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Vulnerable options pricing under uncertain volatility model

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Abstract

In this paper, we consider the pricing problem of options with counterparty default risks. We study the asymptotic behavior of vulnerable option prices in the worst case scenario under an uncertain volatility model which contains both corporate assets and underlying assets. We propose a method to estimate the price of vulnerable options when the volatility of the underlying assets is within a small interval. By imposing additional conditions on the boundary condition and cutting the obtained Black–Scholes–Barenblatt equation into two Black–Scholes-like equations, we obtain an approximate method for solving the fully nonlinear partial differential equation satisfied by the price of vulnerable options under the uncertain volatility model.

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Keywords: Uncertain volatility model; Vulnerable option; Nonlinear Black–Scholes–Barenblatt partial differential equation; Stochastic control

1 Introduction

With the continuous opening and development of China's financial market in recent years, the domestic option market has made a major breakthrough in practical terms. With the footsteps of the 2015 New Year, the Shanghai Stock Exchange's 50 ETF options are registered with the Shanghai Stock Exchange. This means that the mainland China stock market has ushered in the "option era" since its establishment 24 years ago. After that, some other kinds of options will enter the financial market one after another, and the scale of the transaction will also expand, and then comes the over-the-counter (OTC for short) market. When an option is traded in an OTC market, the absence of a supervisory body such as a clearing house to supervise the short side of options obligations at maturity will result in the bearer of the option simultaneously bearing market and credit risk, which will inevitably lead to defaults in the transaction process. Up to now, the theoretical research on market credit default risk has been relatively mature, but the research on market pricing with default risk is relatively scarce, and the conditions taken into account are few, which can not describe the change of option price well. At present, there is not much literature on vulnerable options. Johnson et al. [1] first introduced default risk into option pricing and proposed the definition of vulnerable option. Hull et al. [2] not only gave the pricing formula of vulnerable options, but also compared the pricing methods of standard European options, American options, and European vulnerable options by numerical methods. Jarrow et al. [3] studied the pricing and hedging of derivatives with credit risk using no-arbitrage pricing method. Don [4] studied the pricing of European vulnerable options considering default time and uncertain recovery value. Klein [5] assumed that default occurred when the assets of the company with open options were lower than a fixed default boundary. The pricing formula of vulnerable options was obtained by martingale method. Ammann [6] extended Klein's model using structured method. Under the assumption of random interest rate and random default boundary, the explicit solution of vulnerable options was obtained. Further research on vulnerable options can be found in [7–11].

In the previous literature, most of the models assume that the volatility of underlying assets is constant, but it is not constant in the real world. However, it is widely believed that continued volatility does not explain the observed market price of an option. After the work of Black, Scholes, and Merton, some scholars studied the stochastic volatility option pricing model. In a series of papers, several stochastic volatility models were introduced, such as Hull–White stochastic volatility model [12] and Heston stochastic volatility model [13]. Compared with the simple model, the stochastic volatility model has more state variables, which makes it more difficult to give the analytic solution of option price. The uncertain volatility model just overcomes this shortcoming. Thus, we study the uncertain volatility model in this paper.

The uncertain volatility model was independently developed by Lyons [14] and Avellaneda et al. [15]. Under these circumstances, volatility is assumed to lie within a range of values. Therefore, the price obtained under no-arbitrage analysis is no longer unique. All that can be calculated are the best case scenario prices and the worst case scenario prices. We can see some related results about uncertain volatility in [14–18]. The option pricing under uncertain volatility model by nonlinear partial differential equation (PDE for short) has been studied in their papers. In Avellaneda et al. [15] and Pooley [19], some numerical methods have been proposed.

In 2014, Fouque [20] studied the worst case scenario prices of European derivatives under the uncertain volatility model. They provided an approximate method of derivative pricing based on wavelet fluctuation intervals. In addition, they also argue that when it comes to simple options with convex gains, the solution can be attributed to a constant volatility problem.

In this paper, the pricing problem of vulnerable options is studied. We assume that the volatility of underlying assets is in a small interval, and the volatility of the counterparty's asset value is deterministic. In the process of finding the estimation of the worst case scenario vulnerable option prices, the first difficulty that we encountered was to obtain the Hamilton–Jacobi–Bellman (HJB for short) equation of the prices. The HJB equation is called Black–Scholes–Barenblatt (BSB for short) equation in the financial mathematics. We can get the BSB equation by the stochastic control theory. The next problem is to prove the convergence of the estimation. That is to say, we have to control the error term. When analyzing the error term, we obtain its expectation by the Dynkin formula and find the conditions we should impose on the payoff function. Finally, we obtain the approximation procedure for the vulnerable option prices. Compared with Fouque's paper [20], our paper adds an equation about counterparty's assets in the stochastic control system, which can also be reflected in the BSB equation. Since the underlying asset has nothing to do with the value of the counterparty's assets, our analysis will be a little simpler.

This paper is organized as follows: in Sect. 2, the vulnerable options under the uncertain volatility model are briefly introduced, and the BSB equations of the option prices are given. In Sect. 3, we divide the estimation of vulnerable option prices which are in the worst case scenario into two Black–Scholes-like PDEs, thus, we obtain the estimation of vulnerable option prices. Then, the main results of this paper are given, and the rationality of the estimation is illustrated. In Sect. 4, we give the proof of the main results. By applying conditions to the payoff function, the convergence of the error term is obtained. Through the analysis of the error term, the expected form of the error term is obtained and decomposed into three parts. These three parts of control are given by the stochastic control theory and the character of the worst-case vulnerable option price process. Finally, the conclusions of this paper are given.

2 Pricing vulnerable options under uncertain volatility model

In this section, we introduce the vulnerable options under uncertain volatility model. Then we give the BSB equation for the price of the vulnerable option. Suppose that $\mathcal{X}$ is a vulnerable option with maturity T and payoff $\varphi(\cdot,\cdot)$. The results of this paper cover generalized vulnerable options.

Assumption 2.1 Let X denote the market value of the asset underlying the option. The dynamics of X are given by the following process:

$$dX_t = rX_t\,dt + \sigma_t X_t\,dW_t^x, \tag{2.1}$$

where r is the constant risk-free interest rate, W_t^x is a standard Brownian motion on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and the volatility process $\sigma_t \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]$ for each $t \in [0, T]$ is a family of progressively measurable and $[\underline{\sigma}, \overline{\sigma}]$-valued processes. According to the above definition, we know that the volatility in an uncertain volatility model is not a stochastic process with a probability distribution, but a family of stochastic processes with unknown prior information. Thus, what we can use to distinguish the difference between uncertain volatility model is the model ambiguity.

Assumption 2.2 Let Y denote the total value of the firm's assets. The dynamics of Y are given by the following process:

$$dY_t = \mu_2 Y_t\,dt + \sigma_2 Y_t\,dW_t^y, \tag{2.2}$$

where μ_2 represents the expected rate of return on the asset value of the counterparty company, σ_2 represents the volatility of the asset value, and both μ_2 and σ_2 are constants (σ_2 is constant because only insiders can understand the value of the counterparty's assets). W_t^y represents the standard Brownian motion. We have

$$\operatorname{Cov}(dW_t^x, dW_t^y) = \rho_{XY}\,dt.$$

For computational convenience, we suppose that

$$\rho_{XY} = 0. \tag{2.3}$$

Assumption 2.3 Markets are perfect and frictionless. There are no transaction costs or taxes and securities are continuously traded.

