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λ-Analogues of Stirling polynomials of the first kind and their applications

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Abstract

Recently, λ-analogues of Stirling numbers of the first kind were studied. In this paper, we introduce, as natural extensions of these numbers, λ-Stirling polynomials of the first kind and *r*-truncated λ-Stirling polynomials of the first kind. We give recurrence relations, explicit expressions, some identities, and connections with other special polynomials for those polynomials. Further, as applications, we show that both of them appear in an expression of the probability mass function of a suitable discrete random variable, constructed from λ-logarithmic and negative λ-binomial distributions.

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1 Introduction

As is known, the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \quad (n \geq 0) \text{ (see [1, 2, 4, 7–25])}, \tag{1}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$ $(n \geq 1)$.

For any real number λ, the λ-analogue of $(x)_n$ is defined as

$$(x)_{0,\lambda} = 1, \qquad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots\bigl(x-(n-1)\lambda\bigr) \quad (n \geq 1). \tag{2}$$

Note that $\lim_{\lambda\to 1}(x)_{n,\lambda} = (x)_n$ $(n \geq 0)$ (see [2, 11, 13, 16, 19]).

The λ-analogues of the Stirling numbers of the first kind are given by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}(n,k)x^k \quad \text{(see [13])}. \tag{3}$$

We recall that the λ-binomial coefficients are defined by the generating function

$$(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{l=0}^{\infty} \binom{x}{l}_{\lambda} t^l = \sum_{l=0}^{\infty} (x)_{l,\lambda}\frac{t^l}{l!} \quad \text{(see [13, 16])}. \tag{4}$$

From (4), we note that

$$\binom{n}{k}_\lambda = \frac{(n!)_\lambda}{k!(n-k\lambda)_{n-k,\lambda}} = \frac{(n)_{k,\lambda}}{k!} \quad (n \geq k \geq 0), \tag{5}$$

where

$$(n!)_\lambda = n(n-\lambda)(n-2\lambda)\cdots\bigl(n-(n-1)\lambda\bigr) = (n)_{n,\lambda} \quad \text{(see [13])}$$

and

$$\sum_{m=0}^{n}\binom{y}{m}_\lambda\binom{x}{n-m}_\lambda = \binom{x+y}{n}_\lambda \quad (n \geq 0). \tag{6}$$

For $r \in \mathbb{N}$, the unsigned r-Stirling numbers of the first kind are defined by

$$(x+r)(x+r+1)\cdots(x+r+n-1) = \sum_{k=0}^{n}\begin{bmatrix} n+r \\ k+r \end{bmatrix}_r x^k \quad \text{(see [10, 13, 16])}. \tag{7}$$

In [16, 18], the r-Stirling numbers of the first kind are given by

$$(x+r)_n = \sum_{k=0}^{n} S_1^{(r)}(n,k)x^k \quad (n \geq 0). \tag{8}$$

By (7) and (8), we get

$$S_1^{(-r)}(n,k) = (-1)^{n-k}\begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \quad (n \geq k \geq 0). \tag{9}$$

It is known that λ-analogues of r-Stirling numbers of the first kind are given by

$$(x+r)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n,k)x^k \quad \text{(see [16])}. \tag{10}$$

From (3), (6), and (10), we note that

$$S_{1,\lambda}^{(r)}(n,k) = \sum_{m=k}^{n}\binom{n}{m} S_{1,\lambda}(m,k)(r)_{n-m,\lambda} \quad (n,k \geq 0). \tag{11}$$

If X is a discrete random variable taking values in the nonnegative integers, then the probability generating function of X is defined as follows:

$$G(t) = E\bigl[t^X\bigr] = \sum_{x=0}^{\infty} p(x)t^x \quad \text{(see [7, 24])}, \tag{12}$$

where $p(x)$ is the probability mass function of X.

Let $X = (X_1, X_2, \dots, X_k)$ be a discrete random variable taking values in the k-dimensional nonnegative integer lattice. Then the probability generating function of X is defined as follows:

$$\begin{aligned} G(t) = G(t_1, t_2, \dots, t_k) &= E\bigl[t_1^{X_1}, t_2^{X_2}, \dots, t_k^{X_k}\bigr] \\ &= \sum_{x_1, x_2, \dots, x_k = 0}^{\infty} p(x_1, x_2, \dots, x_k) t_1^{x_1} t_2^{x_2} \cdots t_k^{x_k}, \end{aligned} \tag{13}$$

where $p(x_1, x_2, \dots, x_k)$ is the probability mass function of X. The power series converges absolutely at least for all convex vectors $t = (t_1, t_2, \dots, t_k) \in \mathbb{C}^k$ with $\max\{|t_1|, |t_2|, \dots, |t_k|\} \leq 1$.

