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Multilinear fractional integral operators on central Morrey spaces with variable exponent

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Abstract
In this paper, we obtain the boundedness of the multilinear fractional integral operators and their commutators on central Morrey spaces with variable exponent.

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Keywords: Multilinear fractional integral operator; Commutator; Variable exponent; λ-central BMO spaces; Central Morrey space

1 Introduction

The study of multilinear integral operators was motivated not only as the generalization of the theory of linear ones but also their natural appearance in analysis. It has increasing attention and much development in recent years, such as the study of the bilinear Hilbert transform by Lacey and Thiele [21, 22] and the systematic treatment of multilinear Calderón–Zygmund operators by Grafokas and Torres [13, 14], Grafokas and Kalton [12]. The importance of fractional integral operators is owing to the fact that they are smooth operators and have been extensively used in various areas such as potential analysis, harmonic analysis and partial differential equations. As one of the most important operators, the multilinear fractional integral operator (also known as the multilinear Riesz potential) has also attracted more attention, see for example [3, 11, 18, 24].

It is well known that function spaces with variable exponent arouse strong interest not only in harmonic analysis but also in applied mathematics. The theory of function spaces with variable exponent has made great progress since some elementary properties were given by Kováčik and Rákosník [19] in 1991. Lebesgue and Sobolev spaces with integrability exponent have been widely studied, see [5, 8] and the references therein. Many applications of these spaces were given, for example, in the modeling of electrorheological fluids [26], in the study of image processing [4], and in differential equations with nonstandard growth [15]. On the other hand, the λ-central bounded mean oscillation spaces, Morrey type spaces and related function spaces have interesting applications in studying boundedness of operators including singular integral operators; see for example [1, 9, 19, 27–29]. In 2015, Mizuta, Ohno and Shimomura introduced the non-homogeneous central Morrey spaces of variable exponent in [23]. Recently, Fu et al. introduced the λ-central BMO spaces and the central Morrey spaces with variable exponent and gave the boundedness of some operators in [10]. In [2, 6, 7, 17] and [31–34], the authors proved the boundedness of some integral operators on variable function spaces, respectively. Meanwhile, some authors gave the boundedness of multilinear integral operators and their commutators on variable exponent function spaces, such as [16, 30, 35].

Motivated by [9, 10, 29], we will study the boundedness of the multilinear fractional integral operators and their commutators on the central Morrey spaces with variable exponent.

Let us explain the outline of this article. In Sect. 2, we first briefly recall some standard notations and lemmas in variable Lebesgue spaces. Then we will recall the definiton of the λ-central BMO spaces and central Morrey spaces with variable exponent. In Sect. 3, we will establish the boundedness for a class of multi-sublinear fractional integral operators on central Morrey spaces with variable exponent. Subsequently the boundedness of multilinear fractional integral commutators on central Morrey spaces with variable exponent will be obtained in Sect. 4. In Sect. 5, we will also consider the boundedness of another multilinear fractional integral commutators.

In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$ by $|A|$ and χ_A, respectively. The notation $f \approx g$ means that there exist constants $C_1, C_2 > 0$ such that $C_1 g \le f \le C_2 g$.

2 Variable exponent function spaces

Firstly we give some notation and basic definitions on variable exponent Lebesgue spaces.

Given an open set $E \subset \mathbb{R}^n$, and a measurable function $p(\cdot): E \to [1,\infty)$. $p'(\cdot)$ is the conjugate exponent defined by $p'(\cdot) = p(\cdot)/(p(\cdot)-1)$.

The set $\mathcal{P}(E)$ consists of all $p(\cdot): E \to [1,\infty)$ satisfying

$$p^- = \operatorname{ess\,inf}\{p(x): x \in E\} > 1,$$
$$p^+ = \operatorname{ess\,sup}\{p(x): x \in E\} < \infty.$$

By $L^{p(\cdot)}(E)$ we denote the space of all measurable functions f on E such that, for some $\lambda > 0$,

$$\int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This is a Banach function space with respect to the Luxemburg–Nakano norm,

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\lambda > 0: \int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

The space $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by $L^{p(\cdot)}_{\mathrm{loc}}(\Omega) := \{f : f \in L^{p(\cdot)}(E)$ for all compact subsets $E \subset \Omega\}$.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \left|f(y)\right| dy,$$

where $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. The set $\mathcal{B}(\mathbb{R}^n)$ consists of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 2.1 ([7]) *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies*

$$\left|p(x) - p(y)\right| \le \frac{C}{-\log(|x-y|)}, \quad |x-y| \le 1/2, \tag{2.1}$$

and

$$\left|p(x) - p(y)\right| \le \frac{C}{\log(|x|+e)}, \quad |y| \ge |x|, \tag{2.2}$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, that is, the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 2.2 ([20] (Generalized Hölder inequality)) *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} \left|f(x)g(x)\right| dx \le r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

Lemma 2.3 ([17]) *Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

Lemma 2.4 ([17]) *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \qquad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}$$

and

$$\frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Lemma 2.5 ([8]) *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (2.1) and (2.2) in Lemma 2.1. Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{p(x)}} & \text{if } |Q| \le 2^n \text{ and } x \in Q, \\ |Q|^{\frac{1}{p(\infty)}} & \text{if } |Q| \ge 1 \end{cases}$$

for every cube (or ball) $Q \subset \mathbb{R}^n$, where $p(\infty) = \lim_{x\to\infty} p(x)$.

Lemma 2.6 ([8]) *Let $p(\cdot), q(\cdot), s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$$

for almost every $x \in \mathbb{R}^n$. Then

$$\|fg\|_{L^{s(\cdot)}(\mathbb{R}^n)} \leq 2\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$.

Now we recall that the central Morrey space with variable exponent and the λ-central bounded mean oscillation space with variable exponent in [10] are defined as follows.

Definition 2.1 ([10]) Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The central Morrey space with variable exponent $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)$ is defined by

$$\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n) = \left\{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.$$

Definition 2.2 ([10]) Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\lambda < 1/n$. The λ-central BMO space with variable exponent $\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbb{R}^n)$ is defined by

$$\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbb{R}^n) = \left\{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|(f - f_{B(0,R)})\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.$$

Remark 2.1 Denote by $\mathcal{B}^{q(\cdot),\lambda}(\mathbb{R}^n)$ and $\mathrm{CMO}^{q(\cdot),\lambda}(\mathbb{R}^n)$ the inhomogeneous versions of the central Morrey space and the λ-central BMO space with variable exponent, which are defined, respectively, by taking the supremum over $R \geq 1$ in Definition 2.1 and Definition 2.2 instead of $R > 0$ there.

Remark 2.2 Our results in this paper remain true for the inhomogeneous versions of λ-central BMO spaces and central Morrey spaces with variable exponent.

3 Multilinear fractional integral operators

Let $m \in \mathbb{N}$ and $K(y_0, y_1, \ldots, y_m)$ be a function defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. We denote by $\vec{f}$ the m-tuple $(f_1, \ldots, f_m)$. Now we consider that T is an m-linear operator defined on the product of test functions such that, for K, the integral representation below is valid:

$$T(\vec{f})(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \ldots, y_m) \prod_{j=1}^{m} f_j(y_j)\, dy_1 \cdots dy_m,$$

whenever f_j, $j = 1,\dots,m$, are smooth functions with compact support and $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$.

Particularly, there is a kind of multilinear operator $T_{\alpha,m}$, which is called multilinear fractional integral operator, whose kernel is

$$K(x,y_1,\dots,y_m) = \left|(x-y_1,\dots,x-y_m)\right|^{\alpha-mn}, \quad 0<\alpha<mn. \tag{3.1}$$

In 1999, Kenig and Stein [18] gave the boundedness of the above multilinear fractional integral operator $T_{\alpha,m}$ on the product of Lebesgue spaces.

Theorem A ([18]) *Let $0<\alpha<mn$ and $T_{\alpha,m}$ be an m-linear fractional integral operator with kernel K satisfying (3.1). Suppose $1 \le p_1,p_2,\dots,p_m \le \infty$, $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n > 0$.*

(1) *If each $p_j > 1$, $j = 1,\dots,m$, then*

$$\left\|T_{\alpha,m}(\vec{f})\right\|_{L^q(\mathbb{R}^n)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

(2) *If each $p_j = 1$ for some j, then*

$$\left\|T_{\alpha,m}(\vec{f})\right\|_{L^{q,\infty}(\mathbb{R}^n)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

In the variable exponent case, Tan, Liu and Zhao [30] gave the following result.

Theorem B ([30]) *Let $m \in \mathbb{N}$, $0<\alpha<mn$, $q(\cdot),p_1(\cdot),\dots,p_m(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1 and $1/q(\cdot) = 1/p_1(\cdot) + \cdots + 1/p_m(\cdot) - \alpha/n$. Then*

$$\left\|T_{\alpha,m}(\vec{f})\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}.$$

Next we will give the boundedness of a class of multi-sublinear fractional integral operators T on the product of central Morrey spaces with variable exponent.