Suppose that the payoff function of the vulnerable option is given by $\varphi(X_T, Y_T)$. Then we get the vulnerable option prices in the worst case scenario at time $t < T$ as follows:

$$F(t, X_t, Y_t) = e^{-r(T-t)} \operatorname*{esssup}_{\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]} \mathbb{E}\big[\varphi(X_T, Y_T) | \mathscr{F}_t\big], \tag{2.4}$$

where esssup is essential supremum. By the ambiguity of the uncertain volatility model, we obtain the definition of price as equation (2.4). It is related to the consistency risk measure, which quantifies the model risk caused by volatility uncertainty (see [21]). In addition, the problem of model ambiguity in financial mathematics has also attracted many people's attention. Therefore, we should pay attention to the importance of price in the worst case.

Through the stochastic control theory (see [22]), we know that $F(t, X_t, Y_t)$ satisfies the BSB equation.

Lemma 2.1 *$F(t, X_t, Y_t)$ satisfies the following BSB equation*:

$$\begin{cases} \partial_t F + r(x\partial_x F - F) + \mu_2 y \partial_y F + \sup_{\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]} \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 F + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy}^2 F = 0, \\ \quad 0 \le t \le T, x \ge 0, y \ge 0, \\ F(T, x, y) = \varphi(x, y), \quad x \ge 0, y \ge 0. \end{cases} \tag{2.5}$$

Proof Notice that the stochastic control system is

$$\begin{cases} dX_t = rX_t\,dt + \sigma_t X_t\,dW_t^x, \quad \sigma_t \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}], \\ dY_t = \mu_2 Y_t\,dt + \sigma_2 Y_t\,dW_t^y. \end{cases}$$

Then, for all $(s, x, y) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+$, we establish the dynamic program frame first

$$\begin{cases} dX_t = rX_t\,dt + \sigma_t X_t\,dW_t^x, \quad \sigma_t \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}], \\ dY_t = \mu_2 Y_t\,dt + \sigma_2 Y_t\,dW_t^y, \\ X_s = x, \\ Y_s = y. \end{cases} \tag{2.6}$$

The cost function is

$$\mathbb{J}(s, x, y; \sigma) = \mathbb{E}_s\big[e^{-r(T-s)}\varphi(X_T, Y_T)\big],$$

where $\mathbb{E}_s[\cdot] = \mathbb{E}[\cdot | \mathscr{F}_s]$. The value function is

$$F(s, x, y) = \operatorname*{esssup}_{\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]} \mathbb{J}(s, x, y; \sigma).$$

For all $0 \le s \le \hat{s} \le T$, $\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]$, we have

$$\begin{aligned} F(s,x,y) &\ge \mathbb{E}_s\left[e^{-r(T-s)}\varphi(X_T, Y_T)\right] \\ &= \mathbb{E}_s\left[\int_s^{\hat{s}} -re^{-r(T-t)}\varphi(X_T, Y_T)\,dt + e^{-r(T-\hat{s})}\varphi(X_T, Y_T)\right]. \end{aligned}$$

For convenience, we use φ to denote $\varphi(X_T, Y_T)$. Then we obtain

$$0 \ge \mathbb{E}_s\left[\int_s^{\hat{s}} -re^{-r(T-t)}\varphi\,dt\right] + F(\hat{s},x,y) - F(s,x,y).$$

Dividing two sides of the above inequality by $\hat{s} - s$, we have that

$$0 \ge \mathbb{E}_s\left[\frac{\int_s^{\hat{s}} -re^{-r(T-t)}\varphi\,dt}{\hat{s}-s}\right] + \frac{F(\hat{s},x,y) - F(s,x,y)}{\hat{s}-s}.$$

Here, we assume that φ is Lipschitz continuous. Then, according to Itô's formula and equations (2.6), we obtain

$$\begin{aligned} dF &= F_t\,dt + F_x\,dX_t + F_y\,dY_t + \frac{1}{2}F_{xx}\,dX_t\,dX_t + \frac{1}{2}F_{yy}\,dY_t\,dY_t + \frac{1}{2}F_{xy}\,dX_t\,dY_t \\ &= \left(F_t + rX_tF_x + \mu_2Y_tF_y + \frac{1}{2}\sigma_t^2X_t^2F_{xx} + \frac{1}{2}\sigma_2^2Y_t^2F_{yy}\right)dt \\ &\quad + \sigma_tX_tF_x\,dW_t^x + \sigma_2Y_tF_y\,dW_t^y. \end{aligned}$$

Let $\hat{s} \to s$. For all $\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]$, we have that

$$\begin{aligned} 0 &\ge -r\mathbb{E}_s\left[e^{-r(T-s)}\varphi\right] + F_t + rX_sF_x + \mu_2Y_sF_y + \frac{1}{2}\sigma_s^2X_s^2F_{xx} + \frac{1}{2}\sigma_2^2Y_s^2F_{yy} \\ &\ge -rF(s,x,y) + F_t(s,x,y) + rxF_x(s,x,y) + \mu_2yF_y(s,x,y) \\ &\quad + \frac{1}{2}\sigma_s^2X_s^2F_{xx}(s,x,y) + \frac{1}{2}\sigma_2^2Y_s^2F_{yy}(s,x,y), \end{aligned}$$

which is

$$0 \ge -rF + F_t + rxF_x + \mu_2yF_y + \sup_{\sigma\in\mathcal{A}[\underline{\sigma},\overline{\sigma}]} \frac{1}{2}\sigma^2x^2F_{xx} + \frac{1}{2}\sigma_2^2y^2F_{yy}. \tag{2.7}$$

On the other hand, according to the character of the supremum, for any $\varepsilon > 0$, there is $\sigma(\varepsilon) \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]$ such that

$$\begin{aligned} F(s,x,y) - \varepsilon(\hat{s}-s) &\le \mathbb{E}_s\left[e^{-r(T-s)}\varphi\right] \\ &= \mathbb{E}_s\left[\int_s^{\hat{s}} -re^{-r(T-t)}\varphi\,dt\right] + \mathbb{E}_s\left[e^{-r(T-\hat{s})}\varphi\right]. \end{aligned}$$

So we have that

$$-\varepsilon \le \mathbb{E}_s\left[\frac{\int_s^{\hat{s}} -re^{-r(T-t)}\varphi\,dt}{\hat{s}-s}\right] + \frac{F(\hat{s},x,y) - F(s,x,y)}{\hat{s}-s}.$$

Using the similar discussion, we can conclude that

$$0 \leq -rF + F_t + rxF_x + \mu_2 yF_y + \sup_{\sigma \in \mathcal{A}[\underline{\sigma},\overline{\sigma}]} \frac{1}{2}\sigma^2 x^2 F_{xx} + \frac{1}{2}\sigma_2^2 y^2 F_{yy}. \tag{2.8}$$

According to (2.7) and (2.8), we can obtain that

$$0 = -rF + F_t + rxF_x + \mu_2 yF_y + \sup_{\sigma \in \mathcal{A}[\underline{\sigma},\overline{\sigma}]} \frac{1}{2}\sigma^2 x^2 F_{xx} + \frac{1}{2}\sigma_2^2 y^2 F_{yy}. \qquad \square$$

Remark 2.1 In this case, adding variable Y to the dynamic system will lead to a more complex stochastic control system, which increases the dimension of the BSB equation.