The logarithmic random variable X with parameter $\alpha \in (0, 1)$ is a discrete random variable on $\mathbb{N}$ with probability mass function $p(x)$ given by

$$P[X = n] = p(n) = -\frac{1}{\log(1-\alpha)} \cdot \frac{\alpha^n}{n} \quad (n \in \mathbb{N}). \tag{14}$$

Note that

$$\sum_{n=1}^{\infty} p(n) = -\frac{1}{\log(1-\alpha)} \sum_{n=1}^{\infty} \frac{\alpha^n}{n} = 1$$

and

$$E[X] = \sum_{n=1}^{\infty} p(n) \cdot n = -\frac{1}{\log(1-\alpha)} \sum_{n=1}^{\infty} \alpha^n = -\frac{1}{\log(1-\alpha)} \cdot \frac{\alpha}{1-\alpha}.$$

In probability and statistics, the logarithmic distribution (also known as the logarithmic series distribution) is a discrete probability distribution derived from the Maclaurin series expansion

$$-\log(1-\alpha) = \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{3} + \cdots.$$

In probability theory and statistics, the negative binomial distribution is a discrete probability distribution of the number of successes in a sequence of independent and identically distributed Bernoulli trials before a specified number of failures (denoted by r) occurs. The negative binomial random variable is sometimes defined in terms of the random variable Y = the number of failures before the rth success. The probability mass function of the negative binomial random variable with parameters r and p is given by

$$P[X = y] = p(y) = \binom{y+r-1}{y} p^y (1-p)^r. \tag{15}$$

In this paper, we consider λ-Stirling polynomials of the first kind and truncated λ- Stirling polynomials of the first kind rising from the λ-analogues of the falling factorial sequence and investigate some properties for these polynomials. In particular, we give some identities, recurrence relations, and explicit expressions for the λ-Stirling polynomials of the first kind and the truncated λ-Stirling polynomials of the first kind. Further, we show that both of them appear in an expression of the probability mass function of a suitable discrete random variable, constructed from λ-logarithmic and negative λ-binomial distributions.

2 λ-Stirling polynomials of the first kind

Let t be a real variable, x be a real number, and let n be a nonnegative integer. The Taylor expansion of the function $(t)_{n,\lambda}$ is given by

$$(t)_{n,\lambda}=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^k}{dt^k}(t)_{n,\lambda}\right]_{t=x}(t-x)^k. \tag{16}$$

Let

$$S_{1,\lambda}^{(x)}(n,k)=\frac{1}{k!}\left[\frac{d^k}{dt^k}(t)_{n,\lambda}\right]_{t=x}\quad (n,k\geq 0). \tag{17}$$

Then, by (16) and (17), we get

$$(t)_{n,\lambda}=\sum_{k=0}^{n}S_{1,\lambda}^{(x)}(n,k)(t-x)^k\quad (n\geq 0). \tag{18}$$

Here $S_{1,\lambda}^{(x)}(n,k)$ will be called the *λ-Stirling polynomials of the first kind.*

It is easy to show that

$$(t)_{n+1,\lambda}=(t-x)(t)_{n,\lambda}+(x-n\lambda)(t)_{n,\lambda}. \tag{19}$$

From (18), we can derive the following equation:

$$\begin{aligned}\sum_{k=0}^{n+1}S_{1,\lambda}^{(x)}(n+1,k)(t-x)^k&=(t)_{n+1,\lambda}=(t-x)(t)_{n,\lambda}+(x-n\lambda)(t)_{n,\lambda}\\&=\sum_{k=0}^{n}S_{1,\lambda}^{(x)}(n,k)(t-x)^{k+1}+(x-n\lambda)\sum_{k=0}^{n}S_{1,\lambda}^{(x)}(n,k)(t-x)^k\\&=\sum_{k=1}^{n+1}S_{1,\lambda}^{(x)}(n,k-1)(t-x)^k+(x-n\lambda)\sum_{k=0}^{n}S_{1,\lambda}^{(x)}(n,k)(t-x)^k\\&=\sum_{k=0}^{n+1}\big(S_{1,\lambda}^{(x)}(n,k-1)+(x-n\lambda)S_{1,\lambda}^{(x)}(n,k)\big)(t-x)^k.\end{aligned} \tag{20}$$

Therefore, by comparing the coefficients on both sides of (20), we obtain the following theorem.