Theorem 3.1 *Let $m \in \mathbb{N}$, $0<\alpha<mn$ and T be a multi-sublinear fractional integral operator such that*

$$\left|T(\vec{f})(x)\right| \le C\int_{(\mathbb{R}^n)^m} \frac{|f_1(y_1)|\cdots|f_m(y_m)|}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}}\,dy_1\cdots dy_m \tag{3.2}$$

for any integrable functions $f_1,\dots,f_m$ with compact support and $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$. Suppose $\lambda_j < -\frac{\alpha}{mn}$, $\lambda = \sum_{j=1}^m \lambda_j + \alpha/n$, $p_j(\cdot)$ $(j = 1,\dots,m)$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1 and $1/q(\cdot) = \sum_{j=1}^m 1/p_j(\cdot) - \alpha/n > 0$. If T is bounded from $L^{p_1(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$, then T is also bounded from $\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n) \times \cdots \times \dot{\mathcal{B}}^{p_m(\cdot),\lambda_m}(\mathbb{R}^n)$ into $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)$.

If $0<\alpha<mn$ and $T_{\alpha,m}$ is an m-linear fractional integral operator, then the condition (3.2) is obviously satisfied by (3.1). By Theorem B we can get the following corollary of Theorem 3.1.

Corollary 3.1 *Let $m \in \mathbb{N}$, $0 < \alpha < mn$ and $T_{\alpha,m}$ be an m-linear fractional integral operator with kernel K satisfying (3.1). Suppose $\lambda_j < -\frac{\alpha}{mn}$, $\lambda = \sum_{j=1}^m \lambda_j + \alpha/n$, $p_j(\cdot)$ $(j = 1, \dots, m)$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1 and $1/q(\cdot) = \sum_{j=1}^m 1/p_j(\cdot) - \alpha/n > 0$. Then $T_{\alpha,m}$ is bounded from $\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n) \times \cdots \times \dot{\mathcal{B}}^{p_m(\cdot),\lambda_m}(\mathbb{R}^n)$ into $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)$.*

Proof of Theorem 3.1 In order to simplify the proof, we consider only the situation when $m = 2$. Actually, a similar procedure works for all $m \in \mathbb{N}$. Let f_1, f_2 be functions in $\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)$ and $\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)$, respectively. For fixed $R > 0$, denote $B(0,R)$ by B. We need to prove

$$\left\| T(f_1,f_2)\chi_B \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C|B|^{\lambda} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)},$$

where C is a constant independent of R.

By the Minkowski inequality we write

$$\begin{aligned} \left\| T(f_1,f_2)\chi_B \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leq \left\| T(f_1\chi_{2B},f_2\chi_{2B})\chi_B \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \left\| T(f_1\chi_{(2B)^c},f_2\chi_{2B})\chi_B \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \left\| T(f_1\chi_{2B},f_2\chi_{(2B)^c})\chi_B \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \left\| T(f_1\chi_{(2B)^c},f_2\chi_{(2B)^c})\chi_B \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.3}$$

We first estimate I_1. Using Lemma 2.4 and the boundedness of T from $L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$, we have

$$\begin{aligned} I_1 &\leq C\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|f_2\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2B|^{\lambda_1} \|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} |2B|^{\lambda_2} \|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C|2B|^{\lambda_1+\lambda_2+\alpha/n} |2B|^{-\alpha/n} \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C|2B|^{\lambda_1+\lambda_2+\alpha/n} \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|\chi_{2B}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leq C|B|^{\lambda} \|\chi_{2B}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}, \end{aligned} \tag{3.4}$$

where

$$\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \approx |2B|^{1/p_1(\cdot)+1/p_2(\cdot)} = |2B|^{1/q(\cdot)+\alpha/n} \approx |2B|^{\alpha/n} \|\chi_{2B}\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Next we estimate I_2. Noting that $|(x-y_1,x-y_2)|^{2n-\alpha} \geq |x-y_1|^{2n-\alpha}$, by using (3.2), $\lambda_1 < -\frac{\alpha}{2n}$, Lemma 2.3, the Minkowski inequality and the generalized Hölder inequality, we have

$$\begin{aligned} I_2 &= \left\| T(f_1\chi_{(2B)^c},f_2\chi_{2B})\chi_B \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leq \sum_{k=1}^{\infty} \left\| T(f_1\chi_{2^{k+1}B\setminus 2^kB},f_2\chi_{2B})\chi_B \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{k=1}^{\infty}\left\|\int_{2B}\int_{2^{k+1}B\setminus 2^kB}\frac{|f_1(y_1)||f_2(y_2)|}{|(\cdot-y_1,\cdot-y_2)|^{2n-\alpha}}\,dy_1\,dy_2\,\chi_B(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_2\chi_{2B}\|_{L^1(\mathbb{R}^n)}\sum_{k=1}^{\infty}\left\|\int_{2^{k+1}B\setminus 2^kB}\frac{|f_1(y_1)|}{|\cdot-y_1|^{2n-\alpha}}\,dy_1\,\chi_B(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left(2^{k-1}R\right)^{-2n+\alpha}\|f_1\chi_{2^{k+1}B}\|_{L^1(\mathbb{R}^n)}\\
&\le C\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\sum_{k=1}^{\infty}\left|2^kB\right|^{-2+\alpha/n}\|f_1\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2B|^{\lambda_2}\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B|^{-2+\alpha/n}\\
&\quad\times\sum_{k=1}^{\infty}2^{kn(-2+\alpha/n)}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\left|2^{k+1}B\right|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|B|^{\lambda_2+1-2+\alpha/n+\lambda_1+1}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\sum_{k=1}^{\infty}2^{-2kn+k\alpha+n\lambda_2+n+(k+1)n\lambda_1+(k+1)n}\\
&\le C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\sum_{k=1}^{\infty}2^{kn(-1+\lambda_1+\alpha/n)}\\
&\le C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\sum_{k=1}^{\infty}2^{kn(\lambda_1+\frac{\alpha}{2n})}\\
&\le C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.
\end{aligned}\tag{3.5}
$$

Similarly, we estimate I_3. Noticing that $|(x-y_1,x-y_2)|^{2n-\alpha}\ge|x-y_2|^{2n-\alpha}$, by (3.2), $\lambda_2<-\frac{\alpha}{2n}$, Lemma 2.3, the Minkowski inequality and the generalized Hölder inequality, we obtain

$$
\begin{aligned}
I_3&=\left\|T(f_1\chi_{2B},f_2\chi_{(2B)^c})\chi_B\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le\sum_{k=1}^{\infty}\left\|T(f_1\chi_{2B},f_2\chi_{2^{k+1}B\setminus 2^kB})\chi_B\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{k=1}^{\infty}\left\|\int_{2^{k+1}B\setminus 2^kB}\int_{2B}\frac{|f_1(y_1)||f_2(y_2)|}{|(\cdot-y_1,\cdot-y_2)|^{2n-\alpha}}\,dy_1\,dy_2\,\chi_B(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_1\chi_{2B}\|_{L^1(\mathbb{R}^n)}\sum_{k=1}^{\infty}\left\|\int_{2^{k+1}B\setminus 2^kB}\frac{|f_2(y_2)|}{|\cdot-y_2|^{2n-\alpha}}\,dy_2\,\chi_B(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left(2^{k-1}R\right)^{-2n+\alpha}\|f_2\chi_{2^{k+1}B}\|_{L^1(\mathbb{R}^n)}\\
&\le C\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{k=1}^{\infty} \left|2^k B\right|^{-2+\alpha/n} \|f_2 \chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)} \\
&\le C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2B|^{\lambda_1} \|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} |B|^{-2+\alpha/n} \\
&\quad \times \sum_{k=1}^{\infty} 2^{kn(-2+\alpha/n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \left|2^{k+1}B\right|^{\lambda_2} \|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)} \\
&\le C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} |B|^{\lambda_1+1-2+\alpha/n+\lambda_2+1} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \times \sum_{k=1}^{\infty} 2^{-2kn+k\alpha+n\lambda_1+n+(k+1)n\lambda_2+(k+1)n} \\
&\le C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} |B|^{\lambda} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \sum_{k=1}^{\infty} 2^{kn(-1+\lambda_2+\alpha/n)} \\
&\le C|B|^{\lambda} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.
\end{aligned} \tag{3.6}
$$