Remark 2.2 Note that (2.5) is a completely nonlinear PDE, which has no solution like Black–Scholes equation. Therefore, we decide to solve this problem by simplifying it to two Black–Scholes-like PDEs.

3 Black–Scholes-like PDEs and the main result

In this section, we first reparameterize the uncertainty volatility model to study the prices in the worst case scenario. We suppose that the price process X_t^ε has a dynamic

$$\begin{cases} dX_t^\varepsilon = rX_t^\varepsilon\,dt + \sigma_t X_t^\varepsilon\,dW_t^x, \\ dY_t = \mu_2 Y_t\,dt + \sigma_2 Y_t\,dW_t^y, \end{cases} \tag{3.1}$$

where $\sigma_t \in \mathcal{A}^\varepsilon = \{\sigma_t | \sigma_t \text{ is a } [\sigma_0, \sigma_0 + \varepsilon]\text{-valued progressively measurable process}\}$ and $\sigma_0 \in [\underline{\sigma}, \overline{\sigma}]$.

The cost function is

$$\mathbb{J}^\varepsilon(t,x,y;\sigma) = e^{-r(T-t)}\mathbb{E}_{txy}\big[\varphi\big(X_T^\varepsilon, Y_T\big)\big],$$

where $\mathbb{E}_{txy}[\cdot]$ means the conditional expectation taken with respect to $X_t^\varepsilon = x$, $Y_t = y$. The value function is as follows:

$$F^\varepsilon(t,x,y;\sigma) = \operatorname*{esssup}_{\sigma \in \mathcal{A}^\varepsilon}\big[\mathbb{J}^\varepsilon(t,x,y;\sigma)\big].$$

By Lemma 2.1, we obtain the following BSB equation for F^ε:

$$\begin{cases} \partial_t F^\varepsilon + r(x\partial_x F^\varepsilon - F^\varepsilon) + \mu_2 y \partial_y F^\varepsilon + \sup_{\sigma \in \mathcal{A}^\varepsilon} \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 F^\varepsilon + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy}^2 F^\varepsilon = 0, \\ \quad 0 \leq t \leq T, x \geq 0, y \geq 0, \\ F^\varepsilon(T,x,y) = \varphi(x,y), \quad x \geq 0, y \geq 0, \end{cases} \tag{3.2}$$

which is equivalent to

$$\begin{cases} \partial_t F^\varepsilon + r(x\partial_x F^\varepsilon - F^\varepsilon) + \mu_2 y \partial_y F^\varepsilon + \sup_{\gamma \in \mathcal{A}[0,1]} \frac{1}{2}(\sigma_0 + \varepsilon\gamma)^2 x^2 \partial_{xx}^2 F^\varepsilon + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy}^2 F^\varepsilon \\ \quad = 0, \quad 0 \leq t \leq T, x \geq 0, y \geq 0, \\ F^\varepsilon(T,x,y) = \varphi(x,y), \quad x \geq 0, y \geq 0, \end{cases} \tag{3.3}$$

where $\mathcal{A}[0,1] = \{\gamma_t | \gamma_t \text{ is a } [0,1]\text{-valued progressively measurable process}\}$.

Obviously, the price in the worst case scenario is larger than any Black–Scholes price with a constant volatility $\sigma_0 \in [\underline{\sigma}, \overline{\sigma}]$. In the following section, we demonstrate that the worst case scenario price of vulnerable option converges to its Black–Scholes price with a constant volatility σ_0. In addition, when the volatility interval is reduced to a certain point, the convergence rate of vulnerable option prices can be obtained. Then, we can get the estimation of the prices through this result when the interval is small enough.

Let F_0 be the Black–Scholes price, $F^0 = F^\varepsilon|_{\varepsilon=0}$, $F_1 = \partial_\varepsilon F^\varepsilon|_{\varepsilon=0}$. Now, we assume that F^ε is continuous with respect to ε. Then, according to the continuity of F^ε and equation (2.4), we have $F_0 = F^0 = F^\varepsilon|_{\varepsilon=0}$. As we all know, F_0 satisfies the following PDE:

$$\begin{cases} \partial_t F_0 + r(x\partial_x F_0 - F_0) + \mu_2 y \partial_y F_0 + \frac{1}{2}\sigma_0^2 x^2 \partial_{xx}^2 F_0 + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy}^2 F_0 = 0, \\ \quad 0 \le t \le T, x \ge 0, y \ge 0, \\ F_0(T,x,y) = \varphi(x,y), \quad x \ge 0, y \ge 0. \end{cases} \tag{3.4}$$

Then we analyze the equation of F_1. We have $F_1 = \partial_\varepsilon F^\varepsilon|_{\varepsilon=0}$, which is the rate of convergence of the vulnerable option prices as ε closes to 0. F_1 is the first order derivative of F^ε with respect to ε, and let $\varepsilon = 0$, then we obtain the equation of F_1 as follows:

$$\begin{cases} \partial_t F_1 + r(x\partial_x F_1 - F_1) + \mu_2 y \partial_y F_1 + \frac{1}{2}\sigma_0^2 x^2 \partial_{xx}^2 F_1 + \sup_{\gamma \in \mathcal{A}[0,1]} \gamma \sigma_0 x^2 \partial_{xx}^2 F_0 \\ \quad + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy}^2 F_1 = 0, \quad 0 \le t \le T, x \ge 0, y \ge 0, \\ F_1(T,x,y) = \varphi(x,y), \quad x \ge 0, y \ge 0. \end{cases} \tag{3.5}$$

Now, we have two Black–Scholes-like PDEs as above. We tend to find the connection between F^ε and F_0, F_1. Then we try to prove whether we can impose additional conditions on the payoff function to make the term $F^\varepsilon - (F_0 + \varepsilon F_1)$ be of order $o(\varepsilon)$. That is to say, with the disappearance of model fuzziness, the worst case estimates of vulnerable option prices will approach the real value. It will also show us a way to estimate the price of a vulnerable option in the worst case scenario. Through our deduction in the next section, we get the following main results of this paper.

Theorem 3.1 *Assume that the payoff function $\varphi(x,y)$ is Lipschitz continuous, and the second-order partial derivatives of $\varphi(x,y)$ are continuous. In addition, we also assume that its second-order partial derivatives have polynomial growth. Then*

$$\lim_{\varepsilon \downarrow 0} \frac{F^\varepsilon - (F_0 + \varepsilon F_1)}{\varepsilon} = 0. \tag{3.6}$$

Remark 3.2 It is difficult to prove Theorem 3.1. The first problem is how to convert the error term into an estimable form. In the next section, we get the expectation of the error term and divide it into three parts. The second problem is how to estimate these three parts. We can use stochastic control theory, the zero set's characteristics of equation (4.1), the characteristics of the sublinear expectation in [23], and the characteristics of vulnerable option price process in the worst case scenario.

