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The partially shared values and small functions for meromorphic functions in a k-punctured complex plane

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Abstract

The main aim of this article is to discuss the uniqueness of meromorphic functions partially sharing some values and small functions in a k-punctured complex plane Ω. We proved the following: Let f_1, f_2 be two admissible meromorphic functions in Ω and α_j $(j = 1, 2, \dots, l)$ be $l(\geq 5)$ distinct small functions with respect to f and g. If $\widetilde{E}(\alpha_j, \Omega, f_1) \subseteq \widetilde{E}(\alpha_j, \Omega, f_2)$ $(j = 1, 2, \dots, l)$ and

$$\liminf_{r\to+\infty} \frac{\sum_{j=1}^{l} \overline{N}_0(r, \frac{1}{f_1-\alpha_j})}{\sum_{j=1}^{l} \overline{N}_0(r, \frac{1}{f_2-\alpha_j})} > \frac{5}{2l-5},$$

then $f_1 \equiv f_2$. Our results are some improvements and extension of previous theorems given by Cao–Yi and Ge–Wu.

MSC: Primary 30D30; secondary 30D35

Keywords: Meromorphic function; Partially sharing; Small function; k-punctured

1 Introduction

This article is devoted to the study of uniqueness of functions which are meromorphic in a multiply-connected domain–k-punctured complex plane Ω. In 1920s, Nevanlinna gave the definition of characterized function $T(r,f)$ of meromorphic function and established the famous first and second main theorem, lemma on the logarithmic derivatives etc. of Nevanlinna theory (see Hayman [1], Yang [2] and Yi and Yang [3]). Nowadays, Nevanlinna theory is a powerful tool in studying the properties of meromorphic functions in the fields of complex analysis. By applying this theory, the following well-known five-value theorem was given by Nevanlinna [4].

Theorem A (see [4]) *If f and g are two nonconstant meromorphic functions that share five distinct values a_1, a_2, a_3, a_4, a_5 IM in $X = \mathbb{C}$, then $f(z) \equiv g(z)$.*

Nevanlinna [4] also pointed out the following question.

Question A (see [4]) *Does Theorem A still hold if the five distinct values a_1, a_2, a_3, a_4, a_5 are replaced by five distinct small functions α_j $(j = 1, 2, \dots, 5)$?*

Around Theorem A and Question A, the value distribution theory of meromorphic functions occupies one of the central places in complex analysis. Moreover, *it is always an interesting topic how to extend and improve some important uniqueness theorems in the complex plane to the subset* $\mathbb{X}$ (including the unit disc, the angular domain, the annulus, etc.). Many scholars have paid significant attention to this topic and obtained lots of meaningful and important results (see [3, 5–9]). For example, Fang [10] in 1999 proved the five-value theorem for meromorphic functions in the unit disc; Zheng [11] in 2003 obtained the five-value theorem for meromorphic functions in an angular domain; Cao, Yi, and Xu [12] in 2009 gave the five-value theorem for meromorphic functions in the annuli with the help of the Nevanlinna theory for meromorphic functions on annuli given by Khrystiyanyn and Kondratyuk [13, 14], or [15] in 2005, or [16] in 2004 (see [12]), etc. including [3, 10, 11, 17–22]; Yi and Yang, Lahiri, and Xu improved a series of uniqueness theorems about weight-shared and partially shared (see [3, 23–26]); there are a series of beautiful and important results related to Question A (see [27–32]). Especially, Yi [31] gave a positive answer to Question A and extended the five-value theorem to the case of sharing five distinct small functions.

Theorem B ([31] The five small functions theorem) *Let f and g be two nonconstant meromorphic functions in a complex plane $\mathbb{C}$ and a_j ($j = 1, 2, 3, 4, 5$) be five distinct small functions with respect to f and g. If f and g share a_j ($j = 1, 2, 3, 4, 5$) IM in $\mathbb{C}$, then $f \equiv g$.*

In 2016, the authors investigated the uniqueness of meromorphic functions sharing some finite sets in a special multiply-connected region—k-punctured complex plane—and obtained an analog of Nevanlinna's famous five-value theorem for meromorphic functions f and g in a k-punctured complex plane [33, 34]. To state the result, some basic notations and a definition about k-punctured complex plane should be introduced as follows, which can be found in [35].

