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Limit theorems for ratios of order statistics from uniform distributions

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Abstract

Let $\{X_{ni}, 1 \leq i \leq m_n, n \geq 1\}$ be an array of independent random variables with uniform distribution on $[0, \theta_n]$, and $\{X_{n(k)}, k = 1, 2, \ldots, m_n\}$ be the kth order statistics of the random variables $\{X_{ni}, 1 \leq i \leq m_n\}$. We study the limit properties of ratios $\{R_{nij} = X_{n(j)}/X_{n(i)}, 1 \leq i < j \leq m_n\}$ for fixed sample size $m_n = m$ based on their moment conditions. For $1 = i < j \leq m$, we establish the weighted law of large numbers, the complete convergence, and the large deviation principle, and for $2 = i < j \leq m$, we obtain some classical limit theorems and self-normalized limit theorems.

MSC: 60F15; 62G30

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1 Introduction

Let $\{X_{ni}, 1 \leq i \leq m_n, n \geq 1\}$ be an array of independent random variables with uniform distribution on $[0, \theta_n]$, and $\{X_{n(k)}, k = 1, 2, \ldots, m_n\}$ be the kth order statistics of the random variables $\{X_{ni}, 1 \leq i \leq m_n\}$ for given n. In contrast to the commonly used models of order statistics, such as the extreme values, the median, the sample range, and the linear functions of order statistics, some researchers are interested in the investigation of limit behaviors for the ratios of these order statistics

$$R_{nij} = \frac{X_{n(j)}}{X_{n(i)}}, \quad 1 \leq i < j \leq m_n.$$

To derive the density function of R_{nij}, we recall that the joint density function of $X_{n(i)}$ and $X_{n(j)}$ is

$$f(x_i, x_j) = \frac{m_n! x_i^{i-1} (x_j - x_i)^{j-i-1} (\theta_n - x_j)^{m_n-j}}{(i-1)!(j-i-1)!(m_n-j)!\theta_n^{m_n}} I(0 \leq x_i < x_j \leq \theta_n).$$

Let $\omega = x_i$, $r = x_j/x_i$. Then the Jacobian of the transformation is ω, so that the joint density function of $X_{n(i)}$ and R_{nij} is

$$f(r, \omega) = \frac{m_n! \omega^{j-1} (r-1)^{j-i-1} (\theta_n - r\omega)^{m_n-j}}{(i-1)!(j-i-1)!(m_n-j)!\theta_n^{m_n}} I(\omega \geq 0, r > 1, r\omega \leq \theta_n).$$

Therefore, the density function of the ratio R_{nij} is

$$
\begin{aligned}
f_{R_{nij}}(r) &= \frac{m_n!(r-1)^{j-i-1}}{(i-1)!(j-i-1)!(m_n-j)!\theta_n^{m_n}} \int_0^{\theta_n/r} \omega^{j-1}(\theta_n - r\omega)^{m_n-j}\, d\omega \\
&= \frac{m_n!}{(i-1)!(j-i-1)!(m_n-j)!\theta_n^{m_n}} \frac{(r-1)^{j-i-1}}{r^j} \int_0^{\theta_n} (\theta_n - u)^{j-1} u^{m_n-j}\, du \\
&= \frac{m_n!}{(i-1)!(j-i-1)!(m_n-j)!} \sum_{k=0}^{j-1} \frac{(-1)^k C_{j-1}^k}{m_n + k - j + 1} \frac{(r-1)^{j-i-1}}{r^j} I(r>1),
\end{aligned}
\tag{1.1}
$$

which is not related to θ_n. Thus, for fixed i, j and the sample size $m_n = m$, $\{R_{nij}, n \geq 1\}$ is a sequence of independent identically distributed (i.i.d.) random variables, which allows us to obtain better limit behaviors for $\{R_{nij}, n \geq 1\}$, although the random variables in different rows of the array $\{X_{ni}, 1 \leq i \leq m_n, n \geq 1\}$ possess different uniform distributions.

In this paper, some generalized results of $\{R_{nij}, n \geq 1\}$ for the fixed sample size $m_n = m$ are established based on the works by Adler [1], Miao et al. [7], and Xu and Miao [11]. In Sect. 2, we first derive the moments of R_{nij}, on which the limit theorems for R_{n1j} and R_{n2j} are established in Sects. 3 and 4, respectively.

Throughout this paper, let $\log x = \ln \max\{e, x\}$, where "ln" is the natural logarithm. Denote by C a generic positive real number, which is not necessarily the same value in each appearance, and by $I(B)$ the indicator function of set B.

2 Moments of R_{nij}

It is known from (1.1) that the density function of R_{n1j} is

$$
f_{R_{n1j}}(r) = \gamma_{m,1,j} \frac{(r-1)^{j-2}}{r^j} I(r>1),
\tag{2.1}
$$

where

$$
\gamma_{m,1,j} = \frac{m!}{(j-2)!(m-j)!} \sum_{k=0}^{j-1} \frac{(-1)^k C_{j-1}^k}{m+k-j+1},
$$

and the density function of R_{n2j} is

$$
f_{R_{n2j}}(r) = \gamma_{m,2,j} \frac{(r-1)^{j-3}}{r^j} I(r>1),
\tag{2.2}
$$

where

$$
\gamma_{m,2,j} = \frac{m!}{(j-3)!(m-j)!} \sum_{k=0}^{j-1} \frac{(-1)^k C_{j-1}^k}{m+k-j+1}.
$$