Theorem 2.1 *For $n,k\geq 0$ with $n\geq k-1$, we have*

$$S_{1,\lambda}^{(x)}(n,k-1)+(x-n\lambda)S_{1,\lambda}^{(x)}(n,k)=S_{1,\lambda}^{(x)}(n+1,k).$$

Note that

$$S_{1,\lambda}^{(x)}(0,0)=1,\qquad S_{1,\lambda}^{(x)}(n,0)=(x)_{n,\lambda},\qquad S_{1,\lambda}^{(x)}(0,k)=0\quad (k>0).$$

From (18), we easily note that $S_{1,\lambda}^{(x)}(n,k)=0$ if $k>n$. Let us take $t=x+1$ and $t=x-1$ in (18). Then we have

$$(x+1)_{n,\lambda}=\sum_{k=0}^{n}S_{1,\lambda}^{(x)}(n,k)(x+1-x)^k=\sum_{k=0}^{n}S_{1,\lambda}^{(x)}(n,k)$$

and

$$(x-1)_{n,\lambda}=\sum_{k=0}^{n}S_{1,\lambda}^{(x)}(n,k)(-1)^k.$$

By (3), we get

$$\begin{aligned}\frac{1}{k!}\left[\frac{d^k}{dt^k}(t)_{n,\lambda}\right]_{t=x}&=\frac{1}{k!}\sum_{l=k}^{n}S_{1,\lambda}(n,l)(l)_k t^{l-k}\Big|_{t=x}\\&=\sum_{l=k}^{n}S_{1,\lambda}(n,l)\binom{l}{k}x^{l-k}.\end{aligned}\tag{21}$$

From (17) and (21), we obtain the following theorem.

Theorem 2.2 *For $n\geq k$, we have*

$$S_{1,\lambda}^{(x)}(n,k)=\sum_{l=k}^{n}S_{1,\lambda}(n,l)\binom{l}{k}x^{l-k}.$$

Note that $S_{1,\lambda}^{(0)}(n,k)=S_{1,\lambda}(n,l)$.

Now, we give an explicit expression for the polynomials $S_{1,\lambda}^{(x)}(n,k)$ and their relations with λ-Stirling numbers of the first kind. First we observe that

$$\begin{aligned}(1+\lambda t)^{\frac{y}{\lambda}}&=\sum_{k=0}^{\infty}(y)_{k,\lambda}\frac{t^k}{k!}\\&=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}\frac{1}{n!}\left[\frac{d^n}{dy^n}(y)_{k,\lambda}\right]_{y=x}(y-x)^n\right)\frac{t^k}{k!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}S_{1,\lambda}^{(x)}(k,n)\frac{t^k}{k!}\right)(y-x)^n.\end{aligned}\tag{22}$$

On the other hand,

$$\begin{aligned}(1+\lambda t)^{\frac{y}{\lambda}} &= e^{\frac{y}{\lambda}\log(1+\lambda t)} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^k e^{\frac{x}{\lambda}\log(1+\lambda t)}(y-x)^k \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^k (1+\lambda t)^{\frac{x}{\lambda}}(y-x)^k. \end{aligned} \tag{23}$$

From (22) and (23), we obtain the generating function for $S_{1,\lambda}^{(x)}(n,k)$ given by

$$\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=k}^{\infty} S_{1,\lambda}^{(x)}(n,k)\frac{t^n}{n!}, \tag{24}$$

where k is a nonnegative integer.

Indeed, we note that

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t)^{\frac{x}{\lambda}} &= \sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}\left(\frac{1}{k!}\sum_{l=k}^{\infty}\left(\sum_{l_1+l_2+\cdots+l_k=l}\frac{(-\lambda)^{l-k}}{l_1 l_2\cdots l_k}\right)t^l\right) \\ &= \sum_{n=k}^{\infty}\left(\frac{1}{k!}\sum_{l=k}^{n}(-\lambda)^{l-k}l!\binom{n}{l}(x)_{n-l,\lambda}\sum_{l_1+l_2+\cdots+l_k=l}\frac{1}{l_1 l_2\cdots l_k}\right)\frac{t^n}{n!}. \end{aligned} \tag{25}$$

Therefore, by (24) and (25), we obtain the following theorem.

Theorem 2.3 *For $n \geq k$, we have*

$$S_{1,\lambda}^{(x)}(n,k) = \frac{1}{k!}\sum_{l=k}^{n}(-\lambda)^{l-k}l!\binom{n}{l}(x)_{n-l,\lambda}\sum_{l_1+l_2+\cdots+l_k=l}\frac{1}{l_1 l_2\cdots l_k},$$

where the inner sum runs over all positive integers $l_1, l_2, \ldots, l_k$ with $l_1 + l_2 + \cdots + l_k = l$.