For the estimate of I_4. Noting that $|(x-y_1,x-y_2)|^{2n-\alpha} \ge |x-y_1|^{n-\alpha/2}|x-y_2|^{n-\alpha/2}$, by (3.2), $\lambda_j < -\frac{\alpha}{2n}$, $j=1,2$, Lemma 2.3, the Minkowski inequality and the generalized Hölder inequality, we have

$$
\begin{aligned}
I_4 &= \left\| T(f_1\chi_{(2B)^c}, f_2\chi_{(2B)^c})\chi_B \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \left\| T(f_1\chi_{2^{k_1+1}B\backslash 2^{k_1}B}, f_2\chi_{2^{k_2+1}B\backslash 2^{k_2}B})\chi_B \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{k=1}^{\infty}\sum_{k_2=1}^{\infty} \left\| \int_{2^{k_2+1}B\backslash 2^{k_2}B} \int_{2^{k_1+1}B\backslash 2^{k_1}B} \frac{|f_1(y_1)||f_2(y_2)|}{|(\cdot-y_1,\cdot-y_2)|^{2n-\alpha}}\,dy_1\,dy_2\,\chi_B(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{k=1}^{\infty}\sum_{k_2=1}^{\infty} \left\| \prod_{j=1}^{2} \int_{2^{k_j+1}B\backslash 2^{k_j}B} \frac{|f_j(y_j)|}{|\cdot-y_j|^{n-\alpha/2}}\,dy_j\,\chi_B(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\prod_{j=1}^{2} \left(\sum_{k_j=1}^{\infty} (2^{k_j-1}R)^{-n+\alpha/2} \int_{2^{k_j+1}B} |f_j(y_j)|\,dy_j \right) \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \prod_{j=1}^{2} \left(\sum_{k_j=1}^{\infty} \left|2^{k_j}B\right|^{-1+\frac{\alpha}{2n}} \|f_j\chi_{2^{k_j+1}B}\|_{L^{p_j(\cdot)}(\mathbb{R}^n)} \|\chi_{2^{k_j+1}B}\|_{L^{p_j'(\cdot)}(\mathbb{R}^n)} \right) \\
&\le C\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \prod_{j=1}^{2} \|f_j\|_{\dot{\mathcal{B}}^{p_j(\cdot),\lambda_j}(\mathbb{R}^n)} \sum_{k_j=1}^{\infty} \left|2^{k_j}B\right|^{\lambda_j-1+\frac{\alpha}{2n}} \|\chi_{2^{k_j+1}B}\|_{L^{p_j(\cdot)}(\mathbb{R}^n)} \|\chi_{2^{k_j+1}B}\|_{L^{p_j'(\cdot)}(\mathbb{R}^n)} \\
&\le C\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \prod_{j=1}^{2} \|f_j\|_{\dot{\mathcal{B}}^{p_j(\cdot),\lambda_j}(\mathbb{R}^n)} \sum_{k_j=1}^{\infty} \left|2^{k_j}B\right|^{\lambda_j-1+\frac{\alpha}{2n}} \left|2^{k_j}B\right| \\
&\le C\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \prod_{j=1}^{2} \|f_j\|_{\dot{\mathcal{B}}^{p_j(\cdot),\lambda_j}(\mathbb{R}^n)} |B|^{\lambda} \sum_{k_j=1}^{\infty} 2^{k_j n(\lambda_j+\frac{\alpha}{2n})} \\
&\le C|B|^{\lambda} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.
\end{aligned} \tag{3.7}
$$

Combining the estimates of (3.3)–(3.7), we have

$$\big\| T(f_1,f_2)\chi_B\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)},$$

that is,

$$\big\| T(f_1,f_2)\big\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)} \leq C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 3.1. □

4 Multilinear fractional integral commutators

Let $m \in \mathbb{N}$, $\vec{b} = (b_1, b_2, \ldots, b_m)$ and $b_i \in \mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)$, $i = 1, \ldots, m$. Then the multilinear commutators of fractional integral operator are defined by

$$[\vec{b}, T_{\alpha,m}](\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^m (b_i(x) - b_i(y_i)f_i(y_i)}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}}\,dy_1\cdots dy_m. \tag{4.1}$$

Theorem 4.1 *Let* $0 < \alpha < mn$, $0 < v_i < 1/n$, $\lambda_i < -\frac{\alpha}{mn}$, $\lambda = \sum_{i=1}^m v_i + \sum_{i=1}^m \lambda_i + \alpha/n$, $v_i + \lambda_i < -\alpha/n$, $p_i(\cdot)$ $(i = 1, \ldots, m)$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfy conditions* (2.1) *and* (2.2) *in Lemma* 2.1 $1/p(\cdot) = \sum_{i=1}^m 1/p_i(\cdot) - \alpha/n > 0$ *and* $1/q(\cdot) = \sum_{i=1}^m 1/u_i(\cdot) + \sum_{i=1}^m 1/p_i(\cdot) - \alpha/n$. *Then* $[\vec{b}, T_{\alpha,m}]$ *is also bounded from* $\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n) \times \cdots \times \dot{\mathcal{B}}^{p_m(\cdot),\lambda_m}(\mathbb{R}^n)$ *into* $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)$ *and the following inequality holds*:

$$\big\|[\vec{b}, T_{\alpha,m}]\vec{f}\big\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)} \leq C\prod_{i=1}^m \big(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\big).$$

Proof Without loss of generality, we still consider only the situation when $m = 2$. Let f_1, f_2 be functions in $\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)$ and $\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)$, respectively. For fixed $R > 0$, denote $B(0,R)$ by B. We have the following decomposition:

$$\begin{aligned}
[\vec{b}, T_{\alpha,2}]\vec{f}(x) &= \big[b_1 - \{b_1\}_B\big]\big[b_2 - \{b_2\}_B\big]T_{\alpha,2}(f_1,f_2)(x) \\
&\quad - \big[b_1 - \{b_1\}_B\big]T_{\alpha,2}\big[f_1, \big(b_2(\cdot) - \{b_2\}_B\big)f_2\big](x) \\
&\quad - \big[b_2 - \{b_2\}_B\big]T_{\alpha,2}\big[\big(b_1(\cdot) - \{b_1\}_B\big)f_1, f_2\big](x) \\
&\quad + T_{\alpha,2}\big[\big(b_1(\cdot) - \{b_1\}_B\big)f_1, \big(b_2(\cdot) - \{b_2\}_B\big)f_2\big](x) \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{4.2}$$

Thus, by the Minkowski inequality we write

$$\big\|[\vec{b}, T_{\alpha,2}]\vec{f}\chi_B\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq \sum_{i=1}^4 \|J_i\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} =: \sum_{i=1}^4 L_i. \tag{4.3}$$

Because of the symmetry of f_1 and f_2, we can see that the estimate of L_2 is analogous to that of L_3. Then we will estimate L_1, L_2 and L_4, respectively.

Next we will decompose f_i as $f_i(y_i) = f_i(y_i)\chi_{2B} + f_i(y_i)\chi_{(2B)^c}$, for $i = 1, 2$.

(i) For L_1, by using the Minkowski inequality we can write

$$\begin{aligned} L_1 &\le \big\| [b_1 - \{b_1\}_B][b_2 - \{b_2\}_B] T_{\alpha,2}(f_1\chi_{2B}, f_2\chi_{2B})(\cdot)\chi_B(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \big\| [b_1 - \{b_1\}_B][b_2 - \{b_2\}_B] T_{\alpha,2}(f_1\chi_{2B}, f_2\chi_{(2B)^c})(\cdot)\chi_B(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \big\| [b_1 - \{b_1\}_B][b_2 - \{b_2\}_B] T_{\alpha,2}(f_1\chi_{(2B)^c}, f_2\chi_{2B})(\cdot)\chi_B(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \big\| [b_1 - \{b_1\}_B][b_2 - \{b_2\}_B] T_{\alpha,2}(f_1\chi_{(2B)^c}, f_2\chi_{(2B)^c})(\cdot)\chi_B(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &=: L_{11} + L_{12} + L_{13} + L_{14}. \end{aligned} \tag{4.4}$$

Firstly we estimate L_{11}. Noticing that $\frac{1}{p(\cdot)} = \sum_{i=1}^2 \frac{1}{p_i(\cdot)} - \frac{\alpha}{n}$, then $\frac{1}{q(\cdot)} = \sum_{i=1}^2 \frac{1}{u_i(\cdot)} + \frac{1}{p(\cdot)}$. By Lemma 2.6 and using the boundedness of $T_{\alpha,2}$ from $L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$ in Theorem B, we get

$$\begin{aligned} L_{11} &\le C\big\| T_{\alpha,2}(f_1\chi_{2B}, f_2\chi_{2B})(\cdot)\chi_B(\cdot)\big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \prod_{i=1}^2 \big\| [b_i - \{b_i\}_B]\chi_B\big\|_{L^{u_i(\cdot)}(\mathbb{R}^n)} \\ &\le C\prod_{i=1}^2 \|f_i\chi_B\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \prod_{i=1}^2 \big\| [b_i - \{b_i\}_B]\chi_B\big\|_{L^{u_i(\cdot)}(\mathbb{R}^n)} \\ &\le C|B|^{\lambda_1+\lambda_2+\nu_1+\nu_2} \|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{u_1(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{u_2(\cdot)}(\mathbb{R}^n)} \\ &\quad \times \prod_{i=1}^2 \big(\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)} \|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}\big) \\ &\le C|B|^{\lambda_1+\lambda_2+\nu_1+\nu_2+\alpha/n+1/q(\cdot)} \prod_{i=1}^2 \big(\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)} \|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}\big) \\ &\le C|B|^{\lambda+1/q(\cdot)} \prod_{i=1}^2 \big(\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)} \|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}\big), \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{u_1(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{u_2(\cdot)}(\mathbb{R}^n)} &\approx |B|^{1/p_1(\cdot)+1/p_2(\cdot)+1/u_1(\cdot)+1/u_2(\cdot)} \\ &= |B|^{1/q(\cdot)+\alpha/n}. \end{aligned}$$