Theorem 2.1

(I) *For $1 < j \leq m$, we have*

$$
\begin{cases}
\mathbb{E} R_{n1j}^\beta < \infty, & 0 < \beta < 1, \\
\mathbb{E} R_{n1j}^\beta = \infty, & \beta \geq 1.
\end{cases}
$$

(II) *For $2 < j \leq m$, we have*

$$\begin{cases} \mathbb{E}R_{n2j}^{\beta} < \infty, & 0 < \beta < 2, \\ \mathbb{E}R_{n2j}^{\beta} = \infty, & \beta \geq 2. \end{cases}$$

(III) *Let $L(x) = \mathbb{E}R_{n2j}^2 I(|R_{n2j}| \leq x)$ and $\widetilde{L}(x) = \mathbb{E}(R_{n2j} - \mathbb{E}R_{n2j})^2 I(|R_{n2j} - \mathbb{E}R_{n2j}| \leq x)$. Then both $L(x)$ and $\widetilde{L}(x)$ are slowly varying functions.*

Proof (I) It is easily known from (2.1) that $f_{R_{n1j}}(r) \leq \gamma_{m,1,j}/r^2$ and $f_{R_{n1j}}(r) \sim \gamma_{m,1,j}/r^2$ as $r \to \infty$. Therefore, $\mathbb{E}R_{n1j}^{\beta} < \infty$ for $0 < \beta < 1$ and $\mathbb{E}R_{n1j}^{\beta} = \infty$ for $\beta \geq 1$, respectively.

(II) Similarly, since $f_{R_{n2j}}(r) \leq \gamma_{m,2,j}/r^3$ and $f_{R_{n2j}}(r) \sim \gamma_{m,2,j}/r^3$ as $r \to \infty$, then we have $\mathbb{E}R_{n2j}^{\beta} < \infty$ for $0 < \beta < 2$ and $\mathbb{E}R_{n2j}^{\beta} = \infty$ for $\beta \geq 2$, respectively.

(III) For any $\lambda > 0$,

$$\begin{aligned} L(\lambda x) &= \int_1^{\lambda x} r^2 f_{R_{n2j}}(r)\, dr \\ &\sim \int_1^{\lambda x} r^2 \frac{\gamma_{m,2,j}}{r^3}\, dr \quad \text{as } x \to \infty. \end{aligned}$$

It is not difficult to prove that $L(\lambda x) \sim \gamma_{m,2,j} \log x \sim L(x)$ as $x \to \infty$, implying that $L(x)$ varies slowly at $x \to \infty$, so does $\widetilde{L}(x)$ as $\widetilde{L}(\lambda x) \sim \gamma_{m,2,j} \log x \sim \widetilde{L}(x)$. $\square$

Remark 2.1 With similar derivations, it can be obtained that the second moment of R_{nij} is finite for all $3 \leq i < j \leq m$. Therefore, a large number of classical limit properties of $\{R_{nij}, n \geq 1\}$ for all $3 \leq i < j \leq m$ can be established easily. Hence, to study the limit behaviors for $\{R_{nij}, n \geq 1\}$, we are only interested in $\{R_{n1j}, n \geq 1\}$ and $\{R_{n2j}, n \geq 1\}$.

3 Limit properties of R_{n1j}

According to part (I) of Theorem 2.1, $\mathbb{E}R_{n1j} = \infty$. It follows that the classical strong law of large numbers for $\{R_{n1j}, n \geq 1\}$ fails. Hence, we give the following weighted strong law of large numbers.

Theorem 3.1 *Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two positive sequences satisfying the following conditions:*

(i) *b_n is nondecreasing, $b_n \to \infty$ as $n \to \infty$, and $\sum_{n=1}^{\infty} \frac{a_n}{b_n} < \infty$;*

(ii) *$\frac{1}{b_N} \sum_{n=1}^{N} a_n \log(\frac{b_n}{a_n}) \to \lambda < \infty$ as $N \to \infty$.*

Then we have

$$\lim_{N \to \infty} \frac{1}{b_N} \sum_{n=1}^{N} a_n R_{n1j} = \gamma_{m,1,j}\lambda \quad \text{almost surely,}$$

where $\gamma_{m,1,j}$ is the same number as that in (2.1).

Proof Let $c_n = b_n/a_n$. Then $c_n \to \infty$ follows from condition (i). Without loss of generality, we assume that $c_n \geq 1$ for all $n \geq 1$. Notice the following partition:

$$\frac{1}{b_N} \sum_{n=1}^{N} a_n R_{n1j} = I_N + II_N + III_N, \tag{3.1}$$

where

$$I_N = \frac{1}{b_N} \sum_{n=1}^{N} a_n \big[R_{n1j} I(1 \leq R_{n1j} \leq c_n) - \mathbb{E} R_{n1j} I(1 \leq R_{n1j} \leq c_n) \big],$$

$$II_N = \frac{1}{b_N} \sum_{n=1}^{N} a_n R_{n1j} I(R_{n1j} > c_n),$$

$$III_N = \frac{1}{b_N} \sum_{n=1}^{N} a_n \mathbb{E} R_{n1j} I(1 \leq R_{n1j} \leq c_n).$$