It is known that

$$\frac{1}{k!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad \text{(see [13, 16])}. \tag{26}$$

From (24) and (26), we have

$$\begin{aligned}\sum_{n=k}^{\infty} S_{1,\lambda}^{(x)}(n,k)\frac{t^n}{n!} &= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\sum_{m=k}^{\infty}S_{1,\lambda}(m,k)\frac{t^m}{m!} \\ &= \sum_{n=k}^{\infty}\left(\sum_{l=0}^{n-k}(x)_{l,\lambda}\binom{n}{l}S_{1,\lambda}(n-l,k)\right)\frac{t^n}{n!} \\ &= \sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}(x)_{n-l,\lambda}\binom{n}{l}S_{1,\lambda}(l,k)\right)\frac{t^n}{n!}. \end{aligned} \tag{27}$$

Comparing the coefficients on both sides of (27), we obtain the following theorem.

Theorem 2.4 *Let n, k be nonnegative integers. Then we have*

$$S_{1,\lambda}^{(x)}(n,k) = \begin{cases} \sum_{l=k}^{n} \binom{n}{l} (x)_{n-l,\lambda} S_{1,\lambda}(l,k), & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

Corollary 2.5 *Let n, k be nonnegative integers. Then we have*

$$S_{1,\lambda}(n,k) = \begin{cases} \sum_{l=k}^{n} \binom{n}{l} (x)_{n-l,\lambda} S_{1,\lambda}^{(x)}(l,k), & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

We now consider the r-truncated λ-Stirling numbers of the first kind.

For $x \in \mathbb{R}$ and $r \in \mathbb{N}$, the *r-truncated λ-Stirling polynomials of the first kind* are defined by

$$(1+\lambda t)^{\frac{x}{\lambda}} \frac{1}{k!} \left(\frac{\log(1+\lambda t)}{\lambda} - \sum_{j=1}^{r-1} (-\lambda)^{j-1} \frac{t^j}{j} \right)^k = \sum_{n=rk}^{\infty} S_{1,\lambda}^{(x)}(n,k|r) \frac{t^n}{n!}. \tag{28}$$

Remark 2.6 The definition of λ-Stirling polynomials of the first kind and that of r-truncated λ-Stirling polynomials of the first kind are similar to the non-central Stirling numbers of the first kind and the generalized non-central Stirling numbers of the first kind, respectively (see [20]). In fact, one replaces α by x and $(x)_n$ by $(x)_{n,\lambda}$, as in [3] and [6], or replaces α by x with setting $\alpha_i = i\lambda$, $i = 0, 1, \ldots, n-1$, as in (1.8) of [5].

From (28), we have

$$\begin{aligned} &(1+\lambda t)^{\frac{x}{\lambda}} \frac{1}{k!} \left(\frac{\log(1+\lambda t)}{\lambda} - \sum_{j=1}^{r-1} (-\lambda)^{j-1} \frac{t^j}{j} \right)^k \\ &= \sum_{m=0}^{\infty} (x)_{m,\lambda} \frac{t^m}{m!} \frac{1}{k!} \sum_{l=rk}^{\infty} \left(\sum_{l_1+l_2+\cdots+l_k=l} \frac{(-\lambda)^{l-k}}{l_1 l_2 \cdots l_k} \right) t^l \\ &= \sum_{n=kr}^{\infty} \left(\frac{n!}{k!} \sum_{l=rk}^{n} \frac{(x)_{n-l,\lambda}}{(n-l)!} \sum_{l_1+\cdots+l_k=l} \frac{(-\lambda)^{l-k}}{l_1 l_2 \cdots l_k} \right) \frac{t^n}{n!}. \end{aligned} \tag{29}$$

Therefore, by (28) and (29), we obtain the following theorem.

Theorem 2.7 *For $n \geq rk$, we have*

$$S_{1,\lambda}^{(x)}(n,k|r) = \frac{n!}{k!} \sum_{l=rk}^{n} \frac{(x)_{n-l,\lambda}}{(n-l)!} \sum_{l_1+\cdots+l_k=l} \frac{(-\lambda)^{l-k}}{l_1 l_2 \cdots l_k}.$$

In particular, if $n < kr$, we have

$$S_{1,\lambda}^{(x)}(n,k|r) = 0.$$

When $x=0$, $S^{(0)}_{1,\lambda}(n,k|r)=S_{1,\lambda}(n,k|r)$ are called the *r-truncated λ-Stirling numbers of the first kind*.

