3$$
$$\leq \prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\prod_{j=1}^{m}\left|f_j(\delta_1,\delta_2,\delta_3)\right|^{1+p_k\alpha_{kj}}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p_k}. \tag{22}$$

(2) *For* $0 < p_s < 1, p_k < 0$ $(k = 1, 2, \dots, s-1)$, $f_j^{1+p_k\alpha_{kj}}$ *is* $\Diamond$*-integrable on* $[\xi, \sigma]_{\mathbb{T}}$, *one has*

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|\prod_{j=1}^{m} f_j(\delta_1,\delta_2,\delta_3)\right|\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3$$
$$\geq \prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\prod_{j=1}^{m}\left|f_j(\delta_1,\delta_2,\delta_3)\right|^{1+p_k\alpha_{kj}}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p_k}. \tag{23}$$

Proof (1) Let

$$g_k(\delta_1,\delta_2,\delta_3) = \left(\prod_{j=1}^{m} f_j^{1+p_k\alpha_{kj}(\delta_1,\delta_2,\delta_3)}\right)^{1/p_k}.$$

Based on the assumptions $\sum_{k=1}^{s}\frac{1}{p_k} = 1$ and $\sum_{k=1}^{s}\alpha_{kj} = 0$, from a direct computation, it is obvious to show that

$$\prod_{k=1}^{s} g_k(\delta_1,\delta_2,\delta_3)$$
$$= g_1 g_2 \cdots g_s$$
$$= \left(\prod_{j=1}^{m} f_j^{1+a_1\alpha_{1j}}(\delta_1,\delta_2,\delta_3)\right)^{1/a_1}\left(\prod_{j=1}^{m} f_j^{1+a_2\alpha_{2j}}(\delta_1,\delta_2,\delta_3)\right)^{1/a_2}\cdots$$

$$\times \left(\prod_{j=1}^{m} f_j^{1+a_s \alpha_{sj}}(\delta_1, \delta_2, \delta_3) \right)^{1/a_s}$$
$$= \prod_{j=1}^{m} f_j^{1/a_1+\alpha_{1j}}(\delta_1, \delta_2, \delta_3) \prod_{j=1}^{m} f_j^{1/a_2+\alpha_{2j}}(\delta_1, \delta_2, \delta_3) \cdots$$
$$\times \prod_{j=1}^{m} f_j^{1/a_s+\alpha_{sj}}(\delta_1, \delta_2, \delta_3)$$
$$= \prod_{j=1}^{m} f_j^{1/a_1+1/a_2+\cdots+1/a_s+\alpha_{1j}+\alpha_{2j}+\cdots+\alpha_{sj}}(\delta_1, \delta_2, \delta_3) = \prod_{j=1}^{m} f_j(\delta_1, \delta_2, \delta_3).$$

From the above result, we can obtain

$$\prod_{k=1}^{s} g_k(\delta_1, \delta_2, \delta_3) = \prod_{j=1}^{m} f_j(\delta_1, \delta_2, \delta_3).$$

Hence, we have

$$\int_{\xi_1}^{\sigma_1} \int_{\xi_2}^{\sigma_2} \int_{\xi_3}^{\sigma_3} \left|h(\delta_1, \delta_2, \delta_3)\right| \left| \prod_{j=1}^{m} f_j(\delta_1, \delta_2, \delta_3) \right| \Diamond\delta_1 \Diamond\delta_2 \Diamond\delta_3$$
$$= \int_{\xi_1}^{\sigma_1} \int_{\xi_2}^{\sigma_2} \int_{\xi_3}^{\sigma_3} \left|h(\delta_1, \delta_2, \delta_3)\right| \left| \prod_{j=1}^{m} g_k(\delta_1, \delta_2, \delta_3) \right| \Diamond\delta_1 \Diamond\delta_2 \Diamond\delta_3.$$

It follows from Hölder's inequality (5) that

$$\int_{\xi_1}^{\sigma_1} \int_{\xi_2}^{\sigma_2} \int_{\xi_3}^{\sigma_3} \left|h(\delta_1, \delta_2, \delta_3)\right| \left| \prod_{j=1}^{m} g_k(\delta_1, \delta_2, \delta_3) \right| \Diamond\delta_1 \Diamond\delta_2 \Diamond\delta_3$$
$$\leq \prod_{k=1}^{s} \left(\int_{\xi_1}^{\sigma_1} \int_{\xi_2}^{\sigma_2} \int_{\xi_3}^{\sigma_3} \left|h(\delta_1, \delta_2, \delta_3)\right| \left|g_k(\delta_1, \delta_2, \delta_3)\right|^{p_k} \Diamond\delta_1 \Diamond\delta_2 \Diamond\delta_3 \right)^{1/p_k}.$$

Thus, we have

$$\int_{\xi_1}^{\sigma_1} \int_{\xi_2}^{\sigma_2} \int_{\xi_3}^{\sigma_3} \left|h(\delta_1, \delta_2, \delta_3)\right| \left| \prod_{j=1}^{m} f_j(\delta_1, \delta_2, \delta_3) \right| \Diamond\delta_1 \Diamond\delta_2 \Diamond\delta_3$$
$$\leq \prod_{k=1}^{s} \left(\int_{\xi_1}^{\sigma_1} \int_{\xi_2}^{\sigma_2} \int_{\xi_3}^{\sigma_3} \left|h(\delta_1, \delta_2, \delta_3)\right| \prod_{j=1}^{m} \left|f_j(\delta_1, \delta_2, \delta_3)\right|^{1+p_k \alpha_{kj}} \Diamond\delta_1 \Diamond\delta_2 \Diamond\delta_3 \right)^{1/p_k}.$$

