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Schur-harmonic convexity related to co-ordinated harmonically convex functions in plane

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Abstract

In this paper, we investigate Schur-harmonic convexity of some functions which are obtained from the co-ordinated harmonically convex functions on a square in a plane.

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1 Introduction

Schur-convexity was introduced by Schur in 1923. Since then many researchers have devoted their efforts to it; see for example [6, 8, 12, 17, 19]. Schur-convexity has many important applications in analytic and geometric inequality, combinatorial analysis, numerical analysis, matrix theory, and so on. We recall some definitions.

Definition 1.1 ([2]) Suppose that $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. x is said to be majorized by y (with symbol $x \prec y$) if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \dots, n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$, denotes the ith largest component in x.

Definition 1.2 ([2]) Let $E \subset \mathbb{R}^n$, $f : E \to \mathbb{R}$ is said to be Schur-convex function on E if $x \prec y$ on E implies $f(x) \leq f(y)$. f is said to be Schur-concave if and only if $-f$ is Schur-convex.

Chu in [4, 5, 7, 18] defined the concept of Schur-harmonically convex function.

Definition 1.3 ([5]) A set $E \subset \mathbb{R}_+^n$ is said to be harmonically convex if $(\frac{2x_1y_1}{x_1+y_1}, \frac{2x_2y_2}{x_2+y_2}, \dots, \frac{2x_ny_n}{x_n+y_n}) \in E$, for every $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in E$.

Definition 1.4 ([5]) A function $f : E \to \mathbb{R}_+$ is said to be Schur-harmonically convex on E, for every $x, y \in E$, if $(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}) \prec (\frac{1}{y_1}, \frac{1}{y_2}, \dots, \frac{1}{y_n})$ implies $f(x) \le f(y)$.

Definition 1.5 ([2, 14])

(i) A set $E \subset \mathbb{R}^n$ is called symmetric, if $x \in E$ implies $Px \in E$ for every $n \times n$ permutation matrix P.

(ii) A function $f : E \to \mathbb{R}$ is said to be a symmetric function if $f(Px) = f(x)$ for every permutation matrix P, and for every $x \in E$.

Recall that a $n \times n$ square matrix P is said to be a permutation matrix if each row and column has a single unit entry, and all other entries are zero. The following theorem, called the Schur condition, is very useful for specifying Schur-convexity or Schur-concavity of functions.

Theorem 1.1 ([2]) *Let $E \subset \mathbb{R}^n$ be a symmetric convex set with nonempty interior* (E° *is the interior of* E), *and $f : E \to \mathbb{R}$ be a symmetric continuous function on E. If f is differentiable on E°, then f is Schur-convex* (*Schur-concave*) *on E° if and only if*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \ge 0 \quad (\le 0),$$

for every $x = (x_1, x_2, \dots, x_n) \in E^\circ$.

In [5] Chu proved the following result, which is useful for determining Schur-harmonic convexity or Schur-harmonic concavity of functions.

Theorem 1.2 ([5]) *Let $E \subset \mathbb{R}_+^n$ be a symmetric and harmonically convex set with nonempty interior* (E° *is the interior of* E), *and $f : E \to \mathbb{R}_+$ is a symmetric continuous function on E. If f is differentiable on E°, then f is Schur-harmonically convex* (*Schur-harmonically concave*) *on E° if and only if*

$$(x_1 - x_2)\left(x_1^2\frac{\partial f}{\partial x_1} - x_2^2\frac{\partial f}{\partial x_2}\right) \ge 0 \quad (\le 0),$$

for every $x = (x_1, x_2, \dots, x_n) \in E^\circ$.

By Definition 1.4 the following simple fact is obvious.

Lemma 1.1 ([5]) *The function $f : E \to \mathbb{R}^+$ is Schur-harmonically convex* (*Schur-harmonically concave*) *if and only if $f(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n})$ is Schur-convex* (*Schur-concave*) *on* $\frac{1}{E} = \{\frac{1}{x} : x \in E\}$.

In [1] harmonical convexity was introduced by Anderson et al. and in [13] İşcan gave the following definition.