Firstly, it can be established by condition (i) that

$$\sum_{n=1}^{\infty} \frac{1}{c_n^2} \mathbb{E} R_{n1j}^2 I(1 \leq R_{n1j} \leq c_n)$$

$$= \sum_{n=1}^{\infty} \frac{\gamma_{m,1,j}}{c_n^2} \int_1^{c_n} \frac{(r-1)^{j-2}}{r^{j-2}} \, dr$$

$$\leq C \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_1^{c_n} 1 \, dr \leq C \sum_{n=1}^{\infty} \frac{1}{c_n}$$

$$= C \sum_{n=1}^{\infty} \frac{a_n}{b_n} < \infty, \tag{3.2}$$

and for any $0 < \varepsilon < 1$,

$$\sum_{n=1}^{\infty} P\big(R_{n1j} I(R_{n1j} > c_n) > \varepsilon \big)$$

$$= \sum_{n=1}^{\infty} P(R_{n1j} > c_n)$$

$$= \sum_{n=1}^{\infty} \gamma_{m,1,j} \int_{c_n}^{\infty} \frac{(r-1)^{j-2}}{r^j} \, dr$$

$$\leq C \sum_{n=1}^{\infty} \int_{c_n}^{\infty} \frac{1}{r^2} \, dr \leq C \sum_{n=1}^{\infty} \frac{1}{c_n} = C \sum_{n=1}^{\infty} \frac{a_n}{b_n} < \infty. \tag{3.3}$$

Consequently, according to (3.2), the Khintchine–Kolmogorov convergence theorem (see Theorem 1 on page 113 of Chow and Teicher [2]), and Kronecker's lemma, it is known that $I_N \to 0$ almost surely. Furthermore, by condition (i) and Kronecker's lemma, we have

$$\frac{1}{b_N} \sum_{n=1}^{N} a_n \to 0 \quad \text{as } N \to \infty.$$

Then it follows from (3.3) and the Borel–Cantelli lemma that $II_N \to 0$ almost surely. Finally, since

$$
\mathbb{E}R_{n1j}I(1 \le R_{n1j} \le c_n)
$$

$$
= \gamma_{m,1,j} \int_1^{c_n} \frac{(r-1)^{j-2}}{r^{j-1}}\, dr
$$

$$
= \gamma_{m,1,j} \sum_{l=0}^{j-2} C_{j-2}^l (-1)^l \int_1^{c_n} r^{-l-1}\, dr
$$

$$
\sim \gamma_{m,1,j} \log c_n \quad \text{as } n \to \infty,
$$

it is obtained by condition (ii) that $III_N \to \gamma_{m,1,j}\lambda$ almost surely.

The proof is then completed. $\qquad\square$

Remark 3.1 In particular, let $j = 2$ in Theorem 3.1, we have

$$
\lim_{N\to\infty} \frac{1}{b_N} \sum_{n=1}^N a_n R_{n12} = \lambda \quad \text{almost surely.}
$$

In this case, the result is independent of the sample size m_n because the density of R_{n12} is independent of m_n. Thus, there is no difference between the assumption $m_n \to \infty$ and for fixed $m_n = m$. Furthermore, if we take $b_n = (\log n)^{\alpha+2}$ and $a_n = (\log n)^\alpha/n$, then for $\alpha > -2$, the conditions of Theorem 3.1 hold and $\lambda = 1/(\alpha + 2)$, demonstrating that Theorem 3.1 in Adler [1] and Theorem 2.3 in Miao et al. [7] are special cases of Theorem 3.1.

Remark 3.2 Except for $b_n = (\log n)^{\alpha+2}$, $a_n = (\log n)^\alpha/n$ with $\alpha > -2$, there are some other sequences satisfying the conditions of Theorem 3.1, for example, (i) $b_n = n^\beta$, $a_n = 1$ for $\beta > 1$; (ii) $b_n = n(\log n)^\beta$, $a_n = 1$ for $\beta > 1$; and (iii) $b_n = (\log n)^2(\log\log n)^\beta$, $a_n = (\log\log n)^\beta/n$ for all $\beta \in R$. As a result, Theorem 2.18 in Miao et al. [7] is also a special case of Theorem 3.1.

To establish the complete convergence of the ratio R_{n1j}, we first introduce the lemma obtained by Sung et al. [10].

Lemma 3.1 *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of row-wise independent random variables and $\{\mu_n, n \ge 1\}$ be a sequence of positive constants with $\sum_{n=1}^\infty \mu_n = \infty$. Suppose that, for every $\varepsilon > 0$ and some $\delta > 0$,*

(i) $\sum_{n=1}^\infty \mu_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty$,

(ii) *there exists $J \ge 2$ such that*

$$
\sum_{n=1}^\infty \mu_n \left(\sum_{i=1}^{k_n} \mathbb{E}X_{ni}^2 I(|X_{ni}| \le \delta) \right)^J < \infty,
$$

(iii) $\sum_{i=1}^{k_n} \mathbb{E}X_{ni}I(|X_{ni}| \le \delta) \to 0$ *as $n \to \infty$.*

Then it is true that

$$
\sum_{n=1}^\infty \mu_n P\left(\left| \sum_{i=1}^{k_n} X_{ni} \right| > \varepsilon \right) < \infty \quad \text{for all } \varepsilon > 0.
$$

Theorem 3.2 *Let $\{a_n, n \geq 1\}$, $\{b_n, n \geq 1\}$, and $\{\mu_n, n \geq 1\}$ be three positive sequences such that*