(2) The proof of inequality (23) is similar to the proof of inequality (22), we have

$$\int_{\xi_1}^{\sigma_1} \int_{\xi_2}^{\sigma_2} \int_{\xi_3}^{\sigma_3} \left|h(\delta_1, \delta_2, \delta_3)\right| \left| \prod_{j=1}^{m} g_k(\delta_1, \delta_2, \delta_3) \right| \Diamond\delta_1 \Diamond\delta_2 \Diamond\delta_3$$
$$\geq \prod_{k=1}^{s} \left(\int_{\xi_1}^{\sigma_1} \int_{\xi_2}^{\sigma_2} \int_{\xi_3}^{\sigma_3} \left|h(\delta_1, \delta_2, \delta_3)\right| \left|g_k(\delta_1, \delta_2, \delta_3)\right|^{p_k} \Diamond\delta_1 \Diamond\delta_2 \Diamond\delta_3 \right)^{1/p_k}.$$

Thus, we have

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\delta_3)\right|\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3$$
$$\geq\prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\prod_{j=1}^{m}\left|f_j(\delta_1,\delta_2,\delta_3)\right|^{1+p_k\alpha_{kj}}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p_k}.$$

Thus, the proof of Theorem 3.8 is completed. □

Corollary 3.9 *Under the assumptions of Theorem 3.8, taking $s=m,\alpha_{kj}=-t/p_k$ for $j\neq k$ and $\alpha_{kk}=t(1-1/p_k)$ with $t\in\mathbb{R}$, the following assertions hold true.*

(1) *For $p_k>1$, one has*

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\delta_3)\right|\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3$$
$$\leq\prod_{k=1}^{m}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left(\prod_{j=1}^{m}\left|f_j(\delta_1,\delta_2,\delta_3)\right|\right)^{1-t}\right.$$
$$\left.\times\left(\left|f_k(\delta_1,\delta_2,\delta_3)\right|^{p_k}\right)^{t}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p_k}.$$

(2) *For $0<p_m<1,p_k<0$ $(k=1,2,\ldots,m-1)$, one has*

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\delta_3)\right|\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3$$
$$\geq\prod_{k=1}^{m}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left(\prod_{j=1}^{m}\left|f_j(\delta_1,\delta_2,\delta_3)\right|\right)^{1-t}\right.$$
$$\left.\times\left(\left|f_k(\delta_1,\delta_2,\delta_3)\right|^{p_k}\right)^{t}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p_k}.$$

Theorem 3.10 *Assume that $\mathbb{T}$ is a time scale, $\xi,\sigma\in\mathbb{T}$ with $\xi<\sigma$ and $p_k>0,\alpha_{kj}\in\mathbb{R}$ $(j=1,2,\ldots,m,k=1,2,\ldots,s)$, $\sum_{k=1}^{s}\frac{1}{p_k}=1$, $\sum_{k=1}^{s}\alpha_{kj}=0$, $f_j,h:\mathbb{T}\to\mathbb{R}$. If h and f_j are $\Diamond$-integrable on $[\xi,\sigma]_{\mathbb{T}}$, then the following assertions hold true.*

(1) *For $p_k>1$, one has*

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\left|h(\delta_1,\delta_2,\ldots,\delta_n)\right|\left|\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\ldots,\delta_n)\right|\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n$$
$$\leq\prod_{k=1}^{s}\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\left|h(\delta_1,\delta_2,\ldots,\delta_n)\right|\right.$$
$$\left.\times\prod_{j=1}^{m}\left|f_j(\delta_1,\delta_2,\ldots,\delta_n)\right|^{1+p_k\alpha_{kj}}\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\right)^{1/p_k}. \tag{24}$$

(2) *For* $0 < p_s < 1, p_k < 0$ $(k = 1, 2, \ldots, s-1)$, $f_j^{1+p_k\alpha_{kj}}$ *is* $\diamondsuit$*-integrable on* $[\xi, \sigma]_{\mathbb{T}}$, *one has*

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\bigl|h(\delta_1,\delta_2,\ldots,\delta_n)\bigr|\left|\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\ldots,\delta_n)\right|\diamondsuit\delta_1\diamondsuit\delta_2\cdots\diamondsuit\delta_n\\
&\quad\geq\prod_{k=1}^{s}\biggl(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\bigl|h(\delta_1,\delta_2,\ldots,\delta_n)\bigr|\\
&\qquad\times\prod_{j=1}^{m}\bigl|f_j(\delta_1,\delta_2,\ldots,\delta_n)\bigr|^{1+p_k\alpha_{kj}}\diamondsuit\delta_1\diamondsuit\delta_2\cdots\diamondsuit\delta_n\biggr)^{1/p_k}.
\end{aligned}\tag{25}$$

Proof Similar to the proof of Theorem 3.8, we get the result of Theorem 3.10. □

Remark 3.11 The inequalities in Theorem 3.8 and Theorem 3.10 are the result of generalization of Theorem 4.1 in Ref. [17].