Definition 1.6 Let $I \subset \mathbb{R} - \{0\}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be HA-convex or harmonically convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le tf(y) + (1-t)f(x), \tag{1}$$

for every $x, y \in I$ and $t \in [0,1]$. If the inequality in (1) is reversed, then f is said to be harmonically concave.

If $I \subset (0,\infty)$ and f is a convex and nondecreasing function then f is harmonically convex. If f is an harmonically convex and nonincreasing function then f is convex. If $[a,b] \subset I \subset (0,\infty)$ then the function $g : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$, defined by $g(t) = f(\frac{1}{t})$, is convex if and only if f is harmonically convex on $[a,b]$ (see [10]).

The following Hermite–Hadamard type inequality for harmonically convex functions was obtained by İşcan [13] and Dragomir [10] in different ways.

Theorem 1.3 *Let $f : I \subset \mathbb{R} - \{0\} \to \mathbb{R}$ ba a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold*:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \quad a, b \in I, a < b.$$

The above inequalities are sharp.

In [9], Dragomir defined convex function on the co-ordinates (or co-ordinated convex functions) on the set $[a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ as follows.

Definition 1.7 A function $f : [a,b] \times [c,d] \to \mathbb{R}$ is said to be convex on the co-ordinates on $[a,b] \times [c,d]$ if for every $y \in [c,d]$ and $x \in [a,b]$, the partial mappings

$$f_y\colon [a,b] \to \mathbb{R}, \quad f_y(u) = f(u,y),$$

and

$$f_x\colon [c,d] \to \mathbb{R}, \quad f_x(v) = f(x,v),$$

are convex. This means that, for every $(x,y), (z,w) \in [a,b] \times [c,d]$ and $t, s \in [0,1]$,

$$\begin{aligned} f\big(tx + (1-t)z, sy + (1-s)w\big) &\leq tsf(x,y) + s(1-t)f(z,y) \\ &\quad + t(1-s)f(x,w) + (1-t)(1-s)f(z,w). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex functions which are not convex. The following Hermite–Hadamard type inequality for co-ordinated convex functions was also proved in [9].

Theorem 1.4 *Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on $[a,b] \times [c,d]$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \right.\end{aligned}$$

$$+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\Big]$$
$$\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

The above inequalities are sharp.

In [16] Set and İşcan defined an harmonically convex and an harmonically convex function on the co-ordinates on the set $[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ as follows.

Definition 1.8 Let $\Delta=[a,b]\times[c,d]\subset(0,\infty)\times(0,\infty)$ with $a<b$ and $c<d$. A function $f:\Delta\to\mathbb{R}$ is said to be harmonically convex on Δ if the following inequality holds:

$$f\left(\frac{xz}{tz+(1-t)x},\frac{yw}{tw+(1-t)y}\right)=f\left(\frac{1}{\frac{t}{x}+\frac{1-t}{z}},\frac{1}{\frac{t}{y}+\frac{1-t}{w}}\right)$$
$$\le tf(x,y)+(1-t)f(z,w),\tag{2}$$

for every $(x,y),(z,w)\in\Delta$ and $t\in[0,1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave on Δ.

Definition 1.9 Let $\Delta=[a,b]\times[c,d]\subset(0,\infty)\times(0,\infty)$ with $a<b$ and $c<d$. A function $f:\Delta\to\mathbb{R}$ is said to be harmonically convex on the co-ordinates on Δ if for every $y\in[c,d]$ and $x\in[a,b]$, the partial mappings,

$$f_y:[a,b]\to\mathbb{R},\quad f_y(u)=f(u,y),$$

and

$$f_x:[c,d]\to\mathbb{R},\quad f_x(v)=f(x,v),$$

are harmonically convex.

Clearly, every harmonically convex function is harmonically convex on the co-ordinates. Furthermore, there exist co-ordinated harmonically convex functions which are not harmonically convex. Note that if f_x and f_y are convex and nondecreasing functions then f_x and f_y are harmonically convex. The following Hermite–Hadamard type inequality for harmonically co-ordinated convex functions was also proved in [16].