For k distinct points $c_j \in \mathbb{C}$, $j \in \{1, 2, \ldots, k\}$, $\Omega = \mathbb{C} \setminus \bigcup_{j=1}^{k} \{c_j\}$ can be called a k-punctured complex plane. Of course, the annulus is a special k-punctured plane as $k = 1$. Let $k \geq 2$, $d = \frac{1}{2} \min\{|c_s - c_j| : j \neq s\}$, and $r_0 = \frac{1}{d} + \max\{|c_j| : j \in \{1, 2, \ldots, k\}\}$, thus it yields that $\frac{1}{r_0} < d$,

$$\overline{D}_{1/r_0}(c_j) \cap \overline{D}_{1/r_0}(c_s) = \emptyset \quad \text{for } j \neq s$$

and

$$\overline{D}_{1/r_0}(c_j) \subset D_{r_0}(0) \quad \text{for } j \in \{1, 2, \ldots, k\},$$

where $D_\delta(c) = \{z : |z - c| < \delta\}$ and $\overline{D}_\delta(c) = \{z : |z - c| \leq \delta\}$. Define

$$\Omega_r = D_r(0) \setminus \bigcup_{j=1}^{k} \overline{D}_{1/r}(c_j) \quad \text{for any } r \geq r_0.$$

Thus, it follows that $\Omega_r \supset \Omega_{r_0}$ for $r_0 < r \leq +\infty$. Obviously, Ω_r is a multiple connected and $k + 1$ connected region.

For a meromorphic function f in the k-punctured plane Ω and $r_0 \le r < +\infty$, let $n_0(r,f)$ denote the counting function of its poles in $\overline{\Omega}_r$, and

$$N_0(r,f) = \int_{r_0}^{r} \frac{n_0(t,f)}{t}\,dt,$$

$$\begin{aligned} m_0(r,f) = {} & \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(re^{i\theta}\right)\right| d\theta + \frac{1}{2\pi}\sum_{j=1}^{m}\int_0^{2\pi} \log^+\left|f\left(c_j + \frac{1}{r}e^{i\theta}\right)\right| d\theta \\ & - \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(r_0e^{i\theta}\right)\right| d\theta - \frac{1}{2\pi}\sum_{j=1}^{m}\int_0^{2\pi} \log^+\left|f\left(c_j + \frac{1}{r_0}e^{i\theta}\right)\right| d\theta, \end{aligned}$$

where $\log^+ x = \max\{\log x, 0\}$, then

$$T_0(r,f) = m_0(r,f) + N_0(r,f)$$

is called the Nevanlinna characteristic of f in the k-punctured complex plane. Besides, we use $S(r,f)$ to denote any quantity satisfying $S(r,f) = o(T_0(r,f))$ for all r outside a possible exceptional set E of finite linear measure.

Definition 1.1 (see [33]) Let f be a nonconstant meromorphic function in a k-punctured plane Ω. The function f is called admissible in a k-punctured plane Ω provided that

$$\limsup_{r\to+\infty} \frac{T_0(r,f)}{\log r} = +\infty, \quad r_0 \le r < +\infty.$$

Remark 1.1 (see [33]) From Theorem 5 in [35], a meromorphic function f in a k-punctured plane is rational if f satisfies

$$\limsup_{r\to+\infty} \frac{T_0(r,f)}{\log r} < +\infty, \quad r_0 \le r < +\infty.$$

Theorem C (see [33, Theorem 3.1]) *Let f and g be two admissible meromorphic functions in Ω; if f,g share five distinct values a_1, a_2, a_3, a_4, a_5 IM in Ω, then $f(z) \equiv g(z)$.*

2 Results

The purpose of this article is to extend and improve some uniqueness results (including Theorems A–C) to a special multiply-connected region—k-punctured complex plane.

By relaxing the form of sharing values *IM* to the partially sharing in Theorem C, we obtain the first result of this article, which is an improvement of Theorem C.

Theorem 2.1 *Let f_1, f_2 be two admissible meromorphic functions in Ω, $a_1, a_2, \ldots, a_l$ be $l(\ge 5)$ distinct values. If $\widetilde{E}(a_j, \Omega, f_1) \subseteq \widetilde{E}(a_j, \Omega, f_2)$ for all $1 \le j \le l$ and*

$$\liminf_{r\to+\infty} \frac{\sum_{j=1}^{l} \widetilde{N}_0(r, \frac{1}{f_1-a_j})}{\sum_{j=1}^{l} \widetilde{N}_0(r, \frac{1}{f_2-a_j})} > \frac{1}{l-3},$$

where $\widetilde{E}(a, \Omega, h) = \{z \mid h(z) - a = 0, z \in \Omega\}$ for a meromorphic function $h(z)$ in Ω, where each zero is counted only once, then $f_1 \equiv f_2$.

From Theorem 2.1, we can obtain the following corollary immediately.