Theorem 3.12 *Assume that* $\mathbb{T}$ *is a time scale*, $\xi, \sigma \in \mathbb{T}$ *with* $\xi < \sigma$ *and* $p_k > 0, r \in \mathbb{R}, \alpha_{kj} \in \mathbb{R}$ $(j = 1, 2, \ldots, m, k = 1, 2, \ldots, s)$, $\sum_{k=1}^{s}\frac{1}{p_k} = r$, $\sum_{k=1}^{s}\alpha_{kj} = 0$, $f_j, h : \mathbb{T} \to \mathbb{R}$. *If* f_j *and* h *are* $\diamondsuit$*-integrable on* $[\xi, \sigma]_T$, *then the following assertions hold true.*

(1) *For* $rp_k > 1$, *one has*

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\left|\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\delta_3)\right|\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\\
&\quad\leq\prod_{k=1}^{s}\biggl(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\prod_{j=1}^{m}\bigl|f_j(\delta_1,\delta_2,\delta_3)\bigr|^{1+rp_k\alpha_{kj}}\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\biggr)^{1/rp_k}.
\end{aligned}\tag{26}$$

(2) *For* $0 < rp_k < 1, rp_k < 0$ $(k = 1, 2, \ldots, s-1)$, $f_j^{1+rp_k\alpha_{kj}}$ *is* $\diamondsuit$*-integrable on* $[\xi, \sigma]_{\mathbb{T}}$, *one has*

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\left|\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\delta_3)\right|\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\\
&\quad\geq\prod_{k=1}^{s}\biggl(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\prod_{j=1}^{m}\bigl|f_j(\delta_1,\delta_2,\delta_3)\bigr|^{1+rp_k\alpha_{kj}}\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\biggr)^{1/rp_k}.
\end{aligned}\tag{27}$$

Proof (1) Since $rp_k > 1$ and $\sum_{k=1}^{s}\frac{1}{rp_k} = 1$. Then by inequality (22) we can obtain inequality (26).

(2) Since $0 < rp_s < 1, rp_k < 0$ and $\sum_{k=1}^{s}\frac{1}{rp_k} = 1$, by inequality (23), we can obtain inequality (27).

The proof of Theorem 3.12 is completed. □

Theorem 3.13 *Assume that* $\mathbb{T}$ *is a time scale*, $\xi, \sigma \in \mathbb{T}$ *with* $\xi < \sigma$ *and* $p_k > 0, r \in \mathbb{R}, \alpha_{kj} \in \mathbb{R}$ $(j = 1, 2, \ldots, m, k = 1, 2, \ldots, s)$, $\sum_{k=1}^{s}\frac{1}{p_k} = r$, $\sum_{k=1}^{s}\alpha_{kj} = 0$, $f_j, h : \mathbb{T} \to \mathbb{R}$. *If* f_j *and* h *are* $\diamondsuit$*-integrable on* $[\xi, \sigma]_T$, *then the following assertions hold true.*

(1) *For* $rp_k > 1$, *one has*

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\left|h(\delta_1,\delta_2,\ldots,\delta_n)\right|\left|\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\ldots,\delta_n)\right|\diamondsuit\delta_1\diamondsuit\delta_2\cdots\diamondsuit\delta_n$$
$$\leq\prod_{k=1}^{s}\Bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\left|h(\delta_1,\delta_2,\ldots,\delta_n)\right|$$
$$\times\prod_{j=1}^{m}\left|f_j(\delta_1,\delta_2,\ldots,\delta_n)\right|^{1+rp_k\alpha_{kj}}\diamondsuit\delta_1\diamondsuit\delta_2\cdots\diamondsuit\delta_n\Bigg)^{1/rp_k}. \tag{28}$$

(2) *For* $0 < rp_k < 1, rp_k < 0\ (k = 1,2,\ldots,s-1), f_j^{1+rp_k\alpha_{kj}}$ *is* $\diamondsuit$*-integrable on* $[\xi,\sigma]_{\mathbb{T}}$, *one has*

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\left|h(\delta_1,\delta_2,\ldots,\delta_n)\right|\left|\prod_{j=1}^{m}f_j(\delta_1,\delta_2,\ldots,\delta_n)\right|\diamondsuit\delta_1\diamondsuit\delta_2\cdots\diamondsuit\delta_n$$
$$\geq\prod_{k=1}^{s}\Bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\left|h(\delta_1,\delta_2,\ldots,\delta_n)\right|$$
$$\times\prod_{j=1}^{m}\left|f_j(\delta_1,\delta_2,\ldots,\delta_n)\right|^{1+rp_k\alpha_{kj}}\diamondsuit\delta_1\diamondsuit\delta_2\cdots\diamondsuit\delta_n\Bigg)^{1/rp_k}. \tag{29}$$

Proof Similar to the proof of Theorem 3.12, we get the result of Theorem 3.13. □

Theorem 3.14 *Let* $f,g,h:\mathbb{T}\to\mathbb{R}$ *be* $\diamondsuit$*-integrable on* $[\xi,\sigma]_T$, *and* $s,t\in\mathbb{R}$, *and let* $p = (s-t)/(1-t), q = (s-t)/(s-1)$.

(1) *If* $s < 1 < t$ *or* $s > 1 > t$, *then*

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)g(\delta_1,\delta_2,\delta_3)\right|\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3$$
$$\leq\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^{sp}\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\right)^{1/p^2}$$
$$\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^{tq}\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\right)^{1/q^2}$$
$$\times\Bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^{tp}\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3$$
$$\times\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^{sq}\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\Bigg)^{1/pq}. \tag{30}$$

(2) *If* $s > t > 1$ *or* $s < t < 1$; $t > s > 1$ *or* $t < s < 1$, *then*

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)g(\delta_1,\delta_2,\delta_3)\right|\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3$$
$$\geq\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^{sp}\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\right)^{1/p^2}$$

$$\times \left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^{tq}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/q^2}$$
$$\times \left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^{tp}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right.$$
$$\left.\times \int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^{sq}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/pq}. \tag{31}$$

Proof (1) Let $p = \frac{s-t}{1-t}$ and in view of $s < 1 < t$ or $s > 1 > t$, we have

$$p = \frac{s-t}{1-t} > 1,$$

by Hölder's inequality (5) with indices $\frac{s-t}{1-t}$ and $\frac{s-t}{s-1}$, we have