　　(i)　*b_n is nondecreasing, $b_n \to \infty$ as $n \to \infty$, and $\sum_{n=1}^{\infty} \frac{a_n}{b_n} < \infty$;*

　　(ii)　*$\frac{1}{b_N} \sum_{n=1}^{N} a_n \log(\frac{b_n}{a_n}) \to \lambda < \infty$ as $N \to \infty$;*

　　(iii)　*$\sum_{n=1}^{\infty} \mu_n = \infty$ and $\sum_{N=1}^{\infty} \mu_N \sum_{n=1}^{N} \frac{a_n}{b_n} < \infty$.*

Then, for every $\varepsilon > 0$, we have

$$\sum_{N=1}^{\infty} \mu_N P\left(\left| \sum_{n=1}^{N} \frac{a_n}{b_N} R_{n1j} - \gamma_{m,1,j}\lambda \right| \geq \varepsilon \right) < \infty,$$

where $\gamma_{m,1,j}$ is the same number as that in (2.1).

Proof Let $c_n = b_n/a_n$. Then $c_n \to \infty$ according to condition (i). Without loss of generality, we assume that $c_n \geq 1$ for all $n \geq 1$. Notice the following partition:

$$\sum_{n=1}^{N} \frac{a_n}{b_N} R_{n1j} = I_N + II_N + III_N, \tag{3.4}$$

where

$$I_N = \sum_{n=1}^{N} \frac{a_n}{b_N} \left[R_{n1j} I(1 \leq R_{n1j} \leq c_n) - \mathbb{E} R_{n1j} I(1 \leq R_{n1j} \leq c_n) \right],$$

$$II_N = \sum_{n=1}^{N} \frac{a_n}{b_N} R_{n1j} I(R_{n1j} > c_n),$$

$$III_N = \sum_{n=1}^{N} \frac{a_n}{b_N} \mathbb{E} R_{n1j} I(1 \leq R_{n1j} \leq c_n).$$

According to condition (i), Markov's inequality, and the following inequality

$$\mathbb{E} R_{n1j}^2 I(1 \leq R_{n1j} \leq c_n) = \gamma_{m,1,j} \int_{1}^{c_n} \frac{(r-1)^{j-2}}{r^{j-2}} \, dr \leq \gamma_{m,1,j} c_n,$$

we have, for any $\varepsilon > 0$, that

$$P(|I_N| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} \sum_{n=1}^{N} \frac{a_n^2}{b_N^2} \mathbb{E} R_{n1j}^2 I(1 \leq R_{n1j} \leq c_n) \leq C \sum_{n=1}^{N} \frac{a_n}{b_n}. \tag{3.5}$$

Moreover, with condition (ii) and the following estimation

$$\mathbb{E} R_{n1j} I(1 \leq R_{n1j} \leq c_n)$$

$$= \gamma_{m,1,j} \int_{1}^{c_n} \frac{(r-1)^{j-2}}{r^{j-1}} \, dr$$

$$= \gamma_{m,1,j} \sum_{l=0}^{j-2} C_{j-2}^{l} (-1)^l \int_{1}^{c_n} r^{-l-1} \, dr$$

$$\sim \gamma_{m,1,j} \log c_n \quad \text{as } n \to \infty,$$

we obtain

$$III_N \sim \gamma_{m,1,j} \sum_{n=1}^{N} \frac{a_n \log c_n}{b_N} = \gamma_{m,1,j} \lambda.$$

It follows from (3.5) and condition (iii) that

$$\sum_{N=1}^{\infty} \mu_N P\big(|I_N + III_N - \gamma_{m,1,j}\lambda| \geq \varepsilon\big) < \infty. \tag{3.6}$$

Next, denote $X_{Nn} = (a_n/b_N)R_{n1j}I(R_{n1j} > c_n)$. For any $\varepsilon > 0$, we have

$$\sum_{n=1}^{N} P(X_{Nn} > \varepsilon) \leq \sum_{n=1}^{N} P\left(R_{n1j} > \max\left\{c_n, \frac{b_N\varepsilon}{a_n}\right\}\right) \leq C \sum_{n=1}^{N} \int_{\xi}^{\infty} \frac{(r-1)^{j-2}}{r^j} \, dr$$

$$\leq C \sum_{n=1}^{N} \int_{\xi}^{\infty} \frac{1}{r^2} \, dr \leq C \sum_{n=1}^{N} \frac{1}{\xi} \leq C \sum_{n=1}^{N} \frac{a_n}{b_n},$$

where $\xi = \max\{c_n, b_N\varepsilon/a_n\}$. It follows from condition (iii) that

$$\sum_{N=1}^{\infty} \mu_N \sum_{n=1}^{N} P(X_{Nn} > \varepsilon) \leq C \sum_{N=1}^{\infty} \mu_N \sum_{n=1}^{N} \frac{a_n}{b_n} < \infty. \tag{3.7}$$