Theorem 1.5 *Let* $f:\Delta=[a,b]\times[c,d]\subset(0,\infty)\times(0,\infty)\to\mathbb{R}$ *is harmonically convex on the co-ordinates on* Δ*. Then*

$$f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)\le\frac{1}{2}\left[\frac{ab}{b-a}\int_a^b\frac{f(x,\frac{2cd}{c+d})}{x^2}\,dx+\frac{cd}{d-c}\int_c^d\frac{f(\frac{2ab}{a+b},y)}{y^2}\,dy\right]$$
$$\le\frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d\frac{f(x,y)}{(xy)^2}\,dy\,dx$$
$$\le\frac{1}{4}\Big[\frac{ab}{b-a}\int_a^b\frac{f(x,c)}{x^2}\,dx+\frac{ab}{b-a}\int_a^b\frac{f(x,d)}{x^2}\,dx$$

$$+ \frac{cd}{d-c}\int_c^d \frac{f(a,y)}{y^2}\,dy + \frac{cd}{d-c}\int_c^d \frac{f(b,y)}{y^2}\,dy\Bigg]$$
$$\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

The above inequalities are sharp.

In [11] Elezović and Pečarić investigated the Schur-convexity on the upper and the lower limit of the integral for the mean of convex function and proved the following important result by using the Hermite–Hadamard inequality.

Theorem 1.6 *Let f be a continuous function on an interval I, and*

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)\,dt, & x,y \in I, x \neq y, \\ f(x), & x = y \in I. \end{cases}$$

Then $F(x,y)$ is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

Let $I \subset \mathbb{R}$ be an open interval and $f \in C^2(I)$. In [6] Chu et al. proved the following theorems.

Theorem 1.7 *Let $f: I \to \mathbb{R}$ be a continuous function. The function*

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)\,dt - f(\frac{x+y}{2}), & x,y \in I, x \neq y, \\ 0, & x = y \in I, \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

Theorem 1.8 *Let $f: I \to \mathbb{R}$ be a continuous function. The function*

$$F(x,y) = \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)\,dt, & x,y \in I, x \neq y, \\ 0, & x = y \in I, \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

We recall the following lemma from [3], which is known as Leibniz's formula.

Lemma 1.2 *Suppose that $f: \Delta = [a,b] \times [c,d] \to \mathbb{R}$ and $\frac{\partial f}{\partial t}: [a,b] \times [c,d] \to \mathbb{R}$ are continuous and $\alpha_1, \alpha_2: [c,d] \to [a,b]$ are differentiable functions. Then the function $\varphi: [c,d] \to \mathbb{R}$ defined by*

$$\varphi(t) = \int_{\alpha_1(t)}^{\alpha_2(t)} f(x,t)\,dx,$$

has a derivative for each $t \in [c,d]$, which is given by

$$\varphi'(t) = f\big(\alpha_2(t),t\big)\alpha_2'(t) - f\big(\alpha_1(t),t\big)\alpha_1'(t) + \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{\partial f}{\partial t}(x,t)\,dx.$$

Moreover, in [15] we proved the following lemma which will be useful in the sequel. A version of the following lemma was proved in [17].

Lemma 1.3 *Let $F(u,v) = \int_u^v \int_u^v f(x,y)\,dx\,dy$, where $f(x,y)$ is continuous on the rectangle $[a,p] \times [a,q]$, $u = u(b)$ and $v = v(b)$ are differentiable with $a \le u(b) \le p$ and $a \le v(b) \le q$. Then*

$$\frac{\partial F}{\partial b} = \left(\int_u^v f(x,v)\,dx + \int_u^v f(v,y)\,dy\right)v'(b) - \left(\int_u^v f(x,u)\,dx + \int_u^v f(u,y)\,dy\right)u'(b).$$

2 Main result

In this section we prove new theorems like Theorem 1.6 and Theorems 1.7 and 1.8 for harmonically convex functions and co-ordinated harmonically convex functions.