Remark 3.3 According to Theorem 3.1, we can compute vulnerable option price $F^\varepsilon(t, X_t^\varepsilon, Y_t)$ with its approximate value, $F_0(t, X_t^\varepsilon, Y_t) + \varepsilon F_1(t, X_t^\varepsilon, Y_t)$, where $F_0(t, X_t^\varepsilon, Y_t)$ is the Black–

Scholes price of vulnerable option and $F_1(t, X_t^\varepsilon, Y_t)$ can be computed by a simple difference scheme in terms of (3.5) (see [19]).

Remark 3.4 We notice that (3.4) and (3.5) are independent of ε. So when we calculate F^ε with different ε for all small values of ε, we just need to compute F_0 and F_1 only once by Theorem 3.1.

4 The proof of the main result

In this section, the processes and details of our thinking are presented. We attempt to control the error term to prove that we can use its estimate $F_0 + \varepsilon F_1$ to calculate F^ε. Under the conditions imposed on φ, which were mentioned in Theorem 3.1, we have the following proof.

4.1 The Lipschitz continuity of payoff function

From Sect. 3 we know that only with the continuity of F^ε can we obtain the PDEs of F_0 (= $F^\varepsilon|_{\varepsilon=0}$) and F_1 (= $\partial_\varepsilon F^\varepsilon|_{\varepsilon=0}$). In order to get the continuity of F^ε, we assume that $\varphi(x, y)$ is Lipschitz continuous. Then there exists a constant K_1 such that

$$\bigl|\varphi(x,t) - \varphi(y,t)\bigr| \le K_1|x-y| \quad \text{for all } x \neq y, x, y \in \mathbb{R}^+. \tag{4.1}$$

Thus, we have the following lemma.

Lemma 4.1 *We suppose that $\varphi(x, y)$ is Lipschitz continuous. Then F^ε is continuous with respect to ε.*

Proof Let $0 \le \varepsilon_0 \le \varepsilon < 1$. Notice that

$$F^\varepsilon(t,x,y;\sigma) = \operatorname*{esssup}_{\sigma\in\mathcal{A}^\varepsilon}\bigl\{e^{-r(T-t)}\mathbb{E}_{txy}\bigl[\varphi\bigl(X_T^\varepsilon, Y_T\bigr)\bigr]\bigr\}.$$

We can conclude that

$$\begin{aligned} e^{-r(T-t)}F^{\varepsilon_0}(t,x,y;\sigma) &= \operatorname*{esssup}_{\sigma\in\mathcal{A}^{\varepsilon_0}}\mathbb{E}_{txy}\bigl[\varphi\bigl(X_T^{\varepsilon_0}(\sigma), Y_T\bigr)\bigr] \\ &= \operatorname*{esssup}_{\sigma\in\mathcal{A}^{\varepsilon}}\mathbb{E}_{txy}\bigl[\varphi\bigl(X_T^{\varepsilon}\bigl(\sigma\wedge(\sigma_0+\varepsilon_0)\bigr), Y_T\bigr)\bigr]. \end{aligned}$$

By the Lipschitz continuity of $\varphi(x, y)$ and equation (4.1), we conclude that there is a constant K_1 such that

$$\begin{aligned} &e^{-r(T-t)}\bigl|F^\varepsilon(t,x,y;\sigma) - F^{\varepsilon_0}(t,x,y;\sigma)\bigr| \\ &\quad\le \operatorname*{esssup}_{\sigma\in\mathcal{A}^{\varepsilon}}\bigl|\mathbb{E}_{txy}\bigl[\varphi\bigl(X_T^\varepsilon(\sigma), Y_T\bigr)\bigr] - \mathbb{E}_{txy}\bigl[\varphi\bigl(X_T^{\varepsilon}\bigl(\sigma\wedge(\sigma_0+\varepsilon_0)\bigr), Y_T\bigr)\bigr]\bigr| \\ &\quad\le K_1 \operatorname*{esssup}_{\sigma\in\mathcal{A}^{\varepsilon}}\bigl(\mathbb{E}_{txy}\bigl|X_T^\varepsilon(\sigma) - X_T^\varepsilon\bigl(\sigma\wedge(\sigma_0+\varepsilon_0)\bigr)\bigr|^2\bigr)^{\frac{1}{2}}. \end{aligned}$$

With the estimates of the moments of solutions of stochastic differential equations (Theorem 9 of Sect. 2.9 and Corollary 12 of Sect. 2.5 in [24]), there are constants $N = N(q, r, \sigma_0)$,

$N' = N'(q, r, \sigma_0)$ and $C = \max\{NN', N + N'\}$ such that

$$\begin{aligned}
&\mathbb{E}_{txy}\big[\sup_{s\le t}\big|X_s^\varepsilon(\sigma) - X_s^\varepsilon\big(\sigma\wedge(\sigma_0+\varepsilon_0)\big)\big|^{2q}\big] \\
&\quad\le Nt^{q-1}e^{Nt}\mathbb{E}_{txy}\bigg[\int_0^t \big|X_s^\varepsilon(\sigma)\big|^{2q}\cdot\big|\sigma_s - \sigma_s\wedge(\sigma_s+\varepsilon_0)\big|^{2q}\,ds\bigg] \\
&\quad\le Nt^{q-1}e^{Nt}N'e^{N't}t\big(1+x^{2q}\big)|\varepsilon-\varepsilon_0|^{2q} \\
&\quad= Ct^q e^{Ct}\big(1+x^{2q}\big)|\varepsilon-\varepsilon_0|^{2q}.
\end{aligned}$$

So we can obtain that

$$\begin{aligned}
e^{-r(T-t)}\big|F^\varepsilon(t,x,y) - F^{\varepsilon_0}(t,x,y)\big| &\le K_1 \operatorname*{esssup}_{\sigma\in\mathcal{A}^\varepsilon}\big(\mathbb{E}_{txy}\big|X_T^\varepsilon(\sigma) - X_T^\varepsilon\big(\sigma\wedge(\sigma_0+\varepsilon_0)\big)\big|^2\big)^{\frac{1}{2}} \\
&\le K_1 \operatorname*{esssup}_{\sigma\in\mathcal{A}^\varepsilon}\big(Cte^{Ct}\big(1+x^2\big)|\varepsilon-\varepsilon_0|^2\big)^{\frac{1}{2}} \\
&\le K_1'\big(1+x^2\big)^{\frac{1}{2}}|\varepsilon-\varepsilon_0|,
\end{aligned}$$

where $K_1' = K_1'(K_1, C, T)$.

Let $\varepsilon \to \varepsilon_0$. We have that $|F^\varepsilon(t,x,y) - F^{\varepsilon_0}(t,x,y)| \to 0$.

So, when $\varepsilon \le \varepsilon_0$, the continuity of F^ε with respect to ε can be proved similarly. □

4.2 Expectation form of the error term

Before proving the convergence of $F_0 + \varepsilon F_1$, we analyze the error term and give its expectation form.