Then, for any $\delta > 0$, the following partial sum of truncated second moment holds:

$$\sum_{n=1}^{N} \mathbb{E} X_{Nn}^2 I(X_{Nn} \leq \delta) \leq \sum_{n=1}^{N} \frac{a_n^2}{b_N^2} \int_{c_n}^{\frac{b_N\delta}{a_n}} \gamma_{m,1,j}\frac{(r-1)^{j-2}}{r^{j-2}} \, dr \leq C \sum_{n=1}^{N} \frac{a_n}{b_n},$$

which is bounded according to condition (i). This, combined with condition (iii), leads to the inequality that, for every $J \geq 2$,

$$\sum_{N=1}^{\infty} \mu_N \left(\sum_{n=1}^{N} \mathbb{E} X_{Nn}^2 I(X_{Nn} \leq \delta)\right)^J \leq C \sum_{N=1}^{\infty} \mu_N \left(\sum_{n=1}^{N} \frac{a_n}{b_n}\right)^J \leq C \sum_{N=1}^{\infty} \mu_N \sum_{n=1}^{N} \frac{a_n}{b_n} < \infty. \tag{3.8}$$

Noting that (1) $\frac{1}{b_N} \sum_{n=1}^{N} a_n \to 0$ as $N \to \infty$ by condition (i) and Kronecker's lemma; and (2) $\log(b_N\delta/b_n)$ is bounded by $\log C$ via taking $\delta = Cb_1/b_N$, we obtain, for any $\delta > 0$, that

$$\sum_{n=1}^{N} \mathbb{E} X_{Nn} I(X_{Nn} \leq \delta)$$

$$= \sum_{n=1}^{N} \frac{a_n}{b_N} \int_{c_n}^{\frac{b_N\delta}{a_n}} \gamma_{m,1,j}\frac{(r-1)^{j-2}}{r^{j-1}} \, dr$$

$$\leq \frac{C}{b_N} \sum_{n=1}^{N} a_n \log\left(\frac{b_N\delta}{b_n}\right) \to 0 \quad \text{as } N \to \infty. \tag{3.9}$$

Thus, from Lemma 3.1 and (3.7)–(3.9), we have

$$\sum_{N=1}^{\infty} \mu_N P\big(|II_N| \geq \varepsilon\big) < \infty. \tag{3.10}$$

The combination of (3.4), (3.6), and (3.10) yields the desired result. □

Remark 3.3 As pointed out in Remark 3.2, the conditions of Theorem 3.2 are also easy to be satisfied. For example, it can be easily proved that the following sequences all satisfy the conditions of Theorem 3.2: (1) $b_n = n^\beta$, $a_n = 1$, $\mu_n = 1/n$, $\beta > 1$; (2) $b_n = n(\log n)^\beta$, $a_n = 1$, $\mu_n = 1/[n(\log)^\delta]$, $\beta > 1$, $\delta > 0$; (3) $b_n = (\log n)^2(\log\log n)^\beta$, $a_n = (\log\log n)^\beta/n$, $\mu_n = 1/[n(\log)^\delta]$, $\beta \in R$, $\delta > 0$; and (4) $b_n = (\log n)^{\alpha+2}$, $a_n = (\log n)^\alpha/n$, $\mu_n = 1/[n(\log)^\delta]$, $\alpha > -2$, $\delta > 0$. When the sequences are taken to be those in (4), Theorem 3.2 becomes Theorem 2.1 and Theorem 2.6 in Xu and Miao [11], respectively.

In what follows, we give the weighted weak law of large numbers.

Theorem 3.3 *Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two positive sequences satisfying the following conditions*:
  (i) $\frac{1}{b_N}\sum_{n=1}^{N} a_n \to 0$ *as $N \to \infty$*;
  (ii) $\frac{1}{b_N}\sum_{n=1}^{N} a_n \log(\frac{b_N}{a_n}) \to \tilde{\lambda} < \infty$ *as $N \to \infty$*.
Then

$$\frac{1}{b_N}\sum_{n=1}^{N} a_n R_{n1j} \xrightarrow{P} \gamma_{m,1,j}\tilde{\lambda} \quad \text{as } N \to \infty,$$

where $\gamma_{m,1,j}$ is the same number as that in (2.1).

Proof We use the so-called weak law (see Theorem 1 on page 356 of Chow and Teicher [2]) for the proof of Theorem 3.3. From condition (i), we have the following two inequalities:

$$\frac{1}{b_N^2}\sum_{n=1}^{N} a_n^2 \mathbb{E}R_{n1j}^2 I(1 \leq R_{n1j} \leq b_N/a_n)$$
$$= \frac{1}{b_N^2}\sum_{n=1}^{N} a_n^2 \int_1^{\frac{b_N}{a_n}} \gamma_{m,1,j}\frac{(r-1)^{j-2}}{r^{j-2}}\,dr$$
$$\leq \frac{C}{b_N^2}\sum_{n=1}^{N} a_n^2 \int_1^{\frac{b_N}{a_n}} 1\,dr \leq \frac{C}{b_N}\sum_{n=1}^{N} a_n \to 0$$

and, for any $\varepsilon > 0$,

$$\sum_{n=1}^{N} P(a_n R_{n1j} > b_N \varepsilon)$$
$$= \sum_{n=1}^{N} \int_{\frac{b_N\varepsilon}{a_n}}^{\infty} \gamma_{m,1,j}\frac{(r-1)^{j-2}}{r^j}\,dr$$

$$\le C \sum_{n=1}^{N} \int_{\frac{b_N \varepsilon}{a_n}}^{\infty} \frac{1}{r^2}\, dr = \frac{C}{b_N} \sum_{n=1}^{N} a_n \to 0,$$

which is sufficient to prove that the weighted sum converges to some real number.