Theorem 2.1 *Let $I \subset (0,\infty)$ be an open interval, and the function $f : I \to \mathbb{R}_+$ be continuously differentiable on I. Suppose that the function $F : I^2 \to \mathbb{R}_+$ is defined by*

$$F(x,y) := \begin{cases} \frac{xy}{y-x}\int_x^y \frac{f(t)}{t^2}\,dt, & x,y \in I, x \neq y, \\ f(x), & x = y \in I. \end{cases} \tag{3}$$

Then F is Schur-harmonically convex on I^2 if and only if f is harmonically convex on I.

Proof According to Lemma 1.1 it is sufficient to show that the function $F(\frac{1}{x},\frac{1}{y})$ is Schur-convex on $\frac{1}{I} \times \frac{1}{I}$. From (3) we have

$$F\left(\frac{1}{x},\frac{1}{y}\right) = \frac{\frac{1}{xy}}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} \frac{f(t)}{t^2}\,dt = \frac{1}{x-y}\int_{\frac{1}{x}}^{\frac{1}{y}} \frac{f(t)}{t^2}\,dt,$$

for every $x,y \in I$, with $x \neq y$. Using the change of variable $s = \frac{1}{t}$, then $F(\frac{1}{x},\frac{1}{y}) = \frac{1}{y-x}\int_x^y f(\frac{1}{s})\,ds$. Thus by Theorem 1.6 the function

$$F\left(\frac{1}{x},\frac{1}{y}\right) = \begin{cases} \frac{1}{y-x}\int_x^y f(\frac{1}{t})\,dt, & x,y \in I, x \neq y, \\ f(\frac{1}{x}), & x = y \in I, \end{cases}$$

is Schur-convex if and only if the function $f(\frac{1}{t})$ is convex on $\frac{1}{I}$. This implies that the function $f(t)$ is harmonically convex on I. Therefore by Theorem 1.6 the result follows. □

The proofs of the following two theorems are similar to the one for Theorem 2.1, hence we omit them.

Theorem 2.2 *Let $I \subset (0,\infty)$ be an open interval, and the function $f : I \to \mathbb{R}$ has continuous second order derivatives on I. Suppose that the function $G : I^2 \to \mathbb{R}_+$ is defined by*

$$G(x,y) := \begin{cases} \frac{xy}{y-x}\int_x^y \frac{f(t)}{t^2}\,dt - f(\frac{2xy}{x+y}), & x,y \in I, x \neq y, \\ 0, & x = y \in I. \end{cases}$$

Then G is Schur-harmonically convex on I^2 if and only if f is harmonically convex on I.

Theorem 2.3 *Let $I \subset (0,\infty)$ be an open interval, and the function $f : I \to \mathbb{R}$ has continuous second order derivatives on I. Suppose that the function $H : I^2 \to \mathbb{R}_+$ is defined by*

$$H(x,y) := \begin{cases} \frac{f(x)+f(y)}{2} - \frac{xy}{y-x}\int_x^y \frac{f(t)}{t^2}\,dt, & x,y \in I, x \neq y, \\ 0, & x = y \in I. \end{cases}$$

Then H is Schur-harmonically convex on I^2 if and only if f is harmonically convex on I.

To reach our main results, we need the following two lemmas.

Lemma 2.1 *Let $D = [a_1,b_1] \times [a_1,b_1]$ be a square in $\mathbb{R}^2 - \{(0,0)\}$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ be continuous, and have continuous second order partial derivatives on D°. Choose $a,b \in (a_1,b_1)$, with $a < b$, and let $\Delta = [a,b] \times [a,b]$. Suppose that the function $F : \Delta \to \mathbb{R}$ is defined by*

$$F(x,y) := \begin{cases} \frac{x^2y^2}{(y-x)^2}\int_x^y\int_x^y \frac{f(t,s)}{t^2s^2}\,dt\,ds, & x \neq y, x,y \in [a,b], \\ f(x,x), & x = y, x,y \in [a,b]. \end{cases}$$