Now we estimate L_{12}. Noticing that $|(x-y_1, x-y_2)|^{2n-\alpha} \ge |x-y_2|^{2n-\alpha}$. By $\lambda_2 < -\frac{\alpha}{2n}$, Lemma 2.3 and the generalized Hölder inequality, we obtain

$$\begin{aligned} &\big| T_{\alpha,2}(f_1\chi_{2B}, f_2\chi_{(2B)^c})(x)\big| \\ &\quad = \left| \int_{(\mathbb{R}^n)^2} \frac{[f_1(y_1)\chi_{2B}(y_1)][f_2(y_2)\chi_{(2B)^c}(y_2)]}{|(x-y_1, x-y_2)|^{2n-\alpha}}\, dy_1\, dy_2 \right| \\ &\quad \le C\int_{2B} |f_1(y_1)|\, dy_1 \int_{(2B)^c} \frac{|f_2(y_2)|}{|x-y_2|^{2n-\alpha}}\, dy_2 \\ &\quad \le C\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \int_{2^{k+1}B\backslash 2^kB} \frac{|f_2(y_2)|}{|x-y_2|^{2n-\alpha}}\, dy_2 \\ &\quad \le C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2B|^{\lambda_1} \|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{k=1}^{\infty} \left(2^{k-1}R\right)^{-2n+\alpha} \int_{2^{k+1}B} \left|f_2(y_2)\right| dy_2 \\
&\leq C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2B|^{\lambda_1+1} \sum_{k=1}^{\infty} \left|2^k B\right|^{-2+\alpha/n} \|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} |B|^{\lambda_1+1-2+\alpha/n+1+\lambda_2} \sum_{k=1}^{\infty} 2^{kn(-2+\alpha/n+\lambda_2+1)} \\
&\leq C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} |B|^{\lambda_1+\lambda_2+\alpha/n} \sum_{k=1}^{\infty} 2^{kn(\frac{\alpha}{2n}+\lambda_2)} \\
&\leq C|B|^{\lambda_1+\lambda_2+\alpha/n} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}.
\end{aligned}
$$

Let $\frac{1}{u(\cdot)} = \frac{1}{u_1(\cdot)} + \frac{1}{u_2(\cdot)}$, by the fact $1/p(\cdot) = \sum_{i=1}^{2} 1/p_i(\cdot) - \alpha/n > 0$ and $1/q(\cdot) = \sum_{i=1}^{m} 1/u_i(\cdot) + \sum_{i=1}^{m} 1/p_i(\cdot) - \alpha/n$, then $u(\cdot) > q(\cdot)$. Thus by Lemma 2.5 and Lemma 2.6 we get

$$
\begin{aligned}
&\left\| \left[b_1 - \{b_1\}_B\right]\left[b_2 - \{b_2\}_B\right]\chi_B \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad\leq C\left\| \left[b_1 - \{b_1\}_B\right]\chi_B \right\|_{L^{u_1(\cdot)}(\mathbb{R}^n)} \left\| \left[b_2 - \{b_2\}_B\right]\chi_B \right\|_{L^{u_2(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\leq C\|b_1\|_{\mathrm{CBMO}^{u_1(\cdot),v_1}(\mathbb{R}^n)} |B|^{v_1} \|\chi_B\|_{L^{u_1(\cdot)}(\mathbb{R}^n)} \\
&\qquad\times \|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)} |B|^{v_2} \|\chi_B\|_{L^{u_2(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad\leq C|B|^{v_1+v_2+\frac{1}{u_1(\cdot)}+\frac{1}{u_2(\cdot)}+\frac{1}{p(\cdot)}} \prod_{i=1}^{2} \|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)} \\
&\quad\leq C|B|^{v_1+v_2+\frac{1}{q(\cdot)}} \prod_{i=1}^{2} \|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
L_{12} &\leq C|B|^{\lambda_1+\lambda_2+\alpha/n+v_1+v_2+\frac{1}{q(\cdot)}} \prod_{i=1}^{2} \left(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)} \|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)} \right) \\
&\leq C|B|^{\lambda+\frac{1}{q(\cdot)}} \prod_{i=1}^{2} \left(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)} \|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)} \right). \qquad (4.6)
\end{aligned}
$$

Similarly, we have

$$
L_{13} \leq C|B|^{\lambda+\frac{1}{q(\cdot)}} \prod_{i=1}^{2} \left(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)} \|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)} \right). \qquad (4.7)
$$

Now for the estimate of L_{14}. Note that $|(x-y_1,x-y_2)|^{2n-\alpha} \geq |x-y_1|^{n-\alpha/2}|x-y_2|^{n-\alpha/2}$. Using $\lambda_j < -\frac{\alpha}{2n}$, $j = 1, 2$ and the generalized Hölder inequality, we have

$$
\begin{aligned}
&\left| T_{\alpha,2}(f_1\chi_{(2B)^c}, f_2\chi_{(2B)^c})(x) \right| \\
&\quad\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_1(y_1)\chi_{(2B)^c}(y_1)||f_2(y_2)\chi_{(2B)^c}(y_2)|}{|(x-y_1,x-y_2)|^{2n-\alpha}} \, dy_1 \, dy_2
\end{aligned}
$$

$$\begin{aligned}
&\leq C\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\int_{2^{k_2+1}B\setminus 2^{k_2}B}\int_{2^{k_1+1}B\setminus 2^{k_1}B}\frac{|f_1(y_1)||f_2(y_2)|}{|(x-y_1,x-y_2)|^{2n-\alpha}}\,dy_1\,dy_2\\
&\leq C\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}\int_{2^{k_i+1}B\setminus 2^{k_i}B}\frac{|f_i(y_i)|}{|(x-y_i)|^{n-\alpha/2}}\,dy_i\\
&\leq C\prod_{i=1}^{2}\sum_{k_i=1}^{\infty}\bigl(2^{k_i-1}R\bigr)^{-n+\alpha/2}\int_{2^{k_i+1}B}\bigl|f_i(y_i)\bigr|\,dy_i\\
&\leq C\prod_{i=1}^{2}\sum_{k_i=1}^{\infty}\bigl|2^{k_j}B\bigr|^{-1+\frac{\alpha}{2n}}\|f_j\chi_{2^{k_j+1}B}\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k_j+1}B}\|_{L^{p_j'(\cdot)}(\mathbb{R}^n)}\\
&\leq C\prod_{i=1}^{2}\|f_j\|_{\dot{\mathcal{B}}^{p_j(\cdot),\lambda_j}(\mathbb{R}^n)}\sum_{k_i=1}^{\infty}\bigl|2^{k_j}B\bigr|^{-1+\frac{\alpha}{2n}+\lambda_j+1}\\
&\leq C|B|^{\lambda_1+\lambda_2+\alpha/n}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}.
\end{aligned}$$

Similar to the estimates for L_{12}, we have

$$L_{14}\leq C|B|^{\lambda+\frac{1}{q(\cdot)}}\prod_{i=1}^{2}\bigl(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\bigr). \tag{4.8}$$

By the estimates of L_{1j}, $j=1,2,3,4$, we get

$$L_{1}\leq C|B|^{\lambda+\frac{1}{q(\cdot)}}\prod_{i=1}^{2}\bigl(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\bigr). \tag{4.9}$$

(ii) For L_2, we obtain

$$\begin{aligned}
L_2 &\leq \bigl\|\bigl[b_1-\{b_1\}_B\bigr]T_{\alpha,2}\bigl[f_1\chi_{2B},\bigl(b_2(\cdot)-\{b_2\}_B\bigr)f_2\chi_{2B}\bigr]\chi_B\bigr\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad+\bigl\|\bigl[b_1-\{b_1\}_B\bigr]T_{\alpha,2}\bigl[f_1\chi_{(2B)^c},\bigl(b_2-\{b_2\}_B\bigr)f_2\chi_{2B}\bigr]\chi_B\bigr\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad+\bigl\|\bigl[b_1-\{b_1\}_B\bigr]T_{\alpha,2}\bigl[f_1\chi_{2B},\bigl(b_2-\{b_2\}_B\bigr)f_2\chi_{(2B)^c}\bigr]\chi_B\bigr\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad+\bigl\|\bigl[b_1-\{b_1\}_B\bigr]T_{\alpha,2}\bigl[f_1\chi_{(2B)^c},\bigl(b_2-\{b_2\}_B\bigr)f_2\chi_{(2B)^c}\bigr]\chi_B(\cdot)\bigr\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&=: L_{21}+L_{22}+L_{23}+L_{24}.
\end{aligned} \tag{4.10}$$