Corollary 2.1 *Let f_1,f_2 be two admissible meromorphic functions in Ω, $a_1,a_2,\ldots,a_l$ be $l(\geq 5)$ distinct values. If $\widetilde{E}(a_j,\Omega,f_1)\subseteq\widetilde{E}(a_j,\Omega,f_2)$ for all $1\leq j\leq l$ and $f_1\not\equiv f_2$, then*

$$\liminf_{r\to+\infty}\frac{\sum_{j=1}^{l}\widetilde{N}_0(r,\frac{1}{f_1-a_j})}{\sum_{j=1}^{l}\widetilde{N}_0(r,\frac{1}{f_2-a_j})}\leq\frac{1}{l-3}.$$

Remark 2.1 When $l=5$ and $\widetilde{E}(a_j,\Omega,f_1)=\widetilde{E}(a_j,\Omega,f_2)$ for all $1\leq j\leq l$, in this case,

$$\liminf_{r\to+\infty}\frac{\sum_{j=1}^{5}\widetilde{N}_0(r,\frac{1}{f_1-a_j})}{\sum_{j=1}^{5}\widetilde{N}_0(r,\frac{1}{f_2-a_j})}=1>\frac{1}{2}.$$

Then it follows $f_1\equiv f_2$ by Theorem 2.1. Thus, this shows that Theorem 2.1 is an improvement of Theorem C.

Inspired by Question A, Theorem B, and Theorem 2.1, *the second purpose of this paper is to investigate the uniqueness of meromorphic functions concerning small functions, and we obtain an analog of Nevanlinna's five-value theorem for meromorphic functions in a k-punctured complex plane.*

Theorem 2.2 *Let f_1,f_2 be two admissible meromorphic functions in Ω, $\alpha_1,\alpha_2,\ldots,\alpha_l$ be $l(\geq 5)$ distinct small functions with respect to f_1,f_2. If $\widetilde{E}(\alpha_j,\Omega,f_1)\subseteq\widetilde{E}(\alpha_j,\Omega,f_2)$ for all $1\leq j\leq l$ and*

$$\liminf_{r\to+\infty}\frac{\sum_{j=1}^{l}\widetilde{N}_0(r,\frac{1}{f_1-\alpha_j})}{\sum_{j=1}^{l}\widetilde{N}_0(r,\frac{1}{f_2-\alpha_j})}>\frac{5}{2l-5},$$

where $\widetilde{E}(\alpha,\Omega,h)=\{z|h(z)-\alpha(z)=0,z\in\Omega\}$ for a meromorphic function $h(z)$ in Ω, where each zero is counted only once, then $f_1\equiv f_2$.

Remark 2.2 Let f be a nonconstant meromorphic function in a k-punctured plane Ω, we denote by $S(f)$ a set of meromorphic function $a(z)$ in a k-punctured plane Ω satisfying $T_0(r,a)=S(r,f)$, and such a meromorphic function $a(z)$ in a k-punctured plane Ω is called a small function with respect to f.

From Theorem 2.2, the following results can be obtained immediately.

Corollary 2.2 *Let f_1,f_2 be two admissible meromorphic functions in Ω, and let $\alpha_1,\alpha_2,\ldots,\alpha_l$ be $l(\geq 5)$ distinct small functions with respect to f_1,f_2. If $\widetilde{E}(\alpha_j,\Omega,f_1)\subseteq\widetilde{E}(\alpha_j,\Omega,f_2)$ for all $1\leq j\leq l$ and $f_1\not\equiv f_2$, then*

$$\liminf_{r\to+\infty}\frac{\sum_{j=1}^{l}\widetilde{N}_0(r,\frac{1}{f_1-\alpha_j})}{\sum_{j=1}^{l}\widetilde{N}_0(r,\frac{1}{f_2-\alpha_j})}\leq\frac{5}{2l-5}.$$

Corollary 2.3 *Let f_1, f_2 be two admissible meromorphic functions in Ω, and let $\alpha_1, \alpha_2, \dots, \alpha_6$ be six distinct small functions with respect to f_1, f_2. If $\widetilde{E}(\alpha_j, \Omega, f_1) = \widetilde{E}(\alpha_j, \Omega, f_2)$ for all $1 \le j \le 6$, then $f_1 \equiv f_2$.*

3 The proof of Theorem 2.1

To prove Theorem 2.1, we require the following lemmas.

Lemma 3.1 (see [35, Theorem 3]) *Let f, f_1, f_2 be meromorphic functions in a k-punctured plane Ω. Then:*

(i) *the function $T_0(r,f)$ is nonnegative, continuous, nondecreasing, and convex with respect to $\log r$ on $[r_0, +\infty)$, $T_0(r_0, f) = 0$;*

(ii) *if f identically equals a constant, then $T_0(r,f)$ vanishes identically;*

(iii) *if f is not identically equal to zero, then $T_0(r,f) = T_0(r, 1/f), r_0 \le r < +\infty$;*

(iv) *$T_0(r, f_1 f_2) \le T_0(r, f_1) + T_0(r, f_2) + O(1)$ and $T_0(r, f_1 + f_2) \le T_0(r, f_1) + T_0(r, f_2) + O(1)$ for $r_0 \le r < +\infty$;*

(v) *$T_0(r, \frac{1}{f-a}) = T_0(r,f) + O(1)$ for any fixed $a \in \mathbb{C}$.*

By using Lemma 6 in [35], we can get the following lemma easily.