Let $\hat{\sigma}_t$ be volatility process in the worst case scenario and $\hat{X}_t^\varepsilon$ be risky asset process in the worst case scenario. Then equations (3.1) can be rewritten as follows:

$$\begin{cases}
d\hat{X}_t^\varepsilon = r\hat{X}_t^\varepsilon\,dt + \hat{\sigma}_t\hat{X}_t^\varepsilon\,dW_t^x, \\
dY_t = \mu_2 Y_t\,dt + \sigma_2 Y_t\,dW_t^y.
\end{cases}$$

We can get the representation of $\hat{\sigma}$ by equations (3.3) and $\hat{\sigma}(\varepsilon) = \sigma_0 + \varepsilon\hat{\gamma}$, where

$$\hat{\gamma}(t,x,y;\varepsilon) = \begin{cases} 1, & \partial_{xx}^2 F^\varepsilon(t,x,y) \ge 0, \\ 0, & \partial_{xx}^2 F^\varepsilon(t,x,y) < 0. \end{cases} \tag{4.2}$$

Similarly, by solving equation (3.5) of F_1, we have the volatility process: $\bar{\sigma}(\varepsilon) = \sigma_0 + \varepsilon\bar{\gamma}$, where

$$\bar{\gamma}(t,x,y) = \begin{cases} 1, & \partial_{xx}^2 F_0(t,x,y) \ge 0, \\ 0, & \partial_{xx}^2 F_0(t,x,y) < 0. \end{cases} \tag{4.3}$$

Here, we use short symbols $\hat{\gamma}_t$ and $\bar{\gamma}_t$ to denote $\hat{\gamma}(t,x,y;\varepsilon)$ and $\bar{\gamma}(t,x,y)$.

Let $Z^\varepsilon = F^\varepsilon - (F_0 + \varepsilon F_1)$. In order to estimate the error term Z^ε, we define a kind of operator $L(\sigma) = \partial_t + rx\partial_x - r + \mu_2 y\partial_y + \frac{1}{2}\sigma^2x^2\partial_{xx}^2 + \frac{1}{2}\sigma_2^2y^2\partial_{yy}^2$. According to PDEs (3.2), (3.4), and (3.5), we can conclude that

$$\begin{aligned}
L(\hat{\sigma}_t)Z^\varepsilon &= L(\hat{\sigma}_t)\big(F^\varepsilon - (F_0 + \varepsilon F_1)\big) \\
&= 0 - L(\hat{\sigma}_t)(F_0 + \varepsilon F_1) \\
&= -\big(L(\hat{\sigma}_t) - L(\sigma_0)\big)F_0 - L(\sigma_0)F_0 - \varepsilon\big(L(\hat{\sigma}_t) - L(\sigma_0)\big)F_1 - \varepsilon L(\sigma_0)F_1 \\
&= \varepsilon(\bar{\gamma}_t - \hat{\gamma}_t)\sigma_0 x^2 \partial^2_{xx}F_0 - \frac{\varepsilon^2}{2}\big((\hat{\gamma}_t)^2 x^2 \partial^2_{xx}F_0 + 2\sigma_0\hat{\gamma}_t x^2 \partial^2_{xx}F_1\big) - \frac{\varepsilon^3}{2}(\hat{\gamma}_t)^2 x^2 \partial^2_{xx}F_1 \\
&= -f^\varepsilon(t,x,y),
\end{aligned}$$

with the boundary condition $Z^\varepsilon(T) = F^\varepsilon(T) - F_0(T) - \varepsilon F_1(T) = 0$. The last equality means that we let the above formula equal $-f^\varepsilon(t,x,y)$.

According to the Dynkin formula, we can get the expectation form of Z^ε as follows:

$$\begin{aligned}
Z^\varepsilon &= \mathbb{E}_{txy}\left[\int_t^T f^\varepsilon(s,x,y)\,ds\right] \\
&= \varepsilon\mathbb{E}_{txy}\left[\int_t^T (\hat{\gamma}_s - \bar{\gamma}_s)\cdot\sigma_0\cdot\big(\hat{X}^\varepsilon_s\big)^2\partial^2_{xx}F_0\big(s,\hat{X}^\varepsilon_s,Y_s\big)\,ds\right] \\
&\quad + \varepsilon^2\mathbb{E}_{txy}\left[\int_t^T\left\{\frac{1}{2}(\hat{\gamma}_s)^2\big(\hat{X}^\varepsilon_s\big)^2\partial^2_{xx}F_0\big(s,\hat{X}^\varepsilon_s,Y_s\big) + \sigma_0\hat{\gamma}_s\big(\hat{X}^\varepsilon_s\big)^2\partial^2_{xx}F_1\big(s,\hat{X}^\varepsilon_s,Y_s\big)\right\}ds\right] \\
&\quad + \varepsilon^3\mathbb{E}_{txy}\left[\int_t^T \frac{1}{2}(\hat{\gamma}_s)^2\big(\hat{X}^\varepsilon_s\big)^2\partial^2_{xx}F_1\big(s,\hat{X}^\varepsilon_s,Y_s\big)\,ds\right] \\
&= \varepsilon I_1 + \varepsilon^2 I_2 + \varepsilon^3 I_3,
\end{aligned}$$

where

$$I_1 = \mathbb{E}_{txy}\left[\int_t^T (\hat{\gamma}_s - \bar{\gamma}_s)\cdot\sigma_0\cdot\big(\hat{X}^\varepsilon_s\big)^2\partial^2_{xx}F_0\big(s,\hat{X}^\varepsilon_s,Y_s\big)\,ds\right], \tag{4.4}$$

$$I_2 = \mathbb{E}_{txy}\left[\int_t^T\left\{\frac{1}{2}(\hat{\gamma}_s)^2\big(\hat{X}^\varepsilon_s\big)^2\partial^2_{xx}F_0\big(s,\hat{X}^\varepsilon_s,Y_s\big) + \sigma_0\hat{\gamma}_s\big(\hat{X}^\varepsilon_s\big)^2\partial^2_{xx}F_1\big(s,\hat{X}^\varepsilon_s,Y_s\big)\right\}ds\right], \tag{4.5}$$

$$I_3 = \mathbb{E}_{txy}\left[\int_t^T \frac{1}{2}(\hat{\gamma}_s)^2\big(\hat{X}^\varepsilon_s\big)^2\partial^2_{xx}F_1\big(s,\hat{X}^\varepsilon_s,Y_s\big)\,ds\right]. \tag{4.6}$$

Thus, we can conclude that

$$\big|Z^\varepsilon\big| \le \varepsilon|I_1| + \varepsilon^2|I_2| + \varepsilon^3|I_3|. \tag{4.7}$$

By controlling $|I_1|$, $|I_2|$ and $|I_3|$, we can estimate F^ε.

4.3 The polynomial growth condition of payoff function

In this part, we need to analyze the three parts of $|Z|^\varepsilon$ to control the error term. By (4.7), we have that

$$\frac{|Z|^\varepsilon}{\varepsilon} \le |I_1| + \varepsilon\big(|I_2| + \varepsilon|I_3|\big).$$

Therefore, it is sufficient to prove

$$\lim_{\varepsilon\downarrow 0}|I_1|+\varepsilon\bigl(|I_2|+\varepsilon|I_3|\bigr)=0.$$

Obviously, it is necessary to control I_2 and I_3. Because $|I_1|$ is somewhat complex, let us first consider how to control $|I_2|$ and $|I_3|$.

By observing the expressions of I_2 and I_3, we can see that partial derivatives of F_0 and F_1 are involved. Therefore, we should consider how to estimate them before giving the controls of I_2 and I_3.