Then

$$\begin{aligned} \frac{\partial F}{\partial x}\bigg|_{(t_0,t_0)} = \frac{\partial F}{\partial y}\bigg|_{(t_0,t_0)} &= \frac{1}{6}\bigg[2t_0\big(g(t_0,t_0)+h(t_0,t_0)\big) \\ &\quad + t_0^2\bigg(g_1(t_0,t_0)+h_1(t_0,t_0)+\frac{\partial g}{\partial t}(t,t)\bigg|_{t_0}+\frac{\partial h}{\partial t}(t,t)\bigg|_{t_0}\bigg)\bigg], \end{aligned} \tag{4}$$

for all $t_0 \in [a,b]$, where

$$g(u,t_0+t) = \frac{f(u,t_0+t)}{u^2}, \qquad h(t_0+t,v) = \frac{f(t_0+t,v)}{v^2},$$

and

$$g_1(u,t_0+t) = \frac{\partial g}{\partial t}(u,t_0+t), \qquad h_1(t_0+t,v) = \frac{\partial h}{\partial t}(t_0+t,v).$$

Proof Fix $t_0 \in [a,b]$. By using L'Hopital's rule and Lemmas 1.2, and 1.3 we see that

$$\begin{aligned} \frac{\partial F}{\partial x}\bigg|_{(t_0,t_0)} &= \lim_{t\to 0}\frac{F(t_0+t,t_0)-F(t_0,t_0)}{t} \\ &= \lim_{t\to 0}\frac{1}{t^3}\bigg[t_0^2(t_0+t)^2\int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t}\frac{f(u,v)}{u^2v^2}\,du\,dv - t^2f(t_0,t_0)\bigg] \\ &= \lim_{t\to 0}\frac{1}{3t^2}\bigg[2t_0^2(t_0+t)\int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t}\frac{f(u,v)}{u^2v^2}\,du\,dv \\ &\quad + t_0^2\bigg(\int_{t_0}^{t_0+t}\frac{f(u,t_0+t)}{u^2}\,du + \int_{t_0}^{t_0+t}\frac{f(t_0+t,v)}{v^2}\,dv\bigg) - 2tf(t_0,t_0)\bigg]. \end{aligned} \tag{5}$$

Again, using L'Hopital's rule we obtain

$$\begin{aligned}\frac{\partial F}{\partial x} = \lim_{t\to 0}\frac{1}{6t}\bigg[&2t_0^2\int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t}\frac{f(u,v)}{u^2v^2}\,du\,dv + \frac{2t_0^2}{t_0+t}\bigg(\int_{t_0}^{t_0+t}\frac{f(u,t_0+t)}{u^2}\,du \\ &+ \int_{t_0}^{t_0+t}\frac{f(t_0+t,v)}{v^2}\,dv\bigg) + t_0^2\bigg(g(t_0+t,t_0+t) + \int_{t_0}^{t_0+t}\frac{\partial g}{\partial t}(u,t_0+t)\,du \\ &+ h(t_0+t,t_0+t) + \int_{t_0}^{t_0+t}\frac{\partial h}{\partial t}(t_0+t,v)\,dv\bigg) - 2f(t_0,t_0)\bigg].\end{aligned}$$

By a similar computation it follows that

$$\begin{aligned}\frac{\partial F}{\partial x} &= \lim_{t\to 0}\frac{1}{6}\bigg[\frac{2t_0^2}{(t_0+t)^2}\bigg(\int_{t_0}^{t_0+t}\frac{f(u,t_0+t)}{u^2}\,du + \int_{t_0}^{t_0+t}\frac{f(t_0+t,v)}{v^2}\,dv\bigg) \\ &\quad + \frac{2t_0^2}{t_0+t}\bigg(g(t_0+t,t_0+t) + \int_{t_0}^{t_0+t}\frac{\partial g}{\partial t}(u,t_0+t)\,du + h(t_0+t,t_0+t) \\ &\quad + \int_{t_0}^{t_0+t}\frac{\partial h}{\partial t}(t_0+t,v)\,dv\bigg) + t_0^2\bigg(\frac{\partial g}{\partial t}(t_0+t,t_0+t) + g_1(t_0+t,t_0+t) \\ &\quad + \int_{t_0}^{t_0+t}\frac{\partial g_1}{\partial t}(u,t_0+t)\,du + \frac{\partial h}{\partial t}(t_0+t,t_0+t) + h_1(t_0+t,t_0+t) \\ &\quad + \int_{t_0}^{t_0+t}\frac{\partial h_1}{\partial t}(t_0+t,v)\,dv\bigg)\bigg] \\ &= \frac{1}{6}\bigg[2t_0\big(g(t_0,t_0)+h(t_0,t_0)\big) + t_0^2\bigg(g_1(t_0,t_0)+h_1(t_0,t_0)+\frac{\partial g}{\partial t}(t,t)\bigg|_{t_0} + \frac{\partial h}{\partial t}(t,t)\bigg|_{t_0}\bigg)\bigg].\end{aligned}$$