Lemma 3.2 *Let f be a nonconstant meromorphic function in a k-punctured plane Ω and p be a positive integer, then*

$$m_0\left(r, \frac{f^{(p)}}{f}\right) = O\big(\log T_0(r,f)\big) + O\big(\log^+ r\big) := S(r,f), \quad r \to +\infty,$$

outside a set E of finite linear measure.

Remark 3.1 Obviously, if f is admissible in a k-punctured plane Ω, then

$$m_0\left(r, \frac{f^{(p)}}{f}\right) = S(r,f) = o\big(T_0(r,f)\big).$$

Lemma 3.3 ([34, Theorem 2.5]) *Let f be a nonconstant meromorphic function in a k-punctured plane Ω, and let $a_1, a_2, \dots, a_q$ $(q \ge 3)$ be distinct complex numbers in the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Then, for $r_0 \le r < +\infty$,*

$$(q-2)T_0(r,f) \le \sum_{\nu=1}^{q} \widetilde{N}_0\left(r, \frac{1}{f - a_\nu}\right) + S(r,f),$$

where $\widetilde{n}_0(r, \frac{1}{f-a})$ is the counting function of zeros of $f - a$ in $\overline{\Omega}_r$ with the multiplicities reduced by 1,

$$\widetilde{N}_0\left(r, \frac{1}{f - a_\nu}\right) = \int_{r_0}^{r} \frac{\widetilde{n}_0(t, \frac{1}{f-a_\nu})}{t}\, dt,$$

$r \ge r_0$ and $S(r,f)$ is stated as in Lemma 3.2.

Proof of Theorem 2.1 Without loss of generality, assume that a_j ($j = 1, 2, \dots, l$) are finite. In view of 3.3, it follows

$$(l-2)T_0(r,f_1) \le \sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_1 - a_j}\right) + S(r,f_1)$$

and

$$(l-2)T_0(r,f_2) \le \sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_2 - a_j}\right) + S(r,f_2).$$

Suppose that $f_1 \not\equiv f_2$. In view of $\widetilde{E}(a_j, \Omega, f_1) \subseteq \widetilde{E}(a_j, \Omega, f_2)$ for all $1 \le j \le l$, then it yields

$$\sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_1 - a_j}\right) \le \widetilde{N}_0\left(r, \frac{1}{f_1 - f_2}\right) \le T_0(r,f_1) + T_0(r,f_2) + O(1).$$

Thus, we have

$$\begin{aligned}
&\sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_1 - a_j}\right) \\
&\quad \le \left(\frac{1}{l-2} + o(1)\right) \sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_1 - a_j}\right) + \left(\frac{1}{l-2} + o(1)\right) \sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_2 - a_j}\right)
\end{aligned}$$

for all $r \notin E$, which implies

$$\left(\frac{l-3}{l-2} + o(1)\right) \sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_1 - a_j}\right) \le \left(\frac{1}{l-2} + o(1)\right) \sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_2 - a_j}\right)$$

for all $r \notin E$. Hence, it follows

$$\liminf_{r \to +\infty} \frac{\sum_{j=1}^{l} \widetilde{N}_0(r, \frac{1}{f_1 - \alpha_j})}{\sum_{j=1}^{l} \widetilde{N}_0(r, \frac{1}{f_2 - \alpha_j})} \le \frac{1}{l-3},$$

which is a contradiction. Thus, $f_1 \equiv f_2$.

Therefore, this completes the proof of Theorem 2.1. □

4 The proof of Theorem 2.2

To prove Theorem 2.2, we will require the following lemmas.

Lemma 4.1 *Let $f_1(z)$ and $f_2(z)$ be two admissible meromorphic functions in a k-punctured plane Ω, $a_t(z) (\not\equiv 0, 1) \in S(r) := S(f_1) \cap S(f_2)$, $t = 1, 2$, be not equal to constants simultaneously, and let*

$$F_s(z) = \begin{vmatrix} f_s f_s' & f_s' & f_s^2 - f_s \\ a_1 a_1' & a_1' & a_1^2 - a_1 \\ a_2 a_2' & a_2' & a_2^2 - a_2 \end{vmatrix} \quad \textit{for } s = 1, 2. \tag{4.1}$$

Then:

(i) $F_s(z) \not\equiv 0$ *for* $s = 1, 2$.