Let $\frac{1}{q(\cdot)}=\frac{1}{u_1(\cdot)}+\frac{1}{q_1(\cdot)}$, $\frac{1}{q_1(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{g(\cdot)}-\frac{\alpha}{n}$, $\frac{1}{g(\cdot)}=\frac{1}{p_2(\cdot)}+\frac{1}{u_2(\cdot)}$. By $\lambda_j<-\frac{\alpha}{2n}$, $j=1,2$ and boundedness of $T_{\alpha,2}$ from $L^{p_1(\cdot)}(\mathbb{R}^n)\times L^{g(\cdot)}(\mathbb{R}^n)$ into $L^{q_1(\cdot)}(\mathbb{R}^n)$, we get

$$\begin{aligned}
L_{21}&\leq C\bigl\|\bigl[b_1-\{b_1\}_B\bigr]\chi_B\bigr\|_{L^{u_1(\cdot)}(\mathbb{R}^n)}\bigl\|T_{\alpha,2}\bigl[f_1\chi_{2B},\bigl(b_2(\cdot)-\{b_2\}_B\bigr)f_2\chi_{2B}\bigr]\chi_B\bigr\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\
&\leq C\bigl\|\bigl[b_1-\{b_1\}_B\bigr]\chi_B\bigr\|_{L^{u_1(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\bigl\|\bigl[b_2-\{b_2\}_B\bigr]f_2\chi_{2B}\bigr\|_{L^{g(\cdot)}(\mathbb{R}^n)}\\
&\leq C\bigl\|\bigl[b_1-\{b_1\}_B\bigr]\chi_B\bigr\|_{L^{u_1(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\bigl\|\bigl[b_2-\{b_2\}_{2B}+\{b_2\}_{2B}-\{b_2\}_B\bigr]\chi_{2B}\bigr\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}\\
&\leq C\|b_1\|_{\mathrm{CBMO}^{u_1(\cdot),v_1}(\mathbb{R}^n)}|B|^{v_1}\|\chi_B\|_{L^{u_1(\cdot)}(\mathbb{R}^n)}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2B|^{\lambda_1}\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$\times \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2B|^{\lambda_2}\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)}|2B|^{v_2}\|\chi_{2B}\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}$$
$$\leq C|B|^{v_1+v_2+\lambda_1+\lambda_2+\frac{1}{u_1(\cdot)}+\frac{1}{u_2(\cdot)}+\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}}\prod_{i=1}^{2}\big(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\big)$$
$$\leq C|B|^{\lambda+1/q(\cdot)}\prod_{i=1}^{2}\big(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\big), \tag{4.11}$$

where

$$\big|\{b_2\}_{2B}-\{b_2\}_B\big| \leq \frac{1}{|B|}\big\|\big(b_2-\{b_2\}_{2B}\big)\chi_{2B}\big\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{u_2'(\cdot)}(\mathbb{R}^n)}$$
$$\leq C\big\|\big(b_2-\{b_2\}_{2B}\big)\chi_{2B}\big\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}\frac{1}{\|\chi_{2B}\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}}.$$

For the estimate of L_{22}, we have $|(x-y_1,x-y_2)|^{2n-\alpha}\approx|x-y_1|^{2n-\alpha}$. Using $\frac{1}{p_2'(\cdot)}=\frac{1}{g'(\cdot)}+\frac{1}{u_2(\cdot)}$, $\lambda_1<-\frac{\alpha}{2n}$, Lemmas 2.3, 2.6 and the generalized Hölder inequality, we have

$$\big|T_{\alpha,2}\big[f_1\chi_{(2B)^c},\big(b_2(\cdot)-\{b_2\}_B\big)f_2\chi_{2B}\big](x)\big|$$
$$\leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_1(y_1)\chi_{(2B)^c}(y_1)||f_2(y_2)\chi_{2B}(y_2)||b_2(y_2)-\{b_2\}_B|}{|(x-y_1,x-y_2)|^{2n-\alpha}}\,dy_1\,dy_2$$
$$\leq C\int_{(2B)^c}\frac{|f_1(y_1)|}{|x-y_1|^{2n-\alpha}}\,dy_1\int_{2B}\big|f_2(y_2)\big|\big|b_2(y_2)-\{b_2\}_B\big|\,dy_2$$
$$\leq C\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\big\|\big[b_2-\{b_2\}_B\big]\chi_{2B}\big\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|f_1(y_1)|}{|x-y_1|^{2n-\alpha}}\,dy_1$$
$$\leq C\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\big\|\big[b_2-\{b_2\}_B\big]\chi_{2B}\big\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{g'(\cdot)}(\mathbb{R}^n)}$$
$$\times\sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|f_1(y_1)|}{|x-y_1|^{2n-\alpha}}\,dy_1$$
$$\leq C\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2B|^{\lambda_2}\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)}|2B|^{v_2}\|\chi_{2B}\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}$$
$$\times\|\chi_{2B}\|_{L^{g'(\cdot)}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\big(2^{k-1}R\big)^{-2n+\alpha}\int_{2^{k+1}B}\big|f_1(y_1)\big|\,dy_1$$
$$\leq C\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2B|^{\lambda_2+v_2+1}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)}$$
$$\times\sum_{k=1}^{\infty}\big|2^kB\big|^{-2+\alpha/n}\|f_1\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}$$
$$\leq C\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|B|^{\lambda_2+v_2+1-2+\alpha/n+1+\lambda_1}$$
$$\times\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)}\sum_{k=1}^{\infty}2^{kn(-2+\alpha/n+\lambda_1+1)}$$
$$\leq C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|B|^{\lambda_1+\lambda_2+v_2+\alpha/n}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)}\sum_{k=1}^{\infty}2^{kn(\frac{\alpha}{2n}+\lambda_1)}$$
$$\leq C|B|^{\lambda_1+\lambda_2+v_2+\alpha/n}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}.$$

Thus, from $\frac{1}{q(\cdot)}=\frac{1}{u_1(\cdot)}+\frac{1}{q_1(\cdot)}$ and Lemmas 2.5, 2.6, we get

$$\begin{aligned}
L_{22} &\le C|B|^{\lambda_1+\lambda_2+\nu_2+\alpha/n}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\big\|\big[b_1-\{b_1\}_B\big]\chi_B\big\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C|B|^{\lambda_1+\lambda_2+\nu_2+\alpha/n}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}\big\|\big[b_1-\{b_1\}_B\big]\chi_B\big\|_{L^{u_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\|\chi_B\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\\
&\le C|B|^{\lambda_1+\lambda_2+\nu_2+\alpha/n}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{u_1(\cdot),\nu_1}(\mathbb{R}^n)}|B|^{\nu_1}\|\chi_B\|_{L^{u_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\|\chi_B\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\\
&\le C|B|^{\lambda+1/q(\cdot)}\prod_{i=1}^{2}\big(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\big).
\end{aligned}\tag{4.12}$$

For L_{23}, noticing that $|(x-y_1,x-y_2)|^{2n}\ge|x-y_2|^{2n}$ and $\frac{1}{p_2'(\cdot)}=\frac{1}{g'(\cdot)}+\frac{1}{u_2(\cdot)}$. By Lemmas 2.3, 2.6, $\nu_2+\lambda_2+\alpha/n<0$, the generalized Hölder inequality and the Minkowski inequality, we get

$$\begin{aligned}
&\big|T_{\alpha,2}\big[f_1\chi_{2B},\big(b_2(\cdot)-\{b_2\}_B\big)f_2\chi_{(2B)^c}\big](x)\big|\\
&\quad\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_1(y_1)\chi_{2B}(y_1)||f_2(y_2)\chi_{(2B)^c}(y_2)||b_2(y_2)-\{b_2\}_B|}{|(x-y_1,x-y_2)|^{2n-\alpha}}\,dy_1\,dy_2\\
&\quad\le C\int_{2B}\big|f_1(y_1)\big|\,dy_1\int_{(2B)^c}\frac{|f_2(y_2)||b_2(y_2)-\{b_2\}_B|}{|x-y_2|^{2n-\alpha}}\,dy_2\\
&\quad\le C\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\int_{(2B)^c}\frac{|f_2(y_2)||b_2(y_2)-\{b_2\}_B|}{|x-y_2|^{2n-\alpha}}\,dy_2\\
&\quad\le C|B|^{\lambda_1+1}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\Bigg(\sum_{k=1}^{\infty}\big|2^kB\big|^{-2+\alpha/n}\\
&\qquad\times\int_{2^{k+1}B}\big|f_2(y_2)\big|\big|b_2(y_2)-\{b_2\}_{2^{k+1}B}+\{b_2\}_{2^{k+1}B}-\{b_2\}_B\big|\,dy_2\Bigg)\\
&\quad\le C|B|^{\lambda_1+1}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\Bigg[\sum_{k=1}^{\infty}\big|2^kB\big|^{-2+\alpha/n}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\qquad\times\Big(\big\|\big(b_2-\{b_2\}_{2^{k+1}B}\big)\chi_{2^{k+1}B}\big\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}+\big|\{b_2\}_{2^{k+1}B}-\{b_2\}_B\big|\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\Big)\Bigg]\\
&\quad\le C|B|^{\lambda_1+1}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\Bigg(\sum_{k=1}^{\infty}k\big|2^kB\big|^{-2+\alpha/n+\lambda_2+\nu_2}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\\
&\qquad\times\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{g'(\cdot)}(\mathbb{R}^n)}\Bigg)\\
&\quad\le C|B|^{\lambda_1+1-2+\alpha/n+\lambda_2+\nu_2+1}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}
\end{aligned}$$

$$\times \|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)} \sum_{k=1}^{\infty} k 2^{kn(-2+\alpha/n+\lambda_2+v_2+1)}$$

$$\leq C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} |B|^{\lambda_1+\lambda_2+v_2+\alpha/n} \|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)} \sum_{k=1}^{\infty} k 2^{-kn}$$