Then, we can get the expectation form of F_0 and F_1 by the classical results. When $\varepsilon = 0$, we have

$$X(T)=x\exp\left\{\left(r-\frac{\sigma_0^2}{2}\right)(T-t)+\sigma_0(W_T-W_t)\right\}.$$

Thus,

$$\begin{aligned}F_0(t,x,y)&=e^{-r(T-t)}\mathbb{E}_{txy}\bigl[\varphi(X_T,Y_T)\bigr]\\&=e^{-r(T-t)}\mathbb{E}_{txy}\bigl[\varphi(x\cdot H,Y_T)\bigr],\end{aligned}\tag{4.8}$$

where $H(=\exp\{(r-\frac{\sigma_0^2}{2})(T-t)+\sigma_0(W_T-W_t^x)\})$ is a random variable for fixed $t\in[0,T]$.

Similarly, there is

$$\begin{aligned}F^\varepsilon(t,x,y)&=e^{-r(T-t)}\operatorname*{esssup}_{\sigma\in\mathcal{A}^\varepsilon}\bigl\{\mathbb{E}_{txy}\bigl[\varphi\bigl(X_T^\varepsilon,Y_T\bigr)\bigr]\bigr\}\\&=e^{-r(T-t)}\mathbb{E}_{txy}\bigl[\varphi(x\cdot G,Y_T)\bigr],\end{aligned}\tag{4.9}$$

where G $(=\exp\{(r-\frac{\hat\sigma_T^2}{2})(T-t)+\hat\sigma_T(W_T-W_t^x)\})$ is a random variable for fixed $t\in[0,T]$.

By equations (4.8) and (4.9), we know that it is necessary to impose polynomial growth conditions on $\varphi(x,y)$ to control $\partial_{xx}^2F_0$ and $\partial_{xx}^2F^\varepsilon$. Then we give the estimates of $\varphi(x,y)$ to control $\partial_{xx}^2F_0(t,x,y)$ and $\partial_{xx}^2F^\varepsilon(t,x,y)$ in the following lemma.

Lemma 4.2 *Suppose that the second-order partial derivatives of payoff function satisfy the polynomial growth condition, i.e., there are constants K_2 and m such that $\partial_{xx}^2\varphi(x,y)\le K_2(1+|x|^m+|y|^m)$. Then, we have constant K_3 such that*

$$\bigl|\partial_{xx}^2F_0(t,x,y)\bigr|\le K_3\bigl(1+|x|^m+|y|^m\bigr),\tag{4.10}$$

where K_3 depends on T, t, $\mathbb{E}_{txy}[|H|^2]$, $\mathbb{E}_{txy}[|H|^{m+2}]$ and K_2.

Furthermore, there is a constant K_4 such that

$$\bigl|\partial_{xx}^2F^\varepsilon(t,x,y)\bigr|\le K_4\bigl(1+|x|^m+|y|^m\bigr).\tag{4.11}$$

where K_4 depends on T, t, $\mathbb{E}_{txy}[|G|^2]$, $\mathbb{E}_{txy}[|G|^{m+2}]$ and K_2.

Proof According to Lemma 4.2, we can conclude that

$$\begin{aligned}\left|\partial_{xx}^2 F_0(t,x,y)\right| &= e^{-r(T-t)}\mathbb{E}_{txy}\left[\varphi''(xH,Y_T)H^2\right]\\ &\le e^{-r(T-t)}\mathbb{E}_{txy}\left[K_2\left(1+|xH|^m+|Y_T|^m\right)H^2\right]\\ &\le K_3\left(1+|x|^m+|y|^m\right). \end{aligned}\tag{4.12}$$

Here, K_3 depends on T, t, $\mathbb{E}_{txy}[|H|^2]$, $\mathbb{E}_{txy}[|H|^{m+2}]$ and K_2.

Indeed, for a constant $m > 0$, we have that

$$\begin{aligned}\mathbb{E}H^m &= \mathbb{E}\left(\exp\left\{\left(r-\frac{\sigma_0^2}{2}\right)(T-t)+\sigma_0\left(W_T-W_t^x\right)\right\}\right)^m\\ &= e^{m(r-\sigma_0^2/2)(T-t)}\mathbb{E}\left[e^{\sigma_0(W_T-W_t)}\right]^m < +\infty.\end{aligned}$$

On the other hand, we get the control of $\partial_{xx}^2 F^\varepsilon$ similarly. Then there is a constant K_4 which depends on T, t, $\mathbb{E}_{txy}[|G|^2]$, $\mathbb{E}_{txy}[|G|^{m+2}]$ and K_2 such that

$$\left|\partial_{xx}^2 F^\varepsilon(t,x,y)\right| \le K_4\left(1+|x|^m+|y|^m\right). \tag{4.13}$$

□

Then, we give a proposition that needs to be used to get the controls of I_2 and I_3.

Proposition 4.1 *Assume that $\varphi(x,y)$ satisfies the Lipschitz continuity condition. Then there exist constants C_1 and p_1 such that I_2, I_3 in equations (4.5) and (4.6) satisfy*

$$|I_2| + |I_3| \le C_1\left(1+|x|^{p_1}+|x|^{p_1}|y|^{p_1}\right).$$

Proof By Lemma 4.2, we have the following inequality from (3.3) and (4.11):

$$\begin{aligned}&\left|\partial_t F^\varepsilon + r\left(x\partial_x F^\varepsilon - F^\varepsilon\right) + \mu_2 y\partial_y F^\varepsilon + \frac{1}{2}\sigma_2^2 y^2\partial_{yy}^2 F^\varepsilon\right|\\ &\quad\le \left|\frac{1}{2}(\sigma_0+\varepsilon)^2 x^2\partial_{xx}^2 F^\varepsilon\right|\\ &\quad\le \frac{K_4}{2}(\sigma_0+\varepsilon)^2\left(|x|^2+|x|^{m+2}+|x|^2|y|^m\right).\end{aligned}$$

By the expression of F_1, we have that

$$\left|\partial_t F_1 + r(x\partial_x F_1 - F_1) + \mu_2 y\partial_y F_1 + \frac{1}{2}\sigma_2^2 y^2\partial_{yy}^2 F_1\right| \le \left|K_4\sigma_0\left(|x|^2+|x|^{m+2}+|x|^2|y|^m\right)\right|.$$

By equations (3.5) and (4.10), we get the control of $x^2\partial_{xx}^2 F_1$

$$\begin{aligned}\left|x^2\partial_{xx}^2 F_1\right| &= \left|\partial_t F_1 + r(x\partial_x F_1 - F_1) + \mu_2 y\partial_y F_1 + \frac{1}{2}\sigma_2^2 y^2\partial_{yy}^2 F_1 + \sup_{\gamma\in\mathcal{A}[0,1]}\gamma\sigma_0 x^2\partial_{xx}^2 F_0\right|\left(\frac{2}{\sigma_0^2}\right)\\ &\le \left(\left|\partial_t F_1 + r(x\partial_x F_1 - F_1) + \mu_2 y\partial_y F_1 + \frac{1}{2}\sigma_2^2 y^2\partial_{yy}^2 F_1\right| + \left|\sigma_0 x^2\partial_{xx}^2 F_0\right|\right)\left(\frac{2}{\sigma_0^2}\right)\end{aligned}$$

$$\leq \left[\left|K_4\sigma_0\left(|x|^2+|x|^{m+2}+|y|^m|x|^2\right)\right|+\sigma_0 x^2 K_3\left(1+|x|^m+|y|^m\right)\right]\left(\frac{2}{\sigma_0^2}\right)$$
$$\leq M_1\left(|x|^2+|x|^{m+2}+|x|^2|y|^m\right), \tag{4.14}$$

where M_1 depends on K_3, K_4 and σ_0.