(ii)

$$2T_0(r,f_s) < \widetilde{N}_0\left(r, \frac{1}{f_s - 1}\right) + \widetilde{N}_0\left(r, \frac{1}{f_s}\right) + \widetilde{N}_0(r,f_s) + \widetilde{N}_0\left(r, \frac{1}{f_s - a_1}\right) \\ + \widetilde{N}_0\left(r, \frac{1}{f_s - a_2}\right) + S(r,f_1) + S(r,f_2) \quad \textit{for } s = 1, 2.$$

Proof (i) Assume that $F_1(z) \equiv 0$. Thus, we can rewrite (4.1) as the following form:

$$\left(\frac{a_1'}{a_1} - \frac{a_2'}{a_2}\right)\left(\frac{f_1'}{f_1 - 1} - \frac{a_2'}{a_2 - 1}\right) - \left(\frac{a_1'}{a_1 - 1} - \frac{a_2'}{a_2 - 1}\right)\left(\frac{f_1'}{f_1} - \frac{a_2'}{a_2}\right) \equiv 0. \tag{4.2}$$

Next, we will divide the proof into four cases as follows.

Case 1. If $\frac{a_1'}{a_1} \equiv \frac{a_2'}{a_2}$, then it follows $a_1 = \eta_1 a_2$, where $\eta_1 \neq 1$ is a constant. From the assumptions of this lemma, it yields $\frac{a_1'}{a_1-1} \not\equiv \frac{a_2'}{a_2-1}$, which implies $\frac{f_1'}{f_1} \equiv \frac{a_2'}{a_2}$. Thus, we have $f_1(z) = \eta_2 a_2(z)$, where η_2 is a constant. Therefore, we get a contradiction.

Case 2. If $\frac{a_1'}{a_1-1} \equiv \frac{a_2'}{a_2-1}$. By using the same argument as in Case 1, we also get a contradiction.

Case 3. If $\frac{a_1'}{a_1} \not\equiv \frac{a_2'}{a_2}$, $\frac{a_1'}{a_1-1} \not\equiv \frac{a_2'}{a_2-1}$ and $\frac{a_1'}{a_1} - \frac{a_2'}{a_2} \equiv \frac{a_1'}{a_1-1} - \frac{a_2'}{a_2-1}$. Thus it follows from (4.1) that

$$\frac{f_1'}{f_1 - 1} - \frac{f_1'}{f_1} \equiv \frac{a_2'}{a_2 - 1} - \frac{a_2'}{a_2}.$$

By a simple integral, we have $\frac{1}{f_1} = 1 - \eta_3(1 - \frac{1}{a_2})$, where η_3 is a constant, a contradiction.

Case 4. If $\frac{a_1'}{a_1} \not\equiv \frac{a_2'}{a_2}$, $\frac{a_1'}{a_1-1} \not\equiv \frac{a_2'}{a_2-1}$ and $\frac{a_1'}{a_1} - \frac{a_2'}{a_2} \not\equiv \frac{a_1'}{a_1-1} - \frac{a_2'}{a_2-1}$. Thus, we can rewrite (4.2) as the following form:

$$\left(\frac{a_1'}{a_1} - \frac{a_2'}{a_2}\right)\frac{f_1'}{f_1 - 1} - \left(\frac{a_1'}{a_1 - 1} - \frac{a_2'}{a_2 - 1}\right)\frac{f_1'}{f_1} \equiv \frac{a_1'}{a_1}\frac{a_2'}{a_2 - 1} - \frac{a_2'}{a_2}\frac{a_1'}{a_1 - 1}. \tag{4.3}$$

By observing (4.3), the zeros of $f_1 - 1$ in Ω can only occur at the zeros, 1-points and poles of $a_1(z)$ and $a_2(z)$, and the zeros of $\frac{a_1'}{a_1} - \frac{a_2'}{a_2}$ in Ω. Thus, we have

$$\begin{aligned}\widetilde{N}_0\left(r, \frac{1}{f_1 - 1}\right) &\le \sum_{j=1}^{2}\left\{N_0(r,a_j) + N_0\left(r, \frac{1}{a_j}\right) + N_0\left(r, \frac{1}{a_j - 1}\right)\right\} \\ &\quad + N_0\left(r, \frac{1}{\frac{a_1'}{a_1} - \frac{a_2'}{a_2}}\right) \\ &= S(r,f_1) + S(r,f_2).\end{aligned} \tag{4.4}$$

Similarly, we have

$$\widetilde{N}_0\left(r, \frac{1}{f_1}\right) = S(r,f_1) + S(r,f_2). \tag{4.5}$$

Further, the poles of f_1 in Ω can only occur at the zeros, 1-points and poles of $a_1(z)$ and $a_2(z)$, and the zeros of $(\frac{a_1'}{a_1}-\frac{a_2'}{a_2})-(\frac{a_1'}{a_1-1}-\frac{a_2'}{a_2-1})$ in Ω. By a simple calculation, we have

$$\widetilde{N}_0(r,f_1)=S(r,f_1)+S(r,f_2). \tag{4.6}$$

By Lemma 3.3 and from (4.4)–(4.6), it follows

$$\begin{aligned}T_0(r,f_1)&<\widetilde{N}_0\left(r,\frac{1}{f_1-1}\right)+\widetilde{N}_0\left(r,\frac{1}{f_1}\right)+\widetilde{N}_0(r,f_1)+S(r,f_1)\\&=S(r,f_1)+S(r,f_2),\end{aligned}$$

a contradiction.