By changing the role of x by y in (5), we obtain the required results in (4). □

The proof of the following lemma is similar to the one in Lemma 2.1, hence we omit it.

Lemma 2.2 *Let $D = [a_1,b_1]\times[a_1,b_1]$ be a square in $\mathbb{R}^2 - \{(0,0)\}$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ be continuous, and have continuous third order partial derivatives on D°. Choose $a,b \in (a_1,b_1)$, with $a<b$, and let $\Delta = [a,b]\times[a,b]$. Suppose that the function $G : \Delta \to \mathbb{R}$ is defined by*

$$G(x,y) := \begin{cases}\frac{x^2y^2}{(y-x)^2}\int_x^y\int_x^y\frac{f(t,s)}{t^2s^2}\,dt\,ds - f(\frac{2xy}{x+y},\frac{2xy}{x+y}), & x\neq y, x,y\in[a,b], \\ 0, & x=y, x,y\in[a,b].\end{cases}$$

Then

$$\begin{aligned}\frac{\partial G}{\partial x}\bigg|_{(t_0,t_0)} &= \frac{\partial G}{\partial y}\bigg|_{(t_0,t_0)} \\ &= \frac{1}{6}\bigg[\frac{4f(t_0,t_0)}{t_0} \\ &\quad + t_0^2\bigg(g_1(t_0,t_0)+h_1(t_0,t_0)+\frac{\partial g}{\partial t}(t,t)\bigg|_{t_0}+\frac{\partial h}{\partial t}(t,t)\bigg|_{t_0}\bigg) - 6\frac{\partial f}{\partial t}(t,t)\bigg|_{t_0}\bigg],\end{aligned}$$

for all $t_0 \in [a,b]$, *where*

$$g(u,t_0+t)=\frac{f(u,t_0+t)}{u^2},\qquad h(t_0+t,v)=\frac{f(t_0+t,v)}{v^2},$$

and

$$g_1(u,t_0+t)=\frac{\partial g}{\partial t}(u,t_0+t),\qquad h_1(t_0+t,v)=\frac{\partial h}{\partial t}(t_0+t,v).$$

We now derive the next results for co-ordinated harmonically convex functions.

Theorem 2.4 *Let* $D=[a_1,b_1]\times[a_1,b_1]$ *be a square in* $\mathbb{R}^2_+$ *with* $a_1<b_1$, *and the function* $f:D\to\mathbb{R}_+$ *be continuous, and have continuous second order partial derivatives on* D°. *Choose* $a,b\in(a_1,b_1)$, *with* $a<b$, *and let* $\Delta=[a,b]\times[a,b]$. *Suppose that* f *is harmonically convex on the co-ordinates on* Δ, *then the function* $F:\Delta\to\mathbb{R}_+$ *defined by*

$$F(x,y):=\begin{cases}\frac{x^2y^2}{(y-x)^2}\int_x^y\int_x^y\frac{f(t,s)}{t^2s^2}\,dt\,ds, & x\neq y, x,y\in[a,b],\\ f(x,x), & x=y, x,y\in[a,b],\end{cases}\tag{6}$$

is Schur-harmonically convex on Δ.