We can obtain the existence and uniqueness of $\hat{X}_t^\varepsilon$ from Theorem 5.2.1 in [25]. Then, according to Corollary 12 of Sect. 2.5 in [24], as for the moment estimates of solutions of stochastic differential equations, there is a constant $N_1(q)$ for fixed $q > 0$ such that

$$\mathbb{E}_{txy}\Big[\sup_{s\in[t,T]}\big|\hat{X}_s^\varepsilon\big|^q\Big] \leq N_1(q)e^{N_1(q)(T-t)}\left(1+|x|^q\right). \tag{4.15}$$

Obviously, we can get

$$\mathbb{E}_{txy}\Big[\sup_{s\in[t,T]}\big|\hat{X}_s^\varepsilon\big|^q\Big]\mathbb{E}_{txy}\Big[\sup_{s\in[t,T]}|Y_s|^p\Big]$$
$$\leq N_1(q)e^{N_1(q)(T-t)}\left(1+|x|^q\right)N_1(p)e^{N_1(p)(T-t)}\left(1+|y|^p\right). \tag{4.16}$$

By (4.6), (4.14), (4.15) and (4.16), we have the following inequality:

$$|I_3| = \left|\mathbb{E}_{txy}\left[\int_t^T \frac{1}{2}(\hat{\gamma}_s)^2\left(\hat{X}_s^\varepsilon\right)^2\partial_{xx}^2 F_1\left(s,\hat{X}_s^\varepsilon,Y_s\right)ds\right]\right|$$
$$\leq \frac{M_1}{2}\mathbb{E}_{txy}\left[\int_t^T \left(\big|\hat{X}_s^\varepsilon\big|^2+\big|\hat{X}_s^\varepsilon\big|^{m+2}+|Y_s|^m\big|\hat{X}_s^\varepsilon\big|^2\right)ds\right]$$
$$\leq M_1'\left(1+|x|^{m+2}+|x|^2|y|^m\right), \tag{4.17}$$

where M_1' depends on M_1, T, t, $N_1(2)$, $N_1(m)$ and $N_1(m+2)$.

By (4.5), (4.10), (4.14), (4.15) and (4.16), we can get the control of $|I_2|$:

$$|I_2| = \left|\mathbb{E}_{txy}\left[\int_t^T \left\{\frac{1}{2}(\hat{\gamma}_s)^2\left(\hat{X}_s^\varepsilon\right)^2\partial_{xx}^2 F_0\left(s,\hat{X}_s^\varepsilon,Y_s\right)+\sigma_0\hat{\gamma}_s\left(\hat{X}_s^\varepsilon\right)^2\partial_{xx}^2 F_1\left(s,\hat{X}_s^\varepsilon,Y_s\right)\right\}ds\right]\right|$$
$$\leq \left(\frac{K_3}{2}+M_1\right)\mathbb{E}_{txy}\left[\int_t^T \left(\big|\hat{X}_s^\varepsilon\big|^2+\big|\hat{X}_s^\varepsilon\big|^{m+2}+|Y_s|^m\big|\hat{X}_s^\varepsilon\big|^2\right)ds\right]$$
$$\leq M_2\left(1+|x|^{p_1}+|x|^{p_1}|y|^{p_1}\right), \tag{4.18}$$

where M_2 depends on $T, t, M_1, K_3, N_1(2), N_1(m), N_1(m+2)$, and $p_1 \geq m+2$.

Combining (4.17) and (4.18), we can conclude that there is a constant C_1 such that

$$|I_2| + |I_3| \leq C_1\left(1+|x|^{p_1}+|x|^{p_1}|y|^{p_1}\right). \qquad \square$$

4.4 Convergence of the term I_1

In Proposition 4.1, we give the controls of I_2 and I_3. Then, for a fixed point $(t,x,y) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+$, it is sufficient to prove that

$$\lim_{\varepsilon\downarrow 0}|I_1| = 0.$$

Let $K_\rho = [-\rho, \rho]$ and $h^\varepsilon(t,x,y) = \hat{\gamma}(t,x,y;\varepsilon) - \bar{\gamma}(t,x,y)$. By (4.2) and (4.3), we can conclude that

$$\left|h^\varepsilon(t,x,y)\right| = \begin{cases} 1, & \partial^2_{xx}F^\varepsilon(t,x,y)\partial^2_{xx}F_0(t,x,y) < 0, \\ 0, & \partial^2_{xx}F^\varepsilon(t,x,y)\partial^2_{xx}F_0(t,x,y) \geq 0. \end{cases}$$

Intuitively, F^ε and its partial derivatives will approach F_0 and its corresponding partial derivatives when ε approaches 0. In order to test this intuitive hypothesis, we decompose the interval of $\hat{X}^\varepsilon_s$ into two parts: a compact set K_ρ and the tail part, where $s \in [t,T]$. Consequently, we can write I_1 as the expectation of a sum of two parts as follows:

i. the compact part (when $\hat{X}^\varepsilon_s$ drop into the K_ρ);
ii. the tail part (when $\hat{X}^\varepsilon_s$ drop out of the K_ρ). So we have that

$$\begin{aligned} I_1 &= \mathbb{E}_{txy}\left[\int_t^T h^\varepsilon \cdot \sigma_0 \cdot \left(\hat{X}^\varepsilon_s\right)^2 \partial^2_{xx}F_0\left(s,\hat{X}^\varepsilon_s,Y_s\right)\mathbb{I}_{K_\rho}\left(\hat{X}^\varepsilon_s\right)ds\right] \\ &\quad + \mathbb{E}_{txy}\left[\int_t^T h^\varepsilon \cdot \sigma_0 \cdot \left(\hat{X}^\varepsilon_s\right)^2 \partial^2_{xx}F_0\left(s,\hat{X}^\varepsilon_s,Y_s\right)\mathbb{I}_{K^c_\rho}\left(\hat{X}^\varepsilon_s\right)ds\right] \\ &= \mathbb{E}_{txy}[\mathrm{i}] + \mathbb{E}_{txy}[\mathrm{ii}]. \end{aligned} \tag{4.19}$$

To prove $\lim_{\varepsilon\downarrow 0}|I_1| = 0$, we use localization arguments to deal with $\mathbb{E}_{txy}[\mathrm{ii}]$ and the problem is reduced onto a compact set. On the compact set, according to Lemma 4.1 and equation (2.4), we can conclude that F^ε and its partial derivatives converge to F_0 and its corresponding partial derivatives. Then we get the convergence of $\mathbb{E}_{txy}[\mathrm{i}]$, and it converges to 0.

Then, we analyze the control of the tail part, given $\rho > 0$. Define a stopping time

$$\tau_\rho = \inf\left\{s \in [t,T] \text{ such that } \left|\hat{X}^\varepsilon_s\right| \geq \rho\right\}.$$

As a rule, $\inf\emptyset = \infty$.