$$\leq C|B|^{\lambda_1+\lambda_2+v_2+\alpha/n} \|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)} \prod_{i=1}^{2} \|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)},$$

where

$$\begin{aligned}
\left|\{b_2\}_{2^{k+1}B} - \{b_2\}_B\right| &\leq \sum_{j=0}^{k} \left|\{b_2\}_{2^{j+1}B} - \{b_2\}_{2^j B}\right| \\
&\leq \sum_{j=0}^{k} \frac{1}{|2^j B|} \int_{2^j B} \left|b_2(y) - \{b_2\}_{2^{j+1}B}\right| dy \\
&\leq C \sum_{j=0}^{k} \frac{1}{|2^j B|} \left\| \left(b_2(\cdot) - \{b_2\}_{2^{j+1}B}\right)\chi_{2^{j+1}B} \right\|_{L^{u_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^{j+1}B}\|_{L^{u_2'(\cdot)}(\mathbb{R}^n)} \\
&\leq C \sum_{j=0}^{k} \frac{1}{|2^j B|} \left\| \left(b_2(\cdot) - \{b_2\}_{2^{j+1}B}\right)\chi_{2^{j+1}B} \right\|_{L^{u_2(\cdot)}(\mathbb{R}^n)} \frac{|2^{j+1}B|}{\|\chi_{2^{j+1}B}\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}} \\
&\leq C\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)} \sum_{j=0}^{k} \left|2^{j+1}B\right|^{v_2} \|\chi_{2^{j+1}B}\|_{L^{u_2(\cdot)}(\mathbb{R}^n)} \frac{1}{\|\chi_{2^{j+1}B}\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}} \\
&\leq C\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),v_2}(\mathbb{R}^n)} (k+1) \left|2^{k+1}B\right|^{v_2},
\end{aligned}$$

for $v_2 > 0$.

Hence,

$$L_{23} \leq C|B|^{\lambda+1/q(\cdot)} \prod_{i=1}^{2} \left(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)} \|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)} \right). \tag{4.13}$$

For L_{24}, using Lemmas 2.3, 2.6, $v_2 + \lambda_2 + \alpha/n < 0$ and the generalized Hölder inequality, we obtain

$$\begin{aligned}
&\left| T_{\alpha,2}\left[f_1\chi_{(2B)^c}, \left(b_2(\cdot) - \{b_2\}_B\right) f_2\chi_{(2B)^c}\right](x) \right| \\
&\quad \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_1(y_1)\chi_{(2B)^c}(y_1)||f_2(y_2)\chi_{(2B)^c}(y_2)||b_2(y_2) - \{b_2\}_B|}{|(x-y_1, x-y_2)|^{2n-\alpha}} \, dy_1 \, dy_2 \\
&\quad \leq C \int_{(2B)^c} \frac{|f_1(y_1)|}{|x-y_1|^{n-\alpha/2}} \, dy_1 \int_{(2B)^c} \frac{|f_2(y_2)||b_2(y_2) - \{b_2\}_B|}{|x-y_2|^{n-\alpha/2}} \, dy_2 \\
&\quad \leq C \sum_{k_1=1}^{\infty} \left|2^{k_1}B\right|^{-1+\frac{\alpha}{2n}} \int_{2^{k_1+1}B} \left|f_1(y_1)\right| dy_1 \\
&\qquad \times \sum_{k_2=1}^{\infty} \left|2^{k_2}B\right|^{-1+\frac{\alpha}{2n}} \int_{2^{k_2+1}B} \left|f_2(y_2)\right| \left|b_2(y_2) - \{b_2\}_{2^{k_2+1}B} + \{b_2\}_{2^{k_2+1}B} - \{b_2\}_B\right| dy_2
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{k_1=1}^{\infty}\left|2^{k_1}B\right|^{-1+\frac{\alpha}{2n}}\left|2^{k_1}B\right|^{\lambda_1+1}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \\
&\quad\times\sum_{k_2=1}^{\infty}k_2\left|2^{k_2}B\right|^{\nu_2+\lambda_2+\frac{\alpha}{2n}}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \\
&\le C|B|^{\lambda_1+\lambda_2+\nu_2+\alpha/n}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}\prod_{i=1}^{2}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus

$$
L_{24}\le C|B|^{\lambda+1/q(\cdot)}\prod_{i=1}^{2}\left(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\right). \tag{4.14}
$$

Combining the estimates of L_{2j}, $j=1,2,3,4$, we can deduce that

$$
L_{2}\le C|B|^{\lambda+1/q(\cdot)}\prod_{i=1}^{2}\left(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\right). \tag{4.15}
$$

(iii) For L_4, we have

$$
\begin{aligned}
L_4 &\le \left\|T_{\alpha,2}\left[\left(b_1-\{b_1\}_B\right)f_1\chi_{2B},\left(b_2(\cdot)-\{b_2\}_B\right)f_2\chi_{2B}\right]\chi_B\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad+\left\|T_{\alpha,2}\left[\left(b_1-\{b_1\}_B\right)f_1\chi_{2B},\left(b_2-\{b_2\}_B\right)f_2\chi_{(2B)^c}\right]\chi_B\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad+\left\|T_{\alpha,2}\left[b_1-\{b_1\}_B\right]\left[f_1\chi_{(2B)^c},\left(b_2-\{b_2\}_B\right)f_2\chi_{2B}\right]\chi_B\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad+\left\|T_{\alpha,2}\left[\left(b_1-\{b_1\}_B\right)f_1\chi_{(2B)^c},\left(b_2-\{b_2\}_B\right)f_2\chi_{(2B)^c}\right]\chi_B(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&=: L_{41}+L_{42}+L_{43}+L_{44}.
\end{aligned} \tag{4.16}
$$

For L_{41}, let $\frac{1}{h_i(\cdot)}=\frac{1}{p_i(\cdot)}+\frac{1}{u_i(\cdot)}$, $i=1,2$, then $1/q(\cdot)=\sum_{i=1}^{2}1/h_i(\cdot)-\alpha/n$. Using Lemmas 2.3, 2.5, 2.6 and the boundedness of $T_{\alpha,2}$ from $L^{h_1(\cdot)}(\mathbb{R}^n)\times L^{h_2(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$, we have

$$
\begin{aligned}
L_{41} &\le C\prod_{i=1}^{2}\left\|\left(b_i(\cdot)-\{b_i\}_B\right)f_i(\cdot)\chi_{2B}(\cdot)\right\|_{L^{h_i(\cdot)}(\mathbb{R}^n)} \\
&\le C\prod_{i=1}^{2}\left\|f_i(\cdot)\chi_{2B}(\cdot)\right\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\left\|\left[b_i-\{b_i\}_{2B}+\{b_i\}_{2B}-\{b_i\}_B\right]\chi_{2B}\right\|_{L^{u_i(\cdot)}(\mathbb{R}^n)} \\
&\le C|B|^{\nu_1+\nu_2+\lambda_1+\lambda_2+\frac{1}{u_1(\cdot)}+\frac{1}{u_2(\cdot)}+\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}}\prod_{i=1}^{2}\left(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\right) \\
&\le C|B|^{\lambda+1/q(\cdot)}\prod_{i=1}^{2}\left(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\right).
\end{aligned} \tag{4.17}
$$

For L_{42}, using Lemmas 2.3, 2.6, $\nu_2 + \lambda_2 + \alpha/n < 0$ and the generalized Hölder inequality, we get

$$\begin{aligned}
&\bigl|T_{\alpha,2}\bigl[\bigl(b_1(\cdot)-\{b_1\}_B\bigr)f_1\chi_{2B},\bigl(b_2(\cdot)-\{b_2\}_B\bigr)f_2\chi_{(2B)^c}\bigr](x)\bigr| \\
&\quad\le C\int_{(2B)^c}\int_{2B}\frac{|b_1(y_1)-\{b_1\}_B||f_1(y_1)||b_2(y_2)-\{b_2\}_B||f_2(y_2)|}{|(x-y_1,x-y_2)|^{2n-\alpha}}\,dy_1\,dy_2 \\
&\quad\le C\biggl(\int_{2B}\bigl|b_1(y_1)-\{b_1\}_B\bigr|\bigl|f_1(y_1)\bigr|\,dy_1\biggr)\biggl(\int_{(2B)^c}\frac{|b_2(y_2)-\{b_2\}_B||f_2(y_2)|}{|x-y_2|^{2n-\alpha}}\,dy_2\biggr) \\
&\quad\le C\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\bigl\|\bigl(b_1-\{b_1\}_B\bigr)\chi_{2B}\bigr\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\int_{(2B)^c}\frac{|b_2(y_2)-\{b_2\}_B||f_2(y_2)|}{|x-y_2|^{2n-\alpha}}\,dy_2 \\
&\quad\le C|B|^{\lambda_1+\nu_1+1}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{u_1(\cdot),\nu_1}(\mathbb{R}^n)}\biggl(\sum_{k=1}^{\infty}\bigl|2^kB\bigr|^{-2+\alpha/n} \\
&\qquad\times\int_{2^{k+1}B}\bigl|b_2(y_2)-\{b_2\}_B\bigr|\bigl|f_2(y_2)\bigr|\,dy_2\biggr) \\
&\quad\le C|B|^{\lambda_1+\nu_1+1-2+\alpha/n+\lambda_2+\nu_2+1}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{u_1(\cdot),\nu_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \\
&\qquad\times\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}\sum_{k=1}^{\infty}k2^{kn(-2+\alpha/n+\lambda_2+\nu_2+1)} \\
&\quad\le C|B|^{\lambda}\prod_{i=1}^{2}\bigl(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\bigr).
\end{aligned}$$