It is not difficult to show that

$$S^{(x)}_{1,\lambda}(n,k|r)=\sum_{l=kr}^{n}\binom{n}{l}(x)_{n-l,\lambda}S_{1,\lambda}(l,k|r)\quad \text{if } n\geq kr$$

and

$$S^{(x)}_{1,\lambda}(n,k|r)=0\quad \text{if } n<kr.$$

Let

$$y^{(r)}_{k,\lambda}(t|x)=(1+\lambda t)^{\frac{x}{\lambda}}\frac{1}{k!}\left[\frac{1}{\lambda}\log 1+\lambda t)-\sum_{j=1}^{r-1}\frac{(-\lambda)^{j-1}}{j}t^{j}\right]^{k}. \tag{30}$$

From (30), we can derive the following differential equation:

$$(1+\lambda t)\frac{d}{dt}y^{(r)}_{k,\lambda}(t|x)=xy^{(r)}_{k,\lambda}(t|x)+(-1)^{r-1}t^{r-1}\lambda^{r-1}y^{(r)}_{k-1,\lambda}(t|x). \tag{31}$$

By (28) and (31), we get

$$\begin{aligned}
&(1+\lambda t)\frac{d}{dt}y^{(r)}_{k,\lambda}(t|x)\\
&\quad=xy^{(r)}_{k,\lambda}(t|x)+(-1)^{r-1}t^{r-1}\lambda^{r-1}y^{(r)}_{k-1,\lambda}(t|x)\\
&\quad=x\sum_{n=rk}^{\infty}S^{(x)}_{1,\lambda}(n,k|r)\frac{t^n}{n!}+(-1)^{r-1}t^{r-1}\lambda^{r-1}\sum_{n=r(k-1)}^{\infty}S^{(x)}_{1,\lambda}(n,k-1|r)\frac{t^n}{n!}\\
&\quad=x\sum_{n=rk}^{\infty}S^{(x)}_{1,\lambda}(n,k|r)\frac{t^n}{n!}+(-1)^{r-1}\lambda^{r-1}\sum_{n=rk-1}^{\infty}S^{(x)}_{1,\lambda}(n-r+1,k-1|r)\frac{t^n}{(n-r+1)!}\\
&\quad=\sum_{n=rk-1}^{\infty}\left(xS^{(x)}_{1,\lambda}(n,k|r)+(-1)^{r-1}\lambda^{r-1}(r-1)!\binom{n}{r-1}S^{(x)}_{1,\lambda}(n-r+1,k-1|r)\right)\frac{t^n}{n!}.
\end{aligned} \tag{32}$$

On the other hand,

$$\begin{aligned}
(1+\lambda t)\frac{d}{dt}y^{(r)}_{k,\lambda}(t|x)&=\sum_{n=rk}^{\infty}S^{(x)}_{1,\lambda}(n,k|r)\frac{t^{n-1}}{(n-1)!}(1+\lambda t)\\
&=\sum_{n=rk-1}^{\infty}S^{(x)}_{1,\lambda}(n+1,k|r)\frac{t^n}{n!}+\sum_{n=rk}^{\infty}n\lambda S^{(x)}_{1,\lambda}(n,k|r)\frac{t^n}{n!}\\
&=\sum_{n=rk-1}^{\infty}\left(S^{(x)}_{1,\lambda}(n+1,k|r)+n\lambda S^{(x)}_{1,\lambda}(n,k|r)\right)\frac{t^n}{n!}.
\end{aligned} \tag{33}$$

Therefore, by (32) and (33), we obtain the following theorem.

Theorem 2.8 *Let n, k be nonnegative integers, and let r be a positive integer. Then we have*

$$\begin{aligned}&\lambda^{r-1}(r-1)!\binom{-n+r-2}{r-1}S_{1,\lambda}^{(x)}(n-r+1,k-1|r)\\&\quad=S_{1,\lambda}^{(x)}(n+1,k|r)+(n\lambda-x)S_{1,\lambda}^{(x)}(n,k|r)\quad(n\geq kr-1).\end{aligned}$$