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\\
&\quad=\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|^{s(1-t)/(s-t)}|fg|^{t(s-1)/(s-t)}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\\
&\quad\le\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|^{s}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(1-t)/(s-t)}\\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|^{t}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(s-1)/(s-t)}.
\end{aligned}$$

On the other hand, from Hölder's inequality (5) again for $p = \frac{s-t}{1-t} > 1$, it follows that the following two inequalities are true:

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|^{s}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\\
&\quad\le\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||f|^{s(s-t)/(1-t)}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(1-t)/(s-t)}\\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||g|^{s(s-t)/(s-1)}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(s-1)/(s-t)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|^{t}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\\
&\quad\le\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||f|^{t(s-t)/(1-t)}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(1-t)/(s-t)}\\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||g|^{t(s-t)/(s-1)}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(s-1)/(s-t)}.
\end{aligned}$$

Thus, we have

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)g(\delta_1,\delta_2,\delta_3)\right|\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3$$

$$
\begin{aligned}
&\leq \left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^{sp}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p^2} \\
&\quad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^{tq}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/q^2} \\
&\quad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^{tp}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right. \\
&\quad\left.\times\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^{sq}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/pq}.
\end{aligned}
$$

(2) Let $p = \frac{s-t}{1-t}$ and in view of $s > t > 1$ or $s < t < 1$, we have

$$
p = \frac{s-t}{1-t} < 0
$$

and $t > s > 1$ or $t < s < 1$, we have $0 < \frac{s-t}{1-t} < 1$, by the reverse Hölder inequality (6) with indices $\frac{s-t}{1-t}$ and $\frac{s-t}{s-1}$, we have

$$
\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3 \\
&\quad=\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|^{s(1-t)/(s-t)}|fg|^{t(s-1)/(s-t)}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3 \\
&\quad\geq\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|^{s}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(1-t)/(s-t)} \\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|^{t}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(s-1)/(s-t)}.
\end{aligned}
$$

On the other hand, from Hölder's inequality (6) again for $0 < p = \frac{s-t}{1-t} < 1$ or $p = \frac{s-t}{1-t} < 0$, it follows that the following two inequalities are true:

$$
\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|^{s}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3 \\
&\quad\geq\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||f|^{s(s-t)/(1-t)}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(1-t)/(s-t)} \\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||g|^{s(s-t)/(s-1)}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(s-1)/(s-t)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||fg|^{t}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3 \\
&\quad\geq\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||f|^{t(s-t)/(1-t)}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(1-t)/(s-t)} \\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h||g|^{t(s-t)/(s-1)}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(s-1)/(s-t)}.
\end{aligned}
$$

Thus, we have

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)g(\delta_1,\delta_2,\delta_3)\big|\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\\
&\quad\geq\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)\big|^{sp}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p^2}\\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|g(\delta_1,\delta_2,\delta_3)\big|^{tq}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/q^2}\\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)\big|^{tp}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right.\\
&\qquad\times\left.\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|g(\delta_1,\delta_2,\delta_3)\big|^{sq}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/pq}.
\end{aligned}$$

Thus, the proof of Theorem 3.14 is completed. □

Theorem 3.15 *Let $f,g,h:\mathbb{T}\to\mathbb{R}$ be $\Diamond$-integrable on $[\xi,\sigma]_T$, and $s,t\in\mathbb{R}$, and let $p=(s-t)/(1-t)$, $q=(s-t)/(s-1)$.*

(1) *If $s<1<t$ or $s>1>t$, then*

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|f(\delta_1,\delta_2,\ldots,\delta_n)g(\delta_1,\delta_2,\ldots,\delta_n)\big|\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\\
&\quad\leq\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|f(\delta_1,\delta_2,\ldots,\delta_n)\big|^{sp}\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\right)^{1/p^2}\\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|g(\delta_1,\delta_2,\ldots,\delta_n)\big|^{tq}\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\right)^{1/q^2}\\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|f(\delta_1,\delta_2,\ldots,\delta_n)\big|^{tp}\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\right.\\
&\qquad\times\left.\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|g(\delta_1,\delta_2,\ldots,\delta_n)\big|^{sq}\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\right)^{1/pq}.
\end{aligned}\tag{32}$$

(2) *If $s>t>1$ or $s<t<1$; $t>s>1$ or $t<s<1$, then*

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|f(\delta_1,\delta_2,\ldots,\delta_n)g(\delta_1,\delta_2,\ldots,\delta_n)\big|\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\\
&\quad\geq\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|f(\delta_1,\delta_2,\ldots,\delta_n)\big|^{sp}\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\right)^{1/p^2}\\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|g(\delta_1,\delta_2,\ldots,\delta_n)\big|^{tq}\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\right)^{1/q^2}\\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|f(\delta_1,\delta_2,\ldots,\delta_n)\big|^{tp}\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\right.\\
&\qquad\times\left.\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|g(\delta_1,\delta_2,\ldots,\delta_n)\big|^{sq}\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\right)^{1/pq}.
\end{aligned}\tag{33}$$

Proof Similar to the proof of Theorem 3.14, we get the result of Theorem 3.15. □

Remark 3.16 For the inequality of Theorem 5.1 in the Reference [17], we put forward Theorem 3.14 and Theorem 3.15 as the generalization results.

4 Main results about diamond-alpha integral Minkowski's inequality

Next, we give some generalizations of diamond-α integral Minkowski's inequality in the following theorems.