This implies that

$$L_{42}\le C|B|^{\lambda+1/q(\cdot)}\prod_{i=1}^{2}\bigl(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\bigr). \tag{4.18}$$

Similarly,

$$L_{43}\le C|B|^{\lambda+1/q(\cdot)}\prod_{i=1}^{2}\bigl(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\bigr). \tag{4.19}$$

For L_{44}, using Lemmas 2.3, 2.6, $\nu_i + \lambda_i + \alpha/n < 0$, $i = 1,2$ and the generalized Hölder inequality, we get

$$\begin{aligned}
&\bigl|T_{\alpha,2}\bigl[\bigl(b_1(\cdot)-\{b_1\}_B\bigr)f_1\chi_{(2B)^c},\bigl(b_2(\cdot)-\{b_2\}_B\bigr)f_2\chi_{(2B)^c}\bigr](x)\bigr| \\
&\quad\le C\int_{(2B)^c}\int_{(2B)^c}\frac{|b_1(y_1)-\{b_1\}_B||f_1(y_1)||b_2(y_2)-\{b_2\}_B||f_2(y_2)|}{|(x-y_1,x-y_2)|^{2n-\alpha}}\,dy_1\,dy_2 \\
&\quad\le C\prod_{i=1}^{2}\biggl(\int_{(2B)^c}\frac{|b_i(y_i)-\{b_i\}_B||f_i(y_i)|}{|x-y_i|^{n-\alpha/2}}\,dy_i\biggr) \\
&\quad\le C\prod_{i=1}^{2}\biggl(\sum_{k_i=1}^{\infty}\bigl|2^{k_i}B\bigr|^{-1+\frac{\alpha}{2n}}\int_{2^{k_i+1}B}\bigl|b_i(y_i)-\{b_i\}_B\bigr|\bigl|f_i(y_i)\bigr|\,dy_i\biggr)
\end{aligned}$$

$$\leq C|B|^{\lambda}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\left(\sum_{k_i=1}^{\infty}k_i2^{k_in(v_i+\lambda_i+\frac{\alpha}{2n})}\right)$$
$$\leq C|B|^{\lambda}\prod_{i=1}^{2}\bigl(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\bigr).$$

So

$$L_{44}\leq C|B|^{\lambda+1/q(\cdot)}\prod_{i=1}^{2}\bigl(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\bigr). \tag{4.20}$$

From the estimates of L_{4j}, $j = 1, 2, 3, 4$, we have

$$L_{4}\leq C|B|^{\lambda+1/q(\cdot)}\prod_{i=1}^{2}\bigl(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\bigr). \tag{4.21}$$

Furthermore, we obtain

$$\bigl\|[\vec{b},T_{\alpha,2}]\vec{f}\chi_B\bigr\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leq C|B|^{\lambda+1/q(\cdot)}\prod_{i=1}^{2}\bigl(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\bigr).$$

Thus, we have

$$\bigl\|[\vec{b},T_{\alpha,2}]\vec{f}\bigr\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)}\leq C\prod_{i=1}^{2}\bigl(\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}\|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}\bigr).$$

This completes the proof of Theorem 4.1. □

5 Multilinear fractional integral commutators of the second kind

There is another kind of multilinear commutators $[\vec{b},T_\alpha]$, which was introduced by Pérez and Trujillo-González [25] in 2002, with the vector symbol $\vec{b}=(b_1,b_2,\dots,b_m)$ defined by

$$[\vec{b},T_\alpha]f(x)=\int_{\mathbb{R}^n}\frac{\prod_{i=1}^{m}(b_i(x)-b_i(y))f(y)}{|(x-y)|^{n-\alpha}}\,dy, \tag{5.1}$$

where $b_i\in\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)$, $i=1,\dots,m$. We have the following result.

Theorem 5.1 *Let* $0<\alpha<n$, $0<v_i<1/n$, $\lambda=\sum_{i=1}^{m}v_i+\mu+\alpha/n<0$, $p(\cdot),q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ *satisfy conditions* (2.1) *and* (2.2) *in Lemma* 2.1 $1/p(\cdot)>\alpha/n$ *and* $1/q(\cdot)=\sum_{i=1}^{m}1/u_i(\cdot)+1/p(\cdot)-\alpha/n$. *Then* $[\vec{b},T_\alpha]$ *is bounded from* $\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)$ *into* $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)$ *and the following inequality holds*:

$$\bigl\|[\vec{b},T_\alpha]f\bigr\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)}\leq C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\prod_{i=1}^{m}\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}.$$

Proof Without loss of generality, we can assume that $m=2$. For any fixed $R>0$, denote $B(0,R)$ by B and $B(0,kR)$ by kB for $k\in\mathbb{N}$. Let $\{b\}_E$ denote the integral average of the function b over the set E. For $f\in\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)$ and any $x\in\mathbb{R}^n$, we write

$$f(x)=f(x)\chi_{2B}+f(x)\chi_{(2B)^c}=:f_1(x)+f_2(x),$$

and then we may decompose $[\vec{b}, T_\alpha]\vec{f}(x)$ into four parts as follows:

$$\begin{aligned}[\vec{b}, T_\alpha]f(x) = {} & \bigl[b_1 - \{b_1\}_B\bigr]\bigl[b_2 - \{b_2\}_B\bigr]T_\alpha(f)(x) \\ & - \bigl[b_1 - \{b_1\}_B\bigr]T_\alpha\bigl[\bigl(b_2(\cdot) - \{b_2\}_B\bigr)f\bigr](x) \\ & - \bigl[b_2 - \{b_2\}_B\bigr]T_\alpha\bigl[\bigl(b_1(\cdot) - \{b_1\}_B\bigr)f\bigr](x) \\ & + T_\alpha\bigl[\bigl(b_1(\cdot) - \{b_1\}_B\bigr)\bigl(b_2(\cdot) - \{b_2\}_B\bigr)f\bigr](x) \\ =: {} & M_1(x) + M_2(x) + M_3(x) + M_4(x). \end{aligned} \tag{5.2}$$

Now we will give the estimates of four functions above, respectively.

(i) For $M_1(x)$, using the Minkowski inequality we write

$$\begin{aligned}\|M_1\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le {} & \bigl\|\bigl[b_1(\cdot) - \{b_1\}_B\bigr]\bigl[b_2(\cdot) - \{b_2\}_B\bigr]T_\alpha f_1(\cdot)\chi_B(\cdot)\bigr\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ & + \bigl\|\bigl[b_1(\cdot) - \{b_1\}_B\bigr]\bigl[b_2(\cdot) - \{b_2\}_B\bigr]T_\alpha f_2(\cdot)\chi_B(\cdot)\bigr\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ =: {} & M_{11} + M_{12}. \end{aligned} \tag{5.3}$$

Firstly we estimate M_{11}. Let $\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, then $\frac{1}{q(\cdot)} = \sum_{i=1}^{2} \frac{1}{u_i(\cdot)} + \frac{1}{r(\cdot)}$. By Lemmas 2.3, 2.6 and the boundedness of T_α from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{r(\cdot)}(\mathbb{R}^n)$ in Theorem 1.8 of [2], we have

$$\begin{aligned}M_{11} & \le C\bigl\|T_\alpha f_1(\cdot)\chi_B(\cdot)\bigr\|_{L^{r(\cdot)}(\mathbb{R}^n)} \prod_{i=1}^{2} \bigl\|\bigl[b_i - \{b_i\}_B\bigr]\chi_B\bigr\|_{L^{u_i(\cdot)}(\mathbb{R}^n)} \\ & \le C|B|^{\lambda+1/q(\cdot)}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)} \prod_{i=1}^{2} \|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}. \end{aligned} \tag{5.4}$$

Since

$$\begin{aligned}\bigl|T_\alpha f_2(x)\bigr| & = \biggl|\int_{\mathbb{R}^n} \frac{f_2(y)}{|x-y|^{n-\alpha}}\,dy\biggr| \\ & \le \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^k B} \bigl|f(y)\bigr|\,dy \\ & \le C\sum_{k=1}^{\infty} \bigl|2^k B\bigr|^{-1+\alpha/n} \|f\chi_{2^{k+1}B}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{2^{k+1}B}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ & \le C|B|^{\mu+\alpha/n}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}, \end{aligned}$$

using Lemma 2.6 and the generalized Hölder inequality, we obtain

$$M_{12} \le C|B|^{\lambda+1/q(\cdot)}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)} \prod_{i=1}^{2} \|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),v_i}(\mathbb{R}^n)}. \tag{5.5}$$