Next, using condition (i) again, we have

$$\mathbb{E}R_{n1j}I(1 \le R_{n1j} \le b_N/a_n)$$

$$= \int_{1}^{\frac{b_N}{a_n}} \gamma_{m,1,j} \frac{(r-1)^{j-2}}{r^{j-1}}\, dr$$

$$= \gamma_{m,1,j} \sum_{l=0}^{j-2} C_{j-2}^{l}(-1)^{l} \int_{1}^{\frac{b_N}{a_n}} r^{-l-1}\, dr$$

$$= \gamma_{m,1,j} \left[\log b_N - \log a_n + \sum_{l=1}^{j-2} C_{j-2}^{l}(-1)^{l+1}\left(\frac{a_n^{l}}{lb_N^{l}} - \frac{1}{l} \right) \right]$$

$$\sim \gamma_{m,1,j} \log\left(\frac{b_N}{a_n} \right) \quad \text{as } N \to \infty.$$

It follows from condition (ii) that

$$\frac{1}{b_N} \sum_{n=1}^{N} a_n \mathbb{E}R_{n1j}I(1 \le R_{n1j} \le b_N/a_n)$$

$$\sim \frac{\gamma_{m,1,j}}{b_N} \sum_{n=1}^{N} a_n \log\left(\frac{b_N}{a_n} \right) \to \gamma_{m,1,j}\tilde{\lambda} \quad \text{as } N \to \infty. \tag{3.11}$$

The proof is then completed. $\qquad\square$

In particular, let $L(x)$ be a slowly varying function. If we take $a_n = n^{\alpha} L(n)$ and $b_n = n^{\alpha+1} L(n) \log n$, in which $\alpha > -1/(j-1)$ and $j \ge 2$, the conditions of Theorem 3.3 hold with $\tilde{\lambda} = 1/(\alpha+1)$. This result will be described as the following corollary, whose proof explains how we verify the conditions.

Corollary 3.1 *Let $L(x)$ be a slowly varying function. Then, for any $\alpha > -1/(j-1)$, we have*

$$\frac{1}{N^{\alpha+1}L(N)\log N} \sum_{n=1}^{N} n^{\alpha} L(n) R_{n1j} \xrightarrow{P} \frac{\gamma_{m,1,j}}{\alpha+1} \quad \text{as } N \to \infty,$$

where $\gamma_{m,1,j}$ is the same number as that in (2.1).

Proof Denote $U(x) = x^{\gamma} L(x)$ and $U_p(x) = \int_0^x t^p U(t)\, dt$. $U(x) = x^{\gamma} L(x)$ varies regularly with the exponent γ as $L(x)$ is slowly varying. Therefore, according to part (b) of Theorem 1 on page 281 of Feller [3], for $p \ge -\gamma - 1$, it is true that

$$\int_0^x t^p t^{\gamma} L(t)\, dt \sim \frac{1}{p+\gamma+1} x^{p+1} x^{\gamma} L(x) \quad \text{as } x \to \infty,$$

which implies that

$$\sum_{n=1}^{N} n^{\alpha} L(n) \sim \frac{1}{\alpha + 1} N^{\alpha+1} L(N) \quad \text{for } \alpha > -1, \text{ as } N \to \infty. \tag{3.12}$$

It is easy to check that $L(x) \log x$, $L(x) \log L(x)$, and $L(x)^l$ for all $l \in N$ are slowly varying functions as $L(x)$ is slowly varying. So, we derive the following similar estimates as (3.12):

$$\sum_{n=1}^{N} n^{\alpha} L(n) \log n \sim \frac{1}{\alpha + 1} N^{\alpha+1} L(N) \log N \quad \text{for } \alpha > -1;$$

$$\sum_{n=1}^{N} n^{\alpha} L(n) \log L(n) \sim \frac{1}{\alpha + 1} N^{\alpha+1} L(N) \log L(N) \quad \text{for } \alpha > -1; \tag{3.13}$$

$$\sum_{n=1}^{N} n^{(l+1)\alpha} L(n)^{l+1} \sim \frac{1}{(l+1)\alpha + 1} N^{(l+1)\alpha+1} L(N)^{l+1} \quad \text{for } \alpha > -\frac{1}{l+1}, 1 \le l \le m - 2.$$

Let $a_n = n^{\alpha} L(n)$, $b_n = n^{\alpha+1} L(n) \log n$. We once again use the so-called weak law, which can be found on page 356 of Chow and Teicher [2]. By (3.12), it is easy to obtain that, for any $\varepsilon > 0$,

$$\sum_{n=1}^{N} P(a_n R_{n1j} > b_N \varepsilon)$$

$$= \sum_{n=1}^{N} \int_{\frac{b_N \varepsilon}{a_n}}^{\infty} \gamma_{m,1,j} \frac{(r-1)^{j-2}}{r^j} \, dr$$

$$\le C \sum_{n=1}^{N} \int_{\frac{b_N \varepsilon}{a_n}}^{\infty} \frac{1}{r^2} \, dr = \frac{C}{b_N} \sum_{n=1}^{N} a_n = \frac{C \sum_{n=1}^{N} n^{\alpha} L(n)}{N^{\alpha+1} \log N L(N)}$$

$$\sim \frac{C N^{\alpha+1} L(N)}{N^{\alpha+1} \log N L(N)} = \frac{C}{\log N} \to 0 \quad \text{as } N \to \infty$$