Theorem 4.1 *Let $f,g,h:\mathbb{T}\to\mathbb{R}$ be $\Diamond$ – integrable on $[\xi,\sigma]_{\mathbb{T}}, p>0, s,t\in\mathbb{R}$, and $s\neq t$.*

(1) *Let $p,s,t\in\mathbb{R}$ be different such that $s,t>1$ and $(s-t)/(p-t)>1$. Then*

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\bigr|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\\
&\leq\Biggl[\biggl(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl|f(\delta_1,\delta_2,\delta_3)\bigr|^s\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\biggr)^{\frac{1}{s}}\\
&\quad+\biggl(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl|g(\delta_1,\delta_2,\delta_3)\bigr|^s\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\biggr)^{\frac{1}{s}}\Biggr]^{s(p-t)/(s-t)}\\
&\quad\times\Biggl[\biggl(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl|f(\delta_1,\delta_2,\delta_3)\bigr|^t\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\biggr)^{\frac{1}{t}}\\
&\quad+\biggl(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl|g(\delta_1,\delta_2,\delta_3)\bigr|^t\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\biggr)^{\frac{1}{t}}\Biggr]^{t(s-p)/(s-t)}.
\end{aligned}\tag{34}$$

(2) *Let $p,s,t\in\mathbb{R}$ be different such that $0<s<1, 0<t<1$ and $(s-t)/(p-t)<1$. Then*

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\bigr|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\\
&\geq\Biggl[\biggl(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl|f(\delta_1,\delta_2,\delta_3)\bigr|^s\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\biggr)^{\frac{1}{s}}\\
&\quad+\biggl(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl|g(\delta_1,\delta_2,\delta_3)\bigr|^s\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\biggr)^{\frac{1}{s}}\Biggr]^{s(p-t)/(s-t)}\\
&\quad\times\Biggl[\biggl(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl|f(\delta_1,\delta_2,\delta_3)\bigr|^t\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\biggr)^{\frac{1}{t}}\\
&\quad+\biggl(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl|g(\delta_1,\delta_2,\delta_3)\bigr|^t\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\biggr)^{\frac{1}{t}}\Biggr]^{t(s-p)/(s-t)}.
\end{aligned}\tag{35}$$

Proof (1) We have $(s-t)/(p-t)>1$, and

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\bigr|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\\
&=\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\bigl|h(\delta_1,\delta_2,\delta_3)\bigr|\bigl(\bigl|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\bigr|^s\bigr)^{(p-t)/(s-t)}\\
&\quad\times\bigl(\bigl|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\bigr|^t\bigr)^{(s-p)/(s-t)}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3,
\end{aligned}$$

by using Hölder's inequality (5) with indices $(s-t)/(p-t)$ and $(s-t)/(s-p)$, we have

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\\
&\quad\leq\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^s\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(p-t)/(s-t)}\\
&\qquad\times\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^t\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{(s-p)/(s-t)}.
\end{aligned}$$

On the other hand, by using Minkowski's inequality (6) for $s>1$ and $t>1$, respectively, we can see that the following assertions hold true:

$$\begin{aligned}
&\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^s\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{\frac{1}{s}}\\
&\quad\leq\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^s\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{\frac{1}{s}}\\
&\qquad+\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^s\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{\frac{1}{s}}
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^t\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{\frac{1}{t}}\\
&\quad\leq\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^t\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{\frac{1}{t}}\\
&\qquad+\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^t\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{\frac{1}{t}}.
\end{aligned}$$

So we get the result

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\\
&\quad\leq\Bigg[\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^s\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{\frac{1}{s}}\\
&\qquad+\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^s\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{\frac{1}{s}}\Bigg]^{s(p-t)/(s-t)}\\
&\qquad\times\Bigg[\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^t\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{\frac{1}{t}}\\
&\qquad+\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^t\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{\frac{1}{t}}\Bigg]^{t(s-p)/(s-t)}.
\end{aligned}$$

(2) We have $(s-t)/(p-t) < 1$ and

$$\begin{aligned}&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^p\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\\&\quad=\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big(\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^s\big)^{(p-t)/(s-t)}\\&\qquad\times\big(\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^t\big)^{(s-p)/(s-t)}\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3,\end{aligned}$$

by using Hölder's inequality (7) with indices $(s-t)/(p-t)$ and $(s-t)/(s-p)$, we have

$$\begin{aligned}&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^p\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\\&\quad\geq\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^s\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\bigg)^{(p-t)/(s-t)}\\&\qquad\times\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^t\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\bigg)^{(s-p)/(s-t)}.\end{aligned}$$

On the other hand, by using Minkowski's inequality (8) for $s > 1$ and $t > 1$, respectively, we can see that the following assertions hold true

$$\begin{aligned}&\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^s\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\bigg)^{\frac{1}{s}}\\&\quad\geq\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)\big|^s\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\bigg)^{\frac{1}{s}}\\&\qquad+\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|g(\delta_1,\delta_2,\delta_3)\big|^s\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\bigg)^{\frac{1}{s}}\end{aligned}$$

and

$$\begin{aligned}&\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^t\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\bigg)^{\frac{1}{t}}\\&\quad\geq\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)\big|^t\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\bigg)^{\frac{1}{t}}\\&\qquad+\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|g(\delta_1,\delta_2,\delta_3)\big|^t\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\bigg)^{\frac{1}{t}}.\end{aligned}$$

So we get the result

$$\begin{aligned}&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^p\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\\&\quad\geq\bigg[\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)\big|^s\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\bigg)^{\frac{1}{s}}\\&\qquad+\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|g(\delta_1,\delta_2,\delta_3)\big|^s\lozenge\delta_1\lozenge\delta_2\lozenge\delta_3\bigg)^{\frac{1}{s}}\bigg]^{s(p-t)/(s-t)}\end{aligned}$$

$$\times\Bigg[\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)\big|^t\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\bigg)^{\frac{1}{t}}$$

$$+\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|g(\delta_1,\delta_2,\delta_3)\big|^t\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\bigg)^{\frac{1}{t}}\Bigg]^{t(s-p)/(s-t)}. \qquad \square$$