If $F_2(z)\equiv 0$, by using the same argument as above, we also get a contradiction. Then we prove (i).

(ii) Let

$$\begin{aligned}&\delta(z)=\frac{1}{3}\min\bigl\{1,\bigl|a_1(z)\bigr|,\bigl|a_2(z)\bigr|,\bigl|a_1(z)-1\bigr|,\bigl|a_2(z)-1\bigr|,\bigl|a_1(z)-a_2(z)\bigr|,z\in\Omega\bigr\},\\&\theta_t(r)=\bigl\{\theta:\bigl|f_1\bigl(re^{i\theta}\bigr)-a_t\bigl(r^{i\theta}\bigr)\bigr|\le\delta\bigl(r^{i\theta}\bigr)\bigr\}\quad(t=1,2),\\&\theta_3(r)=\bigl\{\theta:\bigl|f_1\bigl(re^{i\theta}\bigr)\bigr|\le\delta\bigl(r^{i\theta}\bigr)\bigr\},\\&\theta_4(r)=\bigl\{\theta:\bigl|f_1\bigl(re^{i\theta}\bigr)-1\bigr|\le\delta\bigl(r^{i\theta}\bigr)\bigr\}.\end{aligned}$$

Then it follows

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{\delta(re^{i\theta})}\,d\theta\\&\quad\le\frac{1}{2\pi}\int_0^{2\pi}\log\max\left\{1,\frac{1}{|a_1(z)|},\frac{1}{|a_2(z)|},\frac{1}{|a_1(z)-1|},\frac{1}{|a_2(z)-1|},\right.\\&\qquad\left.\frac{1}{|a_1(z)-a_2(z)|},z\in\Omega\right\}d\theta+\log 3\\&\quad\le m\left(r,\frac{1}{a_1}\right)+m\left(r,\frac{1}{a_2}\right)+m\left(r,\frac{1}{a_1-1}\right)+m\left(r,\frac{1}{a_2-1}\right)\\&\qquad+m\left(r,\frac{1}{a_1-a_2}\right)+4\log 2.\end{aligned}$$

Similarly, for any $c_j, j=1,2,\ldots,k$, we have

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{\delta(c_j+\frac{1}{r}e^{i\theta})}\,d\theta\\&\quad\le\frac{1}{2\pi}\int_0^{2\pi}\log\max\left\{1,\frac{1}{|a_1(c_j+\frac{1}{r}e^{i\theta})|},\frac{1}{|a_2(c_j+\frac{1}{r}e^{i\theta})|},\frac{1}{|a_1(c_j+\frac{1}{r}e^{i\theta})-1|},\right.\\&\qquad\left.\frac{1}{|a_2(c_j+\frac{1}{r}e^{i\theta})-1|},\frac{1}{|a_1(c_j+\frac{1}{r}e^{i\theta})-a_2(c_j+\frac{1}{r}e^{i\theta})|},z\in\Omega\right\}d\theta+\log 3\\&\quad\le m\left(\frac{1}{r},\frac{1}{a_1(c_j+z)}\right)+m\left(r,\frac{1}{a_2(c_j+z)}\right)+m\left(r,\frac{1}{a_1(c_j+z)-1}\right)\end{aligned}$$

$$+ m\left(r, \frac{1}{a_2(c_j+z)-1}\right) + m\left(r, \frac{1}{a_1(c_j+z)-a_2(c_j+z)}\right) + 4\log 2.$$

Further, for $j = 1, 2, \ldots, k$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{\delta(r_0e^{i\theta})}\,d\theta = O(1), \qquad \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{\delta(c_j+\frac{1}{r_0}e^{i\theta})}\,d\theta = O(1).$$

Thus, it follows

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{\delta(re^{i\theta})}\,d\theta + \sum_{j=1}^{k}\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{\delta(c_j+\frac{1}{r}e^{i\theta})}\,d\theta \\
&\quad - \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{\delta(r_0e^{i\theta})}\,d\theta - \sum_{j=1}^{k}\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{\delta(c_j+\frac{1}{r_0}e^{i\theta})}\,d\theta \\
&\le m_0\left(r, \frac{1}{a_1}\right) + m_0\left(r, \frac{1}{a_2}\right) + m_0\left(r, \frac{1}{a_1-1}\right) \\
&\quad + m_0\left(r, \frac{1}{a_2-1}\right) + m\left(r, \frac{1}{a_1-a_2}\right) + O(1) \\
&\le S(r,f_1) + S(r,f_2).
\end{aligned} \tag{4.7}$$