Proof *Case 1*: if $x,y\in[a,b]$, with $x=y$. Then Lemma 2.1 implies that

$$(y-x)\left(y^2\frac{\partial F}{\partial y}-x^2\frac{\partial F}{\partial x}\right)=0.$$

Case 2: if $x,y\in[a,b]$, with $x\neq y$. Then by Lemma 1.3 we have

$$\begin{aligned}\frac{\partial F}{\partial y}&=\frac{-2x^3y}{(y-x)^3}\int_x^y\int_x^y\frac{f(t,s)}{t^2s^2}\,dt\,ds\\&\quad+\frac{x^2}{(y-x)^2}\left(\int_x^y\frac{f(t,y)}{t^2}\,dt+\int_x^y\frac{f(y,s)}{s^2}\,ds\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F}{\partial x}&=\frac{2xy^3}{(y-x)^3}\int_x^y\int_x^y\frac{f(t,s)}{t^2s^2}\,dt\,ds\\&\quad-\frac{y^2}{(y-x)^2}\left(\int_x^y\frac{f(t,x)}{t^2}\,dt+\int_x^y\frac{f(x,s)}{s^2}\,ds\right).\end{aligned}$$

Thus,

$$\begin{aligned}\left(y^2\frac{\partial F}{\partial y}-x^2\frac{\partial F}{\partial x}\right)&=\frac{-4x^3y^3}{(y-x)^3}\int_x^y\int_x^y\frac{f(t,s)}{t^2s^2}\,dt\,ds\\&\quad+\frac{x^2y^2}{(y-x)^2}\left(\int_x^y\frac{f(t,x)+f(t,y)}{t^2}\,dt\right.\\&\quad\left.+\int_x^y\frac{f(x,s)+f(y,s)}{s^2}\,ds\right).\end{aligned}$$

Then $(y-x)(y^2\frac{\partial F}{\partial y}-x^2\frac{\partial F}{\partial x})$ is nonnegative if

$$\begin{aligned}&\frac{xy}{(y-x)^2}\int_x^y\int_x^y f(t,s)\,dt\,ds\\&\quad\le\frac{1}{4(y-x)}\biggl(\int_x^y\frac{f(t,y)+f(t,x)}{t^2}\,dt+\frac{f(y,s)+f(x,s)}{s^2}\,ds\biggr).\end{aligned}$$

The last inequality follows from Theorem 1.5. Therefore by Theorem 1.2 the function F is Schur-harmonically convex. □

The following theorem also holds.

Theorem 2.5 *Let $D=[a_1,b_1]\times[a_1,b_1]$ be a square in $\mathbb{R}^2_+$ with $a_1<b_1$, and the function $f:D\to\mathbb{R}$ be continuous, and have continuous third order partial derivatives on D°. Choose $a,b\in(a_1,b_1)$, with $a<b$, and let $\Delta=[a,b]\times[a,b]$. Suppose that f is harmonically convex on the co-ordinates on Δ, then the function $G:\Delta\to\mathbb{R}_+$ defined by*

$$G(x,y):=\begin{cases}\frac{x^2y^2}{(y-x)^2}\int_x^y\int_x^y\frac{f(t,s)}{t^2s^2}\,dt\,ds-f(\frac{2xy}{x+y},\frac{2xy}{x+y}), & x\neq y, x,y\in[a,b],\\ 0, & x=y, x,y\in[a,b],\end{cases}\tag{7}$$

is Schur-harmonically convex for Δ.

Proof *Case 1*: If $x,y\in[a,b]$, with $x=y$. Then Lemma 2.2 implies that

$$(y-x)\biggl(y^2\frac{\partial G}{\partial y}-x^2\frac{\partial G}{\partial x}\biggr)=0.$$

Case 2: If $x,y\in[a,b]$, with $x\neq y$. Then by Lemma 1.2 we have

$$(y-x)\biggl(y^2\frac{\partial G}{\partial y}-x^2\frac{\partial G}{\partial x}\biggr)\ge 0,$$

if

$$\begin{aligned}&\frac{xy}{(y-x)^2}\int_x^y\int_x^y f(t,s)\,dt\,ds\\&\quad\le\frac{1}{4(y-x)}\biggl(\int_x^y\frac{f(t,y)+f(t,x)}{t^2}\,dt+\frac{f(y,s)+f(x,s)}{s^2}\,ds\biggr).\end{aligned}$$

The result follows from Theorem 1.2 and Theorem 1.5. □

In the following examples we show that the converses of Theorems 2.4 and 2.5 are not true in general.