Remark 4.2 (1) Under the conditions of Theorem 4.1, for $p > 1$, letting $s = p + \varepsilon, t = p - \varepsilon$, when p, s, t are different, $s, t > 1$, and letting $\varepsilon \to 0$, we obtain

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^p\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3$$

$$\leq\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)\big|^p\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\bigg)^{\frac{1}{p}}$$

$$+\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|g(\delta_1,\delta_2,\delta_3)\big|^p\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\bigg)^{\frac{1}{p}}.$$

(2) Under the conditions of Theorem 4.1, for $0 < p < 1$, letting $s = p + \varepsilon, t = p - \varepsilon$, when p, s, t are different, $0 < s, t < 1$, and letting $\varepsilon \to 0$, we obtain

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^p\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3$$

$$\geq\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)\big|^p\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\bigg)^{\frac{1}{p}}$$

$$+\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big|\big|g(\delta_1,\delta_2,\delta_3)\big|^p\diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3\bigg)^{\frac{1}{p}}.$$

Now, on the basis of Theorem 4.1, we give the n-tuple diamond-α integral Minkowski's inequality on time scales.

Theorem 4.3 *Let $f, g, h : \mathbb{T} \to \mathbb{R}$ be $\diamondsuit$-integrable on $[\xi, \sigma]_{\mathbb{T}}, p > 0, s, t \in \mathbb{R}$, and $s \neq t$.*

(1) *Let $p, s, t \in \mathbb{R}$ be different such that $s, t > 1$ and $(s - t)/(p - t) > 1$. Then*

$$\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|f(\delta_1,\delta_2,\ldots,\delta_n)+g(\delta_1,\delta_2,\ldots,\delta_n)\big|^p\diamondsuit\delta_1\diamondsuit\delta_2\cdots\diamondsuit\delta_n$$

$$\leq\Bigg[\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|f(\delta_1,\delta_2,\ldots,\delta_n)\big|^s\diamondsuit\delta_1\diamondsuit\delta_2\cdots\diamondsuit\delta_n\bigg)^{\frac{1}{s}}$$

$$+\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|$$

$$\times\big|g(\delta_1,\delta_2,\ldots,\delta_n)\big|^s\diamondsuit\delta_1\diamondsuit\delta_2\cdots\diamondsuit\delta_n\bigg)^{\frac{1}{s}}\Bigg]^{s(p-t)/(s-t)}$$

$$\times\Bigg[\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|\big|f(\delta_1,\delta_2,\ldots,\delta_n)\big|^t\diamondsuit\delta_1\diamondsuit\delta_2\cdots\diamondsuit\delta_n\bigg)^{\frac{1}{t}}$$

$$+\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\big|h(\delta_1,\delta_2,\ldots,\delta_n)\big|$$

$$\times \left|g(\delta_1,\delta_2,\ldots,\delta_n)\right|^t \Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\Bigg)^{\frac{1}{t}}\Bigg]^{t(s-p)/(s-t)}. \tag{36}$$

(2) *Let* $p, s, t \in \mathbb{R}$ *be different such that* $0 < s < 1, 0 < t < 1$ *and* $(s-t)/(p-t) < 1$. *Then*

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\left|h(\delta_1,\delta_2,\ldots,\delta_n)\right|\left|f(\delta_1,\delta_2,\ldots,\delta_n)+g(\delta_1,\delta_2,\ldots,\delta_n)\right|^p\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\\
&\geq\Bigg[\Bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\left|h(\delta_1,\delta_2,\ldots,\delta_n)\right|\left|f(\delta_1,\delta_2,\ldots,\delta_n)\right|^s\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\Bigg)^{\frac{1}{s}}\\
&\quad+\Bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\left|h(\delta_1,\delta_2,\ldots,\delta_n)\right|\\
&\quad\times\left|g(\delta_1,\delta_2,\ldots,\delta_n)\right|^s\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\Bigg)^{\frac{1}{s}}\Bigg]^{s(p-t)/(s-t)}\\
&\quad\times\Bigg[\Bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\left|h(\delta_1,\delta_2,\ldots,\delta_n)\right|\left|f(\delta_1,\delta_2,\ldots,\delta_n)\right|^t\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\Bigg)^{\frac{1}{t}}\\
&\quad+\Bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}\left|h(\delta_1,\delta_2,\ldots,\delta_n)\right|\\
&\quad\times\left|g(\delta_1,\delta_2,\ldots,\delta_n)\right|^t\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n\Bigg)^{\frac{1}{t}}\Bigg]^{t(s-p)/(s-t)}.
\end{aligned} \tag{37}$$

Proof Similar to the proof of Theorem 4.1, we get the result of Theorem 4.3. □

Remark 4.4 Aiming at the diamond-α integral Minkowski's inequality proposed by Theorem 3.5 in Ref. [17], we generalize it in this paper and obtain the three-tuple and n-tuple diamond-α inequalities (34)–(37).