It is easy to show that

$$\begin{aligned}y_{k,\lambda}^{(r+1)}(t|x)&=(1+\lambda t)^{\frac{x}{\lambda}}\frac{1}{k!}\Biggl[\frac{1}{\lambda}\log(1+\lambda t)-\sum_{j=1}^{r}(-\lambda)^{j-1}\frac{t^j}{j}\Biggr]^k\\&=(1+\lambda t)^{\frac{x}{\lambda}}\frac{1}{k!}\Biggl[\frac{1}{\lambda}\log(1+\lambda t)-\sum_{j=1}^{r-1}(-\lambda)^{j-1}\frac{t^j}{j}+(-1)^r\lambda^{r-1}\frac{t^r}{r}\Biggr]^k\\&=(1+\lambda t)^{\frac{x}{\lambda}}\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}\Biggl(\frac{1}{\lambda}\log(1+\lambda t)-\sum_{j=1}^{r-1}(-\lambda)^{j-1}\frac{t^j}{j}\Biggr)^{k-l}(-1)^{rl}\frac{\lambda^{rl-l}}{r^l}t^{rl}\\&=\sum_{l=0}^{k}\frac{(-1)^{rl}\lambda^{rl-l}}{l!r^l}t^{rl}\frac{1}{(k-l)!}(1+\lambda t)^{\frac{x}{\lambda}}\Biggl(\frac{\log(1+\lambda t)}{\lambda}-\sum_{j=1}^{r-1}(-1)^{j-1}\frac{t^j}{j}\Biggr)^{k-l}\\&=\sum_{l=0}^{k}\frac{(-1)^{rl}\lambda^{rl-l}}{l!r^l}t^{rl}y_{k-l,\lambda}^{(r)}(t|x).\end{aligned}\tag{34}$$

By (34), we get

$$\begin{aligned}\sum_{n=kr}^{\infty}S_{1,\lambda}^{(x)}(n,k|r+1)\frac{t^n}{n!}&=\sum_{n=k(r+1)}^{\infty}S_{1,\lambda}^{(x)}(n,k|r+1)\frac{t^n}{n!}\\&=\sum_{l=0}^{k}\frac{(-1)^{rl}\lambda^{rl-l}}{l!r^l}t^{rl}\sum_{n=(k-l)r}^{\infty}S_{1,\lambda}^{(x)}(n,k-l|r+1)\frac{t^n}{n!}\\&=\sum_{n=kr}^{\infty}\Biggl(\sum_{l=0}^{k}\frac{(-1)^{rl}\lambda^{rl-l}}{l!r^l}S_{1,\lambda}^{(x)}(n-lr,k-l|r)(n)_{lr}\Biggr)\frac{t^n}{n!}.\end{aligned}\tag{35}$$

Comparing the coefficients on both sides of (35), we have

$$S_{1,\lambda}^{(x)}(n,k|r+1)=\sum_{l=0}^{k}\frac{(-1)^{rl}\lambda^{rl-l}}{l!r^l}S_{1,\lambda}^{(x)}(n-lr,k-l|r)(n)_{lr},\tag{36}$$

where $n\geq kr$.

For $\lambda\in(0,1)$, X is a random variable with the λ-logarithmic distribution with parameter $\alpha\in(0,1)$ if the probability mass function of X is given by

$$P_\lambda[X=k]=P_\lambda(k)=-\frac{\lambda}{\log(1-\alpha\lambda)}\cdot\frac{\alpha^k\lambda^{k-1}}{k},\tag{37}$$

where k is a positive integer.

We easily see that

$$\sum_{k=1}^{\infty} P_\lambda(k) = 1, \qquad E[X] = -\frac{1}{\log(1-\alpha\lambda)} \cdot \frac{\alpha\lambda}{1-\alpha\lambda}.$$

Y is the random variable with negative λ-binomial distribution with parameters r, α if the probability mass function of Y is given by

$$P_\lambda[Y=k] = P_\lambda(k) = \binom{\frac{r}{\lambda}+k-1}{k}(\lambda\alpha)^k(1-\lambda\alpha)^{\frac{r}{\lambda}},$$

where r, k, α are respectively the number of failures, the number of successes, and the probability of successes.