(ii) For $M_2(x)$, let $\frac{1}{q_1(\cdot)} = \frac{1}{u_2(\cdot)} + \frac{1}{p(\cdot)} - \frac{\alpha}{n}$ and then $\frac{1}{q(\cdot)} = \frac{1}{u_1(\cdot)} + \frac{1}{q_1(\cdot)}$. Using Lemma 2.5 and Lemma 2.6 we get

$$\begin{aligned}\|M_2\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C|B|^{\nu_1+\frac{1}{u_1(\cdot)}}\|b_1\|_{\mathrm{CBMO}^{u_1(\cdot),\nu_1}(\mathbb{R}^n)}\big\|T_\alpha\big[b_2-\{b_2\}_B\big]f(\cdot)\chi_B(\cdot)\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\ &\le C|B|^{\nu_1+\frac{1}{u_1(\cdot)}}\|b_1\|_{\mathrm{CBMO}^{u_1(\cdot),\nu_1}(\mathbb{R}^n)}(M_{21}+M_{22}),\end{aligned} \tag{5.6}$$

where

$$M_{21} =: \big\|T_\alpha\big[b_2-\{b_2\}_B\big]f_1(\cdot)\chi_B(\cdot)\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$$

and

$$M_{22} =: \big\|T_\alpha\big[b_2-\{b_2\}_B\big]f_2(\cdot)\chi_B(\cdot)\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.$$

Let $\frac{1}{l(\cdot)} = \frac{1}{u_2(\cdot)} + \frac{1}{p(\cdot)}$, then $\frac{1}{q_1(\cdot)} = \frac{1}{l(\cdot)} - \frac{\alpha}{n}$. Using the $(L^{l(\cdot)}, L^{q_1(\cdot)})$-boundedness of T_α, Lemma 2.5 and Lemma 2.6 we have

$$\begin{aligned}M_{21} &\le C\big\|\big(b_2(\cdot)-\{b_2\}_B\big)f_1\big\|_{L^{l(\cdot)}(\mathbb{R}^n)}\\ &\le C\big\|\big(b_2(\cdot)-\{b_2\}_B\big)\chi_{2B}\big\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}\|f\chi_{2B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C|B|^{\mu}\|\chi_{2B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\big\|\big[b_2-\{b_2\}_{2B}+\{b_2\}_{2B}-\{b_2\}_B\big]\chi_{2B}\big\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}\\ &\le C|B|^{\mu}\|\chi_{2B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}|2B|^{\nu_2}\|\chi_{2B}\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}\\ &\le C|B|^{\mu+\nu_2}\|\chi_{2B}\|_{L^{l(\cdot)}(\mathbb{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}.\end{aligned} \tag{5.7}$$

For M_{22}, by using $\frac{1}{u_2'(\cdot)} = \frac{1}{l'(\cdot)} + \frac{1}{p(\cdot)}$, $\mu + \nu_2 + \frac{\alpha}{n} < 0$, Lemmas 2.3, 2.5, 2.6 and the generalized Hölder inequality, we get

$$\begin{aligned}&\big|T_\alpha\big(b_2(\cdot)-\{b_2\}_B\big)f_2(x)\big|\\ &\quad\le C\int_{(2B)^c}\frac{|b_2(y)-\{b_2\}_B||f(y)|}{|x-y|^{n-\alpha}}\,dy\\ &\quad\le C\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{|b_2(y)-\{b_2\}_B||f(y)|}{|x-y|^{n-\alpha}}\,dy\\ &\quad\le C\sum_{k=1}^{\infty}\big|2^kB\big|^{-1+\alpha/n}\big\|\big(b_2(\cdot)-\{b_2\}_B\big)\chi_{2^{k+1}B}\big\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{l'(\cdot)}(\mathbb{R}^n)}\\ &\quad\le C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}\sum_{k=1}^{\infty}k\big|2^kB\big|^{\alpha/n+\mu+\nu_2}\\ &\quad\le C|B|^{\alpha/n+\mu+\nu_2}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}\sum_{k=1}^{\infty}k2^{kn(\alpha/n+\mu+\nu_2)}\\ &\quad\le C|B|^{\alpha/n+\mu+\nu_2}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}.\end{aligned}$$

Thus

$$\begin{aligned} M_{22} &\le C|B|^{\alpha/n+\mu+\nu_2+\frac{1}{q_1(\cdot)}}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)} \\ &\le C|B|^{\mu+\nu_2+\frac{1}{u_2(\cdot)}+\frac{1}{p(\cdot)}}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}. \end{aligned} \tag{5.8}$$

Combining the estimates for (5.6)–(5.8), we obtain

$$\|M_2\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C|B|^{\lambda+1/q(\cdot)}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}. \tag{5.9}$$

(iii) Observing that M_3 is symmetric to M_2, we have

$$\|M_3\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C|B|^{\lambda+1/q(\cdot)}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}. \tag{5.10}$$

(iv) Finally, we split M_4 as follows:

$$\begin{aligned} \|M_4\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le{}& \big\|T_\alpha\big[b_1(\cdot)-\{b_1\}_B\big]\big[b_2(\cdot)-\{b_2\}_B\big]f_1(\cdot)\chi_B(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &+\big\|T_\alpha\big[b_1(\cdot)-\{b_1\}_B\big]\big[b_2(\cdot)-\{b_2\}_B\big]f_2(\cdot)\chi_B(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ =:{}& M_{41}+M_{42}. \end{aligned} \tag{5.11}$$

Let $\frac{1}{t(\cdot)}=\frac{1}{u_1(\cdot)}+\frac{1}{u_2(\cdot)}+\frac{1}{p(\cdot)}$, then by the $(L^{t(\cdot)}(\mathbb{R}^n),L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of T_α we have

$$\begin{aligned} M_{41} &\le C\big\|\big(b_1(\cdot)-\{b_1\}_B\big)\big(b_2(\cdot)-\{b_2\}_B\big)f_1\big\|_{L^{t(\cdot)}(\mathbb{R}^n)} \\ &\le C\big\|\big(b_1(\cdot)-\{b_1\}_B\big)\chi_{2B}\big\|_{L^{u_1(\cdot)}(\mathbb{R}^n)}\big\|\big(b_2(\cdot)-\{b_2\}_B\big)\chi_{2B}\big\|_{L^{u_2(\cdot)}(\mathbb{R}^n)}\|f\chi_{2B}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C|B|^{\mu+\nu_2}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}. \end{aligned} \tag{5.12}$$

For M_{42}, using $\mu+\nu_1+\nu_2+\alpha/n<0$, Lemmas 2.3, 2.5, 2.6 and the generalized Hölder inequality, we get

$$\begin{aligned} &\big|T_\alpha\big(b_1(\cdot)-\{b_1\}_B\big)\big(b_2(\cdot)-\{b_2\}_B\big)f_2(x)\big| \\ &\quad\le C\int_{(2B)^c}\frac{|b_1(y)-\{b_1\}_B||b_2(y)-\{b_2\}_B||f(y)|}{|x-y|^{n-\alpha}}\,dy \\ &\quad\le C\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{|b_1(y)-\{b_1\}_B||b_2(y)-\{b_2\}_B||f(y)|}{|x-y|^{n-\alpha}}\,dy \\ &\quad\le C\sum_{k=1}^{\infty}\big|2^kB\big|^{-1+\alpha/n}\big\|\big(b_1(\cdot)-\{b_1\}_B\big)\chi_{2^{k+1}B}\big\|_{L^{u_1(\cdot)}(\mathbb{R}^n)}\big\|\big(b_2(\cdot)-\{b_2\}_B\big)\chi_{2^{k+1}B}\big\|_{L^{u_2(\cdot)}(\mathbb{R}^n)} \\ &\qquad\times\|f\chi_{2^{k+1}B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{t'(\cdot)}(\mathbb{R}^n)} \\ &\quad\le C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{u_1(\cdot),\nu_1}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)}\sum_{k=1}^{\infty}k^2\big|2^kB\big|^{\alpha/n+\mu+\nu_1+\nu_2} \end{aligned}$$

$$\begin{aligned}
&\leq C|B|^{\alpha/n+\mu+\nu_1+\nu_2}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{u_1(\cdot),\nu_1}(\mathbb{R}^n)}\|b_2\|_{\mathrm{CBMO}^{u_2(\cdot),\nu_2}(\mathbb{R}^n)} \\
&\quad\times\sum_{k=1}^{\infty}k^2 2^{kn(\alpha/n+\mu+\nu_1+\nu_2)} \\
&\leq C|B|^{\lambda}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}.
\end{aligned}$$

Thus

$$M_{42}\leq C|B|^{\lambda+\frac{1}{q(\cdot)}}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}. \tag{5.13}$$

In combination with the estimates of M_{41} and M_{42}, we have

$$\|M_4\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leq C|B|^{\lambda+1/q(\cdot)}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}. \tag{5.14}$$

To sum up, combining the estimates of (i)–(iv),

$$\big\|[\vec{b},T_\alpha]f\big\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)}\leq C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\mu}(\mathbb{R}^n)}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{u_i(\cdot),\nu_i}(\mathbb{R}^n)}. \qquad \square$$

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Authors' contributions
HW put forward the ideas of the paper, and the authors completed the paper together. All authors read and approved the final manuscript.

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