On the other hand, taking

$$\begin{aligned}
&f_1f_1' = (f_1-a_1)(f_1'-a_1') + a_1'(f_1-a_1) + a_1(f_1'-a_1') + a_1a_2' := F_1, \\
&f_1' = (f_1'-a_1') + a_1' := F_2, \\
&f_1^2 - f_1 = (f_1-a_1)^2 + (2a_1-1)(f_1-a_1) + a_1^2 - a_1 := F_3,
\end{aligned}$$

and substituting these into (4.22), by a simple calculation, we have

$$F = \begin{vmatrix} F_1 - a_1a_2' & F_2 - a_1' & F_3 - a_1^2 + a_1 \\ a_1a_1' & a_1' & a_1^2 - a_1 \\ a_2a_2' & a_2' & a_2^2 - a_2 \end{vmatrix}. \tag{4.8}$$

From the definition of $\theta_1(r)$ and $\delta(z)$, we have

$$\left|f_1(re^{i\theta}) - a_1(re^{i\theta})\right| \le \delta(re^{i\theta}) \le 1 + \left|a_1(re^{i\theta})\right| \quad \text{as } \theta \in \theta_1(r) \tag{4.9}$$

and

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\theta_1(r)} \log^+\left|\frac{F}{f_1-a_1}\right|\,d\theta \\
&\le m\left(r, \frac{f_1'-a_1'}{f_1-a_1}\right) + O\left(m(r,a_1) + m(r,a_2) + m(r,a_1') + m(r,a_2')\right) \\
&< S(r,f_1) + S(r,f_2).
\end{aligned} \tag{4.10}$$

On the other hand, we have $|f_1(re^{i\theta}) - a_1(re^{i\theta})| \geq \delta(re^{i\theta})$ as $\theta \notin \theta_1(r)$, that is,

$$\frac{1}{|f_1(re^{i\theta}) - a_1(re^{i\theta})|} \leq \frac{1}{\delta(re^{i\theta})} \quad \text{as } \theta \notin \theta_1(r). \tag{4.11}$$

By combining (4.10) and (4.11), we have

$$\begin{aligned} & m\left(r, \frac{1}{f_1 - a_1}\right) \\ & \quad \leq \frac{1}{2\pi} \int_{\theta_1(r)} \log^+ \left|\frac{F}{f_1 - a_1}\right| d\theta + \frac{1}{2\pi} \int_{\theta_1(r)} \log^+ \left|\frac{1}{F}\right| d\theta \\ & \qquad + \frac{1}{2\pi} \int_{[0,2\pi]-\theta_1(r)} \log \left|\frac{1}{\delta}\right| d\theta \\ & \quad \leq \frac{1}{2\pi} \int_{\theta_1(r)} \log^+ \left|\frac{1}{F(re^{i\theta})}\right| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log \left|\frac{1}{\delta(re^{i\theta})}\right| d\theta \\ & \qquad + S(r,f_1) + S(r,f_2). \end{aligned} \tag{4.12}$$

Similarly, we have

$$\begin{aligned} & m\left(\frac{1}{r}, \frac{1}{f_1(c_j + \frac{1}{r}e^{i\theta}) - a_1(c_j + \frac{1}{r}e^{i\theta})}\right) \\ & \quad \leq \frac{1}{2\pi} \int_{\theta_1(r)} \log^+ \left|\frac{F(c_j + \frac{1}{r}e^{i\theta})}{f_1(c_j + \frac{1}{r}e^{i\theta}) - a_1(c_j + \frac{1}{r}e^{i\theta})}\right| d\theta \\ & \qquad + \frac{1}{2\pi} \int_{\theta_1(r)} \log^+ \left|\frac{1}{F(c_j + \frac{1}{r}e^{i\theta})}\right| d\theta + \frac{1}{2\pi} \int_{[0,2\pi]-\theta_1(r)} \log \left|\frac{1}{\delta(c_j + \frac{1}{r}e^{i\theta})}\right| d\theta \\ & \quad \leq \frac{1}{2\pi} \int_{\theta_1(r)} \log^+ \left|\frac{1}{F(c_j + \frac{1}{r}e^{i\theta})}\right| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log \left|\frac{1}{\delta(c_j + \frac{1}{r}e^{i\theta})}\right| d\theta \\ & \qquad + S(r,f_1) + S(r,f_2). \end{aligned} \tag{4.13}$$