Example 2.1 Consider the non-harmonically co-ordinated convex function,

$$f(t,s):=t^2-\frac{1}{3}s^2,\quad t,s\in[1,2].$$

It is easy to see that for the function F as defined in (6) we have $F(x,x) = \frac{2}{3}x^2$, for every $x \in [1,2]$, and

$$F(x,y) = \frac{x^2y^2}{(y-x)^2}\int_x^y\int_x^y \frac{t^2 - \frac{1}{3}s^2}{t^2s^2}\,dt\,ds = \frac{2}{3}xy,$$

for every $x, y \in [1,2]$, with $x \neq y$. Thus,

$$F(x,y) = \frac{2}{3}xy,$$

for every $x, y \in [1,2]$. Clearly F is symmetric, continuous and differentiable on $[1,2] \times [1,2]$.

If $x, y \in [1,2]$, with $x \neq y$, we have

$$(y-x)\left(y^2\frac{\partial F}{\partial y} - x^2\frac{\partial F}{\partial x}\right) = \frac{2}{3}xy(y-x)^2 \geq 0.$$

Therefore by Theorem 1.2 the function F is Schur-harmonically convex.

Remark 2.1 It is easy to see that for the function f as defined in Example 2.1 we have

$$\begin{aligned}
f\left(\frac{2xy}{x+y}, \frac{2xy}{x+y}\right) &\leq \frac{1}{2}\left[\frac{xy}{y-x}\int_x^y \frac{f(t, \frac{2xy}{x+y})}{t^2}\,dt + \frac{xy}{y-x}\int_x^y \frac{f(\frac{2xy}{x+y}, s)}{s^2}\,ds\right] \\
&\leq \frac{x^2y^2}{(y-x)^2}\int_x^y\int_x^y \frac{f(t,s)}{t^2s^2}\,dt\,ds \\
&\leq \frac{1}{4}\left[\frac{xy}{y-x}\int_x^y \frac{f(t,x)}{t^2}\,dt + \frac{xy}{y-x}\int_x^y \frac{f(t,y)}{t^2}\,dt\right. \\
&\quad \left. + \frac{xy}{y-x}\int_x^y \frac{f(x,s)}{s^2}\,ds + \frac{xy}{y-x}\int_x^y \frac{f(y,s)}{s^2}\,ds\right] \\
&\leq \frac{f(x,x) + f(x,y) + f(y,x) + f(y,y)}{4},
\end{aligned}$$

for every $x, y \in [1,2]$, with $x \neq y$. This means that each of the inequalities in Theorem 1.5 is valid while f is not harmonically convex on co-ordinates.

Example 2.2 Consider the non-harmonically co-ordinated convex function:

$$f(t,s) := 2t^2 - s^2, \quad t, s \in [1,2].$$

It is easy to see that for the function G as defined in (7) we have $G(x,x) = 0$, for every $x \in [1,2]$, and

$$\begin{aligned}
G(x,y) &= \frac{1}{(y-x)^2}\int_x^y\int_x^y \frac{2t^2 - s^2}{t^2s^2}\,dt\,ds - \left(\frac{2xy}{x+y}\right)^2 \\
&= xy - \left(\frac{2xy}{x+y}\right)^2,
\end{aligned}$$

for every $x \neq y$, with $x, y \in [1,2]$. Thus,

$$G(x,y) = xy - \left(\frac{2xy}{x+y}\right)^2,$$

for every $x, y \in [1,2]$. Clearly G is symmetric, continuous and differentiable on $[1,2] \times [1,2]$.

If $x, y \in [1,2]$, we have

$$(y-x)\left(y^2\frac{\partial G}{\partial y} - x^2\frac{\partial G}{\partial x}\right) = xy(y-x)^2 \geq 0.$$

Therefore by Theorem 1.2 the function G is Schur-harmonically convex.

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