By using the estimates of moments of solutions of stochastic differential equations and the Chebyshev inequality, we have that

$$\mathbb{Q}_{txy}(\tau_\rho < T) \leq \mathbb{Q}_{txy}\Big(\sup_{s\in[t,T]}\left|\hat{X}^\varepsilon_s\right| \geq \rho\Big) \leq \frac{Ne^{N(T-t)}(1+|x|)}{\rho}. \tag{4.20}$$

By the Hölder inequality, we conclude that

$$\mathbb{E}_{txy}[\mathrm{ii}] \leq \sigma_0 K_3\left[\mathbb{E}_{txy}\int_t^T\left(\hat{X}^\varepsilon_s\right)^4\left(1+\left|\hat{X}^\varepsilon_s\right|^m + |Y_s|^m\right)^2 ds\right]^{1/2}\left[\mathbb{Q}_{txy}(\tau_\rho < T)\right]^{1/2},$$

where

$$\begin{aligned} &\mathbb{E}_{txy}\int_t^T\left(\hat{X}^\varepsilon_s\right)^4\left(1+\left|\hat{X}^\varepsilon_s\right|^m + |Y_s|^m\right)^2 ds \\ &\quad \leq D_1\left(1+|x|^{p_2}+|y|^{p_2}+|x|^{p_2}|y|^{p_2}\right), \end{aligned} \tag{4.21}$$

where D_1 depends on $\sigma_0, T, t, M_1, K_3, N_1(4), N_1(m), N_1(4+m), N_1(4+2m)$, and $p_2 \geq 4+2m$.

Combining inequalities (4.20) and (4.21), we obtain that

$$\begin{aligned}\mathbb{E}_{txy}[\text{ii}] &\le \sqrt{D_1\bigl(1+|x|^{p_2}+|y|^{p_2}+|x|^{p_2}|y|^{p_2}\bigr)}\cdot\sqrt{\frac{Ne^{N(T-t)}(1+|x|)(T-t)}{\rho}}\\ &\le D_2\frac{(1+|x|^{p_3}+|y|^{p_3}+|x|^{p_3}|y|^{p_3})}{\sqrt{\rho}},\end{aligned}$$

where D_2 depends on $\sigma_0, T, t, M_1, K_3, N, N_1(4), N_1(m), N_1(4+m), N_1(4+2m)$, and $p_3 \ge p_2 + 1$.

When the compact set K_ρ becomes larger, i.e., ρ increases, $\hat{X}_s^\varepsilon$ will be less likely to deviate outside of the set K_ρ.

4.5 The proof of the main result

Now, as the analysis above, we can give a brief proof of Theorem 3.6.

Proof of the main result By the discussion in Sect. 4.4, we have that

$$\lim_{\varepsilon\downarrow 0}|I_1| = 0. \tag{4.22}$$

Then, by inequality (4.7), we have

$$\left|\frac{F^\varepsilon-(F_0+\varepsilon F_1)}{\varepsilon}\right| \le |I_1| + \varepsilon\bigl(|I_2|+\varepsilon|I_3|\bigr).$$

Thus, by Proposition 4.1 and equation (4.22), we obtain the theorem. □

5 Conclusion

In this paper, we analyze the behavior of vulnerable option prices in the worst case scenario. The model studied in this paper is an uncertain volatility model with a volatility interval $[\sigma_0, \sigma_0 + \varepsilon]$. As ε is close to 0, the ambiguity of model vanishes. We can also see that as the interval shrinks, the worst case scenario prices of vulnerable options converge to their Black–Scholes prices. We present an estimation method for solving a fully nonlinear PDE (3.2) by imposing additional conditions to the boundary conditions and cutting it into two Black–Scholes-like equations. So, through this research, we have obtained a method to estimate the price of vulnerable options in the worst case scenario.

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References

1. Johnson, H., Stulz, R.: The pricing of options with default risk. J. Finance **42**(2), 267–280 (1978)
2. Hull, J., White, A.: The impact of default risk on the prices of options and other derivative securities. J. Bank. Finance **19**, 299–322 (1995)
3. Jarrow, R.A., Turnbull, S.: Pricing derivatives on financial securities subject to credit risk. J. Finance **50**(1), 53–86 (1995)
4. Don, R.: The valuation and behavior of Black–Scholes options subject to intertemporal default risk. Rev. Deriv. Res. **1**(1), 25–29 (1996)
5. Klein, P.: Pricing Black–Scholes options with correlated credit risk. J. Bank. Finance **20**, 1211–1229 (1996)
6. Manuel, A.: Credit Risk Valuation-Methods. Models and Applications. Springer, New York (2001)
7. Xu, Y., Li, S.: Pricing formula of reset selling option with credit risk. Math. Res. **38**(1), 103–111 (2005)
8. Wang, B., Liu, J.: Pricing of Barrier Options with Credit Risk. Huazhong University of Science and Technology, Wu Han (2007)
9. Cheng, C.: A vulnerable option pricing model in which the underlying asset price follows a jump-diffusion process. J. Eng. Math. **25**(6), 1129–1133 (2008)
10. Mei, Y., Lin, J.: Research on American Default Options. Shanghai Jiaotong University, Shang Hai (2009)
11. Liu, R., Xu, Y.: Asian option pricing considering credit risk. Math. Theory Appl. **30**(3), 63–67 (2010)
12. Hull, J., White, A.: The price of options on assets with stochastic volatilities. J. Finance **42**(2), 281–300 (1987)
13. Heston, S.L.: A closed-form solution for options with stochastic volatility with applications to bond and currency options. Rev. Financ. Stud. **6**(2), 327–343 (1993)
14. Lyons, T.J.: Uncertain volatility and the risk free synthesis of derivatives. Appl. Math. Finance **2**(2), 117–133 (1995)
15. Avellaneda, M., Levy, A., Paras, A.: Pricing and hedging derivative securities in markets with uncertain volatilities. Appl. Math. Finance **2**(2), 73–88 (1995)
16. Dokuvhaev, N.G., Savkin, A.V.: The pricing of options in a financial market model with transaction costs and uncertain volatility. J. Multinatl. Financ. Manag. **8**(2–3), 353–364 (1998)
17. Forsyth, P.A., Vetzal, K.R.: Implicit solution of uncertain volatility/transaction cost option pricing models with discretely observed barriers. Appl. Numer. Math. **36**(4), 427–445 (2001)
18. Vorbrink, J.: Financial markets with volatility uncertainty. J. Math. Econ. **53**, 64–78 (2014)
19. Pooley, D.M., Forsyth, P.A., Vetzal, K.R.: Numerical convergence properties of option pricing PDEs with uncertain volatility. IMA J. Numer. Anal. **23**, 241–267 (2003)
20. Fouque, J.-P., Ren, B.: Approximation for option prices under uncertain volatility. SIAM J. Financ. Math. **5**, 360–383 (2014)
21. Cont, R.: Model uncertainty and its impact on the pricing of derivative instruments. Math. Finance **16**, 519–547 (2006)
22. Yong, J., Zhou, X.: Stochastic Controls, Hamiltonian Systems and HJB Equations. Springer, New York (1999)
23. Denis, L., Hu, M., Peng, S.: Function spaces and capacity related to a sublinear expectation. Application to g-Brownian motion paths. Potential Anal. **34**, 139–161 (2011)
24. Krylov, N.: Controlled Diffusion Processes. Springer, New York (1980)
25. Oksendal, B.: Stochastic Differential Equations. Springer, New York (2003)