Theorem 4.5 *Let* $f, g, h : \mathbb{T} \to \mathbb{R}$ *and* $0 < r < 1 < p$. *If* f, g *and* h *are* $\Diamond$*-integrable on* $[\xi,\sigma]_{\mathbb{T}}$, *then*

$$\begin{aligned}
&\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)|^r\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}\right)^{1/(p-r)}\\
&\quad\leq\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^r\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}\right)^{1/(p-r)}\\
&\qquad+\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^r\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}\right)^{1/(p-r)}.
\end{aligned} \tag{38}$$

Proof From inequality (5) and inequality (6), we have

$$\begin{aligned}
&\Bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\Bigg)^{1/(p-r)}\\
&\leq\Bigg(\Bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\Bigg)^{1/p}\\
&\quad+\Bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\Bigg)^{1/p}\Bigg)^{p/(p-r)}
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg(\bigg(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^p \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3} \bigg)^{1/p} \\
&\quad \times \bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)\big|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3 \bigg)^{1/p} \\
&\quad + \bigg(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^p \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3} \bigg)^{1/p} \\
&\quad \times \bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \big|h(\delta_1,\delta_2,\delta_3)\big|\big|g(\delta_1,\delta_2,\delta_3)\big|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3 \bigg)^{1/p} \Bigg)^{p/(p-r)} \\
&\le \Bigg(\bigg(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^p \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3} \bigg)^{1/(p-r)} \\
&\quad + \bigg(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^p \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3} \bigg)^{1/(p-r)} \Bigg) \\
&\quad \times \Bigg(\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)\big|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3 \bigg)^{1/r} \\
&\quad + \bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \big|h(\delta_1,\delta_2,\delta_3)\big|\big|g(\delta_1,\delta_2,\delta_3)\big|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3 \bigg)^{1/r} \Bigg)^{r/(p-r)}.
\end{aligned}
$$

From inequality (6), we get

$$
\begin{aligned}
&\Bigg(\bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)\big|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3 \bigg)^{1/r} \\
&\quad + \bigg(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \big|h(\delta_1,\delta_2,\delta_3)\big|\big|g(\delta_1,\delta_2,\delta_3)\big|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3 \bigg)^{1/r} \Bigg)^{r} \\
&\le \int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \big|h(\delta_1,\delta_2,\delta_3)\big|\big|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\big|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\bigg(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)|^p \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3} \bigg)^{1/(p-r)} \\
&\le \bigg(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^p \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3} \bigg)^{1/(p-r)} \\
&\quad + \bigg(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^p \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^r \diamondsuit\delta_1\diamondsuit\delta_2\diamondsuit\delta_3} \bigg)^{1/(p-r)}.
\end{aligned}
$$

The proof of Theorem 4.5 is completed. □

Next, on the basis of Theorem 4.5, we give the n-tuple diamond-α integral Minkowski's inequality on time scales.

Theorem 4.6 *Let $f,g,h:\mathbb{T}\to\mathbb{R}$ and $0<r<1<p$. If f,g and h are $\Diamond$-integrable on $[\xi,\sigma]_{\mathbb{T}}$, then*

$$\begin{aligned}&\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||f(\delta_1,\delta_2,\ldots,\delta_n)+g(\delta_1,\delta_2,\ldots,\delta_n)|^p\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||f(\delta_1,\delta_2,\ldots,\delta_n)+g(\delta_1,\delta_2,\ldots,\delta_n)|^r\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n}\right)^{1/(p-r)}\\&\quad\le\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||f(\delta_1,\delta_2,\ldots,\delta_n)|^p\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||f(\delta_1,\delta_2,\ldots,\delta_n)|^r\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n}\right)^{1/(p-r)}\\&\qquad+\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||g(\delta_1,\delta_2,\ldots,\delta_n)|^p\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||g(\delta_1,\delta_2,\ldots,\delta_n)|^r\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n}\right)^{1/(p-r)}.\end{aligned}\tag{39}$$

Proof Similar to the proof of Theorem 4.5, we get the result of Theorem 4.6. □

Remark 4.7 The inequalities in Theorem 4.5 and Theorem 4.6 are generalized results for Theorem 3.6 in Ref. [17].

Theorem 4.8 *Let $f,g,h:\mathbb{T}\to\mathbb{R}$ and $p\le 0\le r$. If f,g,f^p,g^p and h are $\Diamond$-integrable on $[\xi,\sigma]_{\mathbb{T}}$, then*

$$\begin{aligned}&\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)|^r\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}\right)^{1/(p-r)}\\&\quad\ge\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^r\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}\right)^{1/(p-r)}\\&\qquad+\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^r\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}\right)^{1/(p-r)}.\end{aligned}\tag{40}$$

Proof Let $\alpha_1\ge 0,\alpha_2\ge 0,\beta_1>0$, and $\beta_2>0$, and $-1<\lambda<0$, using the following Radon inequality:

$$\sum_{k=1}^n\frac{a_k^p}{b_k^{p-1}}\le\frac{(\sum_{k=1}^n a_k)^p}{(\sum_{k=1}^n b_k)^{p-1}},\quad a_k\ge 0,b_k>0,0<p<1,$$

we have

$$\frac{\alpha_1^{\lambda+1}}{\beta_1^\lambda}+\frac{\alpha_2^{\lambda+1}}{\beta_2^\lambda}\le\frac{(\alpha_1+\alpha_2)^{\lambda+1}}{(\beta_1+\beta_2)^\lambda}.$$

Let

$$\alpha_1=\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big||f|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p},$$

$$\beta_1=\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big||f|^r\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/r},$$

$$\alpha_2=\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big||g|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p},$$

$$\beta_2=\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\big|h(\delta_1,\delta_2,\delta_3)\big||g|^r\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/r},$$

and let $\lambda = \frac{r}{p-r}$, it follows that

$$\begin{aligned}
&\frac{\alpha_1^{\lambda+1}}{\beta_1^{\lambda}} + \frac{\alpha_2^{\lambda+1}}{\beta_2^{\lambda}} \\
&\quad = \frac{(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f|^p \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3)^{(\lambda+1)/p}}{(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f|^r \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3)^{\lambda/r}} \\
&\qquad + \frac{(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||g|^p \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3)^{(\lambda+1)/p}}{(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||g|^r \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3)^{\lambda/r}} \\
&\quad = \left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f|^p \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||f|^r \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}\right)^{1/(p-r)} \\
&\qquad + \left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||g|^p \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} |h(\delta_1,\delta_2,\delta_3)||g|^r \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}\right)^{1/(p-r)} \\
&\quad \le \frac{(\alpha_1+\alpha_2)^{\lambda+1}}{(\beta_1+\beta_2)^{\lambda}} \\
&\quad = \left[\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \left|h(\delta_1,\delta_2,\delta_3)\right||f|^p \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p}\right. \\
&\qquad \left. + \left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \left|h(\delta_1,\delta_2,\delta_3)\right||g|^p \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p}\right]^{p/(p-r)} \\
&\qquad \Big/ \left[\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \left|h(\delta_1,\delta_2,\delta_3)\right||f|^r \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/r}\right. \\
&\qquad \left. + \left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \left|h(\delta_1,\delta_2,\delta_3)\right||g|^r \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/r}\right]^{r/(p-r)}.
\end{aligned}$$