and

$$\frac{1}{b_N^2} \sum_{n=1}^{N} a_n^2 \mathbb{E} R_{n1j}^2 I(1 \le R_{n1j} \le b_N/a_n)$$

$$= \frac{1}{b_N^2} \sum_{n=1}^{N} a_n^2 \int_{1}^{\frac{b_N}{a_n}} \gamma_{m,1,j} \frac{(r-1)^{j-2}}{r^{j-2}} \, dr$$

$$\le \frac{C}{b_N^2} \sum_{n=1}^{N} a_n^2 \int_{1}^{\frac{b_N}{a_n}} 1 \, dr \le \frac{C}{b_N} \sum_{n=1}^{N} a_n \to 0 \quad \text{as } N \to \infty,$$

which are sufficient to prove that the weighted sum converges to some real number.

Next, notice that

$$\mathbb{E} R_{n1j} I(1 \le R_{n1j} \le b_N/a_n)$$

$$= \int_{1}^{\frac{b_N}{a_n}} \gamma_{m,1,j} \frac{(r-1)^{j-2}}{r^{j-1}} \, dr$$

$$= \gamma_{m,1,j} \sum_{l=0}^{j-2} C_{j-2}^l (-1)^l \int_1^{\frac{b_N}{a_n}} r^{-l-1}\, dr$$

$$= \gamma_{m,1,j} \left[\log b_N - \log a_n + \sum_{l=1}^{j-2} C_{j-2}^l (-1)^{l+1} \left(\frac{a_n^l}{l b_N^l} - \frac{1}{l} \right) \right].$$

In addition, by (3.12) and (3.13), we have

$$\frac{\log b_N \sum_{n=1}^N a_n}{b_N} \sim 1 + \frac{\log L(N) + \log\log N}{(\alpha+1)\log N}; \qquad \frac{\sum_{n=1}^N a_n \log a_n}{b_N} \sim \frac{\alpha \log N + \log L(N)}{(\alpha+1)\log N};$$

$$\frac{\sum_{n=1}^N a_n^{l+1}}{l b_N^{l+1}} \sim \frac{1}{l[(l+1)\alpha+1]N^l(\log N)^{l+1}}; \qquad \frac{\sum_{n=1}^N a_n}{l b_N} \sim \frac{1}{l\log N}.$$

Therefore

$$\frac{1}{b_N} \sum_{n=1}^N a_n \mathbb{E} R_{n1j} I(1 \le R_{n1j} \le b_N/a_n)$$

$$= \frac{\gamma_{m,1,j}}{b_N} \sum_{n=1}^N a_n \left[\log b_N - \log a_n + \sum_{l=1}^{j-2} C_{j-2}^l (-1)^{l+1} \left(\frac{a_n^l}{l b_N^l} - \frac{1}{l} \right) \right]$$

$$= \gamma_{m,1,j} \left[\frac{\log b_N \sum_{n=1}^N a_n}{b_N} - \frac{\sum_{n=1}^N a_n \log a_n}{b_N} \right.$$

$$\left. + \sum_{l=1}^{j-2} C_{j-2}^l (-1)^{l+1} \left(\frac{\sum_{n=1}^N a_n^{l+1}}{l b_N^{l+1}} - \frac{\sum_{n=1}^N a_n}{l b_N} \right) \right]$$

$$\sim \gamma_{m,1,j} \frac{\log N + \log\log N}{(\alpha+1)\log N} \to \frac{\gamma_{m,1,j}}{\alpha+1} \quad \text{as } N \to \infty,$$

where $\alpha > -1/(j-1)$. The proof is then completed. $\qquad\square$

Remark 3.4 It is easy to check that the corresponding strong law of numbers of Theorem 3.3 fails. Hence, the weak law of large numbers is optimal in this case. Corollary 3.1 extends Theorem 3.2 in Adler [1], which proved the same result for R_{n12} of the sample from the uniform distribution $U(0,p)$.

4 Limit properties of R_{n2j}

The following Marcinkiewicz–Zygmund type law of large numbers extends the result for R_{n23} in Xu and Miao [11, Theorem 2.3].

Theorem 4.1 *For any* $\delta \in (0,2)$, *we have*

$$\frac{1}{N^{1/\delta}} \sum_{n=1}^N (R_{n2j} - c) \to 0 \quad \text{almost surely}$$

for some finite constant c, where c takes the value $\mathbb{E} R_{n2j}$ for $\delta \in [1,2)$ and c is arbitrary for $\delta \in (0,1)$. In particular, the Kolmogorov type strong law of large numbers holds when $\delta = 1$.

Proof By part (II) of Theorem 2.1 and the Marcinkiewicz–Zygmund theorem (see Theorem 2 on page 125 in Chow and Teicher[2]), the theorem can be proved.   □

The following central limit theorem and its almost sure version extend the responding result for R_{n23} established by Miao et al. [7, Theorem 2.12] and Xu and Miao [11, Theorem 2.4], respectively.

Theorem 4.2 *Let $L(x) = \mathbb{E}R_{n2j}^2 I(|R_{n2j}| \leq x)$. Then*

$$\frac{1}{\eta_N} \sum_{n=1}^{N} (R_{n2j} - \mathbb{E}R_{n2j}) \xrightarrow{d} \Phi(x) \quad as \ N \to \infty,$$

where $\eta_N = 1 \vee \sup\{x > 0; NL(x) \geq x^2\}$ and $\Phi(x)$ denotes the standard normal distribution function.