Let $X_1, X_2, \ldots, X_k$ be independent random variables with λ-logarithmic distribution with parameter α, and let Y be the random variable with negative λ-binomial distribution with parameters r and α. If Y is independent of $X = X_1 + \cdots + X_k$, then we have

$$E\bigl[t^{X+Y}\bigr] = E\bigl[t^X\bigr]E\bigl[t^Y\bigr] = \left(\prod_{j=1}^{k} E\bigl[t^{X_j}\bigr]\right) \cdot E\bigl[t^Y\bigr]. \tag{38}$$

Now, we observe that

$$E\bigl[t^{X_j}\bigr] = \sum_{x=1}^{\infty} P_\lambda[X_j = x]t^x = \frac{1}{\log(1-\alpha\lambda)} \cdot \log(1-\alpha\lambda t) \tag{39}$$

and

$$\begin{aligned} E\bigl[t^Y\bigr] &= \sum_{y=0}^{\infty} P_\lambda[Y=y]t^y = \sum_{y=0}^{\infty}\binom{\frac{r}{\lambda}+y-1}{y}(\lambda\alpha)^y(1-\lambda\alpha)^{\frac{r}{\lambda}}t^y \\ &= (1-\lambda\alpha)^{\frac{r}{\lambda}}(1-\lambda\alpha t)^{-\frac{r}{\lambda}}. \end{aligned} \tag{40}$$

From (38), (39), and (40), we have

$$\begin{aligned} E\bigl[t^{X+Y}\bigr] &= \left(\prod_{j=1}^{k} E\bigl[t^{X_j}\bigr]\right) \cdot E\bigl[t^Y\bigr] \\ &= \left(\frac{1}{\log(1-\alpha\lambda)}\right)^k \bigl(\log(1-\alpha\lambda t)\bigr)^k (1-\alpha\lambda)^{\frac{r}{\lambda}}(1-\alpha\lambda t)^{-\frac{r}{\lambda}} \\ &= k!\left(\frac{\lambda}{\log(1-\alpha\lambda)}\right)^k (1-\alpha\lambda)^{\frac{r}{\lambda}} \frac{1}{k!}\left(\frac{\log(1-\alpha\lambda t)}{\lambda}\right)^k (1-\alpha\lambda t)^{-\frac{r}{\lambda}} \\ &= k!\left(\frac{\lambda}{\log(1-\alpha\lambda)}\right)^k (1-\alpha\lambda)^{\frac{r}{\lambda}} \sum_{n=k}^{\infty} S_{1,\lambda}^{(-r)}(n,k)(-\alpha)^n \frac{t^n}{n!}. \end{aligned} \tag{41}$$

On the other hand,

$$E\bigl[t^{X+Y}\bigr] = \sum_{n=k}^{\infty} P_\lambda[X+Y=n]t^n. \tag{42}$$

Therefore, by (41) and (42), we obtain the following theorem.

Theorem 2.9 *Let $X_1, X_2, \ldots, X_k$ be independent random variables with λ-logarithmic distribution with parameter α, and let Y be the random variable with negative λ-binomial distribution with parameters r and α. If Y is independent of $X = X_1 + X_2 + \cdots + X_k$, then the probability mass function of $X + Y$ is given by*

$$P_\lambda[X+Y=n] = k!\left(\frac{\lambda}{\log(1-\alpha\lambda)}\right)^k (1-\alpha\lambda)^{\frac{r}{x}} \frac{(-\alpha)^n}{n!} S_{1,\lambda}^{(-r)}(n,k)$$

for $n \geq k$.

For $r \in \mathbb{N}$, X is the random variable with r-truncated λ-logarithmic distribution with parameter α if the probability mass function of X is given by

$$\begin{aligned} P_\lambda[X=x] = P_\lambda(x) &= \frac{\lambda}{-\log(1-\alpha\lambda) - \sum_{i=1}^{r-1} \frac{\lambda^i\alpha^i}{i}} \cdot \frac{\alpha^x \lambda^{x-1}}{x} \\ &= C_\lambda(\alpha, r) \frac{\alpha^x \lambda^{x-1}}{x} \quad (x = r, r+1, \ldots), \end{aligned}$$

where

$$C_\lambda(\alpha, r) = \frac{\lambda}{-\log(1-\alpha\lambda) - \sum_{i=1}^{r-1} \frac{\lambda^i\alpha^i}{i}}.$$

Note that $\sum_{x=r}^{\infty} P_\lambda[X=x] = 1$.

Let $X_1, X_2, \ldots, X_k$ be independent random variables with the r-truncated λ-logarithmic distribution with parameter p, and let Y be the random variable with negative λ-binomial distribution with parameters α, p. If Y is independent of $X = X_1 + X_2 + \cdots + X_k$, then we have

$$E\left[t^{X+Y}\right] = E\left[t^X\right] E\left[t^Y\right] = \left(\prod_{j=1}^{k} E\left[t^{X_j}\right]\right) E\left[t^Y\right]. \tag{43}$$