Since $-1 < \lambda = \frac{r}{p-r} < 0$, we may assume $p < 0 < r$, and $0 < r \le 1$, we obtain

$$\begin{aligned}
&\left[\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \left|h(\delta_1,\delta_2,\delta_3)\right||f|^r \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/r}\right. \\
&\qquad \left. + \left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \left|h(\delta_1,\delta_2,\delta_3)\right||g|^r \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/r}\right]^{r} \\
&\quad \le \int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \left|h(\delta_1,\delta_2,\delta_3)\right||f+g|^r \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3.
\end{aligned}$$

For $p < 0$, we obtain

$$\sum_{k=1}^{n} m_k^p n_k^{1/p-1} \ge \left(\sum_{k=1}^{n} m_k\right)^p \left(\sum_{k=1}^{n} n_k^{1/p}\right)^{1-p}.$$

Assume that $f(\delta)$ and $g(\delta)$ are nonzero, let

$$\begin{aligned}
M &= \int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^p \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3, \\
N &= \int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3} \left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^p \Diamond\delta_1\Diamond\delta_2\Diamond\delta_3,
\end{aligned}$$

$$
\begin{aligned}
W = {} & \left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p} \\
& + \left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p} \\
= {} & M^{1/p}+N^{1/p}.
\end{aligned}
$$

From the above inequality, we have

$$
\begin{aligned}
W = {} & M^{1/p}+N^{1/p} \\
= {} & M^{1/p-1}\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3 \\
& + N^{1/p-1}\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3 \\
= {} & \int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\bigl(\left|f(\delta_1,\delta_2,\delta_3)\right|^p M^{1/p-1} \\
& + \left|g(\delta_1,\delta_2,\delta_3)\right|^p N^{1/p-1}\bigr)\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3 \\
\geq {} & \int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^p \\
& \times\bigl(M^{1/p}+N^{1/p}\bigr)^{1-p}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3 \\
= {} & \int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^p W^{1-p}\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3 \\
= {} & W^{1-p}\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3.
\end{aligned}
$$

That is,

$$
W \geq W^{1-p}\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3.
$$

Hence, we have

$$
W^p \geq \int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3.
$$

Based on the above inequality, we obtain

$$
\begin{aligned}
& \Biggl[\left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p} \\
& \quad + \left(\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|g(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3\right)^{1/p}\Biggr]^p \\
& \geq \int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}\left|h(\delta_1,\delta_2,\delta_3)\right|\left|f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)\right|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3.
\end{aligned}
$$

Through the above inequalities, we finally get the result

$$\begin{aligned}&\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)+g(\delta_1,\delta_2,\delta_3)|^r\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}\right)^{1/(p-r)}\\&\quad\le\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||f(\delta_1,\delta_2,\delta_3)|^r\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}\right)^{1/(p-r)}\\&\qquad+\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^p\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\int_{\xi_3}^{\sigma_3}|h(\delta_1,\delta_2,\delta_3)||g(\delta_1,\delta_2,\delta_3)|^r\Diamond\delta_1\Diamond\delta_2\Diamond\delta_3}\right)^{1/(p-r)}.\end{aligned}$$

Thus, Theorem 4.8 is completely proved. □

Theorem 4.9 *Let $f,g,h:\mathbb{T}\to\mathbb{R}$ and $0<r<1<p$. If f, g and h are $\Diamond$-integrable on $[\xi,\sigma]_{\mathbb{T}}$, then we have the following assertion*

$$\begin{aligned}&\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||f(\delta_1,\delta_2,\ldots,\delta_n)+g(x_1,x_2\cdots,x_n)|^p\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||f(\delta_1,\delta_2,\ldots,\delta_n)+g(\delta_1,\delta_2,\ldots,\delta_n)|^r\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n}\right)^{1/(p-r)}\\&\quad\le\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||f(\delta_1,\delta_2,\ldots,\delta_n)|^p\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||f(\delta_1,\delta_2,\ldots,\delta_n)|^r\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n}\right)^{1/(p-r)}\\&\qquad+\left(\frac{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||g(\delta_1,\delta_2,\ldots,\delta_n)|^p\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n}{\int_{\xi_1}^{\sigma_1}\int_{\xi_2}^{\sigma_2}\cdots\int_{\xi_n}^{\sigma_n}|h(\delta_1,\delta_2,\ldots,\delta_n)||g(\delta_1,\delta_2,\ldots,\delta_n)|^r\Diamond\delta_1\Diamond\delta_2\cdots\Diamond\delta_n}\right)^{1/(p-r)}.\end{aligned}\tag{41}$$

Proof Similar to the proof of Theorem 4.8, we get the result of Theorem 4.9. □

Remark 4.10 For the inequality of Theorem 3.7 in Ref. [17], we generalize it in this paper and obtain the generalized inequalities in Theorem 4.8 and Theorem 4.9.

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Authors' contributions
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