Proof From Theorem 4.17 of Kallenberg [6, page 73], it is known that $L(x) = \mathbb{E}R_{n2j}^2 I(|R_{n2j}| \leq x)$ varies slowly at $x \to \infty$. Therefore, the theorem is proved.   □

Theorem 4.3 *Let $L(x) = \mathbb{E}R_{n2j}^2 I(|R_{n2j}| \leq x)$. Then, for any real number x, we have*

$$\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(\frac{S_n}{\eta_n} \leq x\right) = \Phi(x) \quad almost \ surely,$$

where $S_n = \sum_{i=1}^{n} Y_i$, $Y_n = R_{n2j} - \mathbb{E}R_{n2j}$, $\eta_n = 1 \vee \sup\{x > 0; nL(x) \geq x^2\}$ and $\Phi(x)$ denotes the standard normal distribution function.

The proof of Theorem 4.3 is very similar to that of Xu and Miao [11, Theorem 2.4], so we omit it. For the same reason, the following large deviation principle, which extends the corresponding result built by Xu and Miao [11, Theorem 2.5], is given without proof.

Theorem 4.4 *For any $x > 1/2$,*

$$\lim_{N \to \infty} \frac{1}{\log N} \log P\left(\sum_{n=1}^{N} R_{n2j} > N^x\right) = -2x + 1.$$

As aforementioned, the βth moment of R_{n2j} does not exist for any $\beta \geq 2$. Thus, some other classical limit properties, such as the law of iterated logarithm, do not hold. As is well known, the limit theorems for self-normalized sum usually require much less stringent moment conditions than the classical limit theorems. Therefore, we are interested in the investigation of limit behaviors for the self-normalized sum S_N/V_N, where $S_N = \sum_{n=1}^{N}(R_{n2j} - \mathbb{E}R_{n2j})$ and $V_N^2 = \sum_{n=1}^{N}(R_{n2j} - \mathbb{E}R_{n2j})^2$. Notice that $\{R_{n2j} - \mathbb{E}R_{n2j}, n \geq 1\}$ is a sequence of i.i.d. random variables with mean zero. In addition, the distribution of $R_{n2j} - \mathbb{E}R_{n2j}$ is in the domain of attraction of the normal law, because $\widetilde{L}(x) = \mathbb{E}(R_{n2j} - \mathbb{E}R_{n2j})^2 I(|R_{n2j} - \mathbb{E}R_{n2j}| \leq x)$ is slowly varying as $x \to \infty$. Thus many self-normalized limit properties can be directly established as the corollaries of the corresponding well-known results. We list some of them without the proofs.

From Theorem 3.3 in Gine et al. [4], we obtain the following self-normalized central limit theorem.

Theorem 4.5 $S_N/V_N \xrightarrow{d} \Phi(x)$ *as* $N \to \infty$.

The following self-normalized almost sure central limit theorem can be obtained by Corollary 1 in Zhang [12].

Theorem 4.6 *Denote* $d_n = \exp\{(\log n)^\alpha\}/n$ *and* $D_N = \sum_{n=1}^N d_n$ *for* $1 \le \alpha < \frac{1}{2}$. *Then, for any real x, we have*

$$\lim_{N\to\infty} \frac{1}{D_N} \sum_{n=1}^N d_n I\left(\frac{S_n}{V_n} \le x\right) = \Phi(x) \quad a.s.$$

From Theorem 3 in Robinson and Wang [8], we have the following self-normalized Berry–Esseen bounds.

Theorem 4.7 *Let* $\eta_N = \sup\{x : Nx^{-2}\mathbb{E}R_{n2j}^2 I(|R_{n2j}| \le x) \ge 1\}$. *Then there exists* $0 < \eta < 1$ *such that*

$$\left| P\left(\frac{S_N}{V_N} \le x\right) - \Phi(x) \right| \le A\delta_N \exp\left\{-\frac{\eta x^2}{2}\right\}$$

for all $x \in R$ *and* $N \ge 1$, *where*

$$\delta_N = NP\left(|R_{n2j}| > \eta_N\right) + N\eta_N^{-1}\left|\mathbb{E}R_{n2j}I\left(|R_{n2j}| \le \eta_N\right)\right| + N\eta_N^{-3}\mathbb{E}|R_{n2j}|^3 I\left(|R_{n2j}| \le \eta_N\right)$$

and A is an absolute constant.

From Theorem 3.1 in Shao [9], we obtain the following self-normalized moderate deviation principle.

Theorem 4.8 *Let* $\{x_N, N \ge 1\}$ *be a sequence of positive numbers satisfying*

$$x_N \to \infty \quad and \quad \frac{x_N}{\sqrt{N}} \to 0 \quad as\ N \to \infty.$$

Then

$$\lim_{N\to\infty} \frac{1}{x_N^2} \log P\left(\frac{S_N}{V_N} \ge x_N\right) = -\frac{1}{2}.$$

The following self-normalized law of the iterated logarithm can be established by Theorem 3.1 in Griffin and Kuelbs [5].

Theorem 4.9 *It is true that*

$$\limsup_{N\to\infty} \frac{S_N}{V_N\sqrt{2\log\log N}} = 1 \quad almost\ surely.$$

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