Now, we observe that

$$\begin{aligned} E\left[t^{X_j}\right] &= \sum_{x=r}^{\infty} P_\lambda[X_j = x] t^x \\ &= C_\lambda(p,r) \sum_{x=r}^{\infty} \frac{\lambda^{x-1} p^x}{x} t^x \\ &= C_\lambda(p,r) \left(\sum_{x=1}^{\infty} \frac{\lambda^{x-1} p^x}{x} t^x - \sum_{x=1}^{r-1} \frac{\lambda^{x-1} p^x}{x} t^x\right) \\ &= C_\lambda(p,r) \left(-\frac{1}{\lambda} \log(1-\lambda p t) - \sum_{x=1}^{r-1} \frac{\lambda^{x-1} p^x}{x} t^x\right), \end{aligned} \tag{44}$$

and

$$E\left[t^{Y}\right]=\sum_{y=0}^{\infty}\binom{\frac{\alpha}{\lambda}+y-1}{y}(\lambda p)^{y}(1-\lambda p)^{\frac{\alpha}{\lambda}}t^{y}$$
$$=(1-\lambda pt)^{-\frac{\alpha}{\lambda}}(1-\lambda p)^{\frac{\alpha}{\lambda}}. \tag{45}$$

From (43), (44), and (45), we have

$$E\left[t^{X+Y}\right]=k!C_{\lambda}(p,r)^{k}(-1)^{k}\frac{1}{k!}\left(\frac{\log(1-\lambda pt)}{\lambda}+\sum_{x=1}^{r-1}\frac{\lambda^{x-1}}{x}p^{x}t^{x}\right)^{k}(1-\lambda pt)^{-\frac{\alpha}{\lambda}}(1-\lambda p)^{\frac{\alpha}{\lambda}}$$
$$=k!C_{\lambda}(p,r)^{k}(1-\lambda p)^{\frac{\alpha}{\lambda}}(-1)^{k}\sum_{n=rk}^{\infty}S_{1,\lambda}^{(-\alpha)}(n,k|r)\frac{(-p)^{n}}{n!}t^{n}. \tag{46}$$

On the other hand,

$$E\left[t^{X+Y}\right]=\sum_{n=rk}^{\infty}P_{\lambda}[X+Y=n]t^{n}. \tag{47}$$

Therefore, by (46) and (47), we obtain the following theorem.

Theorem 2.10 *For $r\in\mathbb{N}$, let $X_1,X_2,\ldots,X_k$ be independent random variables with the r-truncated λ-logarithmic distribution with parameter p, and let Y be the random variable with negative λ-binomial distribution with parameters α, p. If Y is independent of $X=X_1+X_2+\cdots+X_k$, then the probability mass function of $X+Y$ is given by*

$$P_{\lambda}[X+Y=n]=k!C_{\lambda}(p,r)^{k}(1-\lambda p)^{\frac{\alpha}{\lambda}}(-1)^{n-k}S_{1,\lambda}^{(-\alpha)}(n,k|r)\frac{p^{n}}{n!}\quad(n\geq kr),$$

where

$$C_{\lambda}(\alpha,r)=\frac{\lambda}{-\log(1-\alpha\lambda)-\sum_{i=1}^{r-1}\frac{\lambda^{i}\alpha^{i}}{i}}.$$

3 Conclusion

Stirling numbers of the first kind appear frequently in combinatorics and number theory. Recently, λ-analogues of Stirling numbers of the first kind were studied in [10].

In this paper, we introduced λ-Stirling polynomials of the first kind which appear as the coefficients in the Taylor expansion of λ-falling factorial sequence and reduce to the Stirling numbers of the first kind when $x=0$ and $\lambda=1$. We obtained recurrence relations, explicit expressions, some identities, and connections with other special polynomials for these polynomials. We showed that they appear in an expression of the probability mass function of a suitable discrete random variable, constructed from λ-logarithmic and negative λ-binomial distributions. Thereby we demonstrated that these polynomials are not out of nowhere but arise naturally.

We also considered r-truncated λ-Stirling polynomials of the first kind whose generating function is obtained from that of the λ-Stirling polynomials of the first kind by truncating first $r-1$ terms in the Taylor expansion of the logarithmic function. We derived several basic properties about these polynomials just in the case of λ-Stirling polynomials of the first kind. Then we showed that they also appear in an expression of the probability mass function of a suitable discrete random variable, constructed from r-truncated λ-logarithmic and negative λ-binomial distributions. Once again, this demonstrates that r-truncated λ-Stirling polynomials of the first kind arise naturally.

As one of our future projects, we would like to continue to find many applications of λ-Stirling polynomials of the first kind and r-truncated λ-Stirling polynomials of the first kind in mathematics, sciences, and engineering.

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