Since

$$\begin{aligned} & m\left(r_0, \frac{1}{f_1 - a_1}\right) = O(1), \qquad \frac{1}{2\pi} \int_{\theta_1(r_0)} \log^+ \left|\frac{1}{F(r_0 e^{i\theta})}\right| d\theta = O(1), \\ & m\left(\frac{1}{r_0}, \frac{1}{f_1(c_j + \frac{1}{r_0}e^{i\theta}) - a_1(c_j + \frac{1}{r_0}e^{i\theta})}\right) = O(1), \end{aligned}$$

and

$$\frac{1}{2\pi} \int_{\theta_1(r_0)} \log^+ \left|\frac{1}{F(c_j + \frac{1}{r_0}e^{i\theta})}\right| d\theta = O(1), \quad j = 1, 2, \ldots, k,$$

by combining (4.7), (4.12), and (4.13), it follows

$$m_0\left(r, \frac{1}{f_1 - a_1}\right)$$

$$\begin{aligned}
&\leq \frac{1}{2\pi}\int_{\theta_1(r)}\log^+\left|\frac{1}{F(re^{i\theta})}\right|d\theta+\sum_{j=1}^{k}\frac{1}{2\pi}\int_{\theta_1(r)}\log^+\left|\frac{1}{F(c_j+\frac{1}{r}e^{i\theta})}\right|d\theta \\
&\quad -\frac{1}{2\pi}\int_{\theta_1(r_0)}\log^+\left|\frac{1}{F(r_0e^{i\theta})}\right|d\theta-\sum_{j=1}^{k}\frac{1}{2\pi}\int_{\theta_1(r_0)}\log^+\left|\frac{1}{F(c_j+\frac{1}{r_0}e^{i\theta})}\right|d\theta \\
&\quad + S(r,f_1)+S(r,f_2).
\end{aligned} \tag{4.14}$$

By using the same argument as above, we have

$$\begin{aligned}
&m_0\left(r,\frac{1}{f_1-a_2}\right) \\
&\leq \frac{1}{2\pi}\int_{\theta_2(r)}\log^+\left|\frac{1}{F(re^{i\theta})}\right|d\theta+\sum_{j=1}^{k}\frac{1}{2\pi}\int_{\theta_2(r)}\log^+\left|\frac{1}{F(c_j+\frac{1}{r}e^{i\theta})}\right|d\theta \\
&\quad -\frac{1}{2\pi}\int_{\theta_2(r_0)}\log^+\left|\frac{1}{F(r_0e^{i\theta})}\right|d\theta-\sum_{j=1}^{k}\frac{1}{2\pi}\int_{\theta_2(r_0)}\log^+\left|\frac{1}{F(c_j+\frac{1}{r_0}e^{i\theta})}\right|d\theta \\
&\quad + S(r,f_1)+S(r,f_2);
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&m_0\left(r,\frac{1}{f_1}\right) \\
&\leq \frac{1}{2\pi}\int_{\theta_3(r)}\log^+\left|\frac{1}{F(re^{i\theta})}\right|d\theta+\sum_{j=1}^{k}\frac{1}{2\pi}\int_{\theta_3(r)}\log^+\left|\frac{1}{F(c_j+\frac{1}{r}e^{i\theta})}\right|d\theta \\
&\quad -\frac{1}{2\pi}\int_{\theta_3(r_0)}\log^+\left|\frac{1}{F(r_0e^{i\theta})}\right|d\theta-\sum_{j=1}^{k}\frac{1}{2\pi}\int_{\theta_3(r_0)}\log^+\left|\frac{1}{F(c_j+\frac{1}{r_0}e^{i\theta})}\right|d\theta \\
&\quad + S(r,f_1)+S(r,f_2);
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
&m_0\left(r,\frac{1}{f_1-1}\right) \\
&\leq \frac{1}{2\pi}\int_{\theta_4(r)}\log^+\left|\frac{1}{F(re^{i\theta})}\right|d\theta+\sum_{j=1}^{k}\frac{1}{2\pi}\int_{\theta_4(r)}\log^+\left|\frac{1}{F(c_j+\frac{1}{r}e^{i\theta})}\right|d\theta \\
&\quad -\frac{1}{2\pi}\int_{\theta_4(r_0)}\log^+\left|\frac{1}{F(r_0e^{i\theta})}\right|d\theta-\sum_{j=1}^{k}\frac{1}{2\pi}\int_{\theta_4(r_0)}\log^+\left|\frac{1}{F(c_j+\frac{1}{r_0}e^{i\theta})}\right|d\theta \\
&\quad + S(r,f_1)+S(r,f_2).
\end{aligned} \tag{4.17}$$

Since $\theta(r) \in [0, 2\pi)$, then from (4.14)–(4.17) it yields

$$\begin{aligned}
&m_0\left(r,\frac{1}{f_1-a_1}\right)+m_0\left(r,\frac{1}{f_1-a_2}\right)+m_0\left(r,\frac{1}{f_1}\right)+m_0\left(r,\frac{1}{f_1-1}\right) \\
&< m_0\left(r,\frac{1}{F}\right)+S(r,f_1)+S(r,f_2).
\end{aligned} \tag{4.18}$$

If z_0 is a zero of f_1 or f_1-1 or f_1-a_1 or f_1-a_2 in Ω of multiplies $p>1$ and not a pole of a_1 or a_2 in Ω, then z_0 must be a zero of $F_1(z)$ in Ω of multiplies $p-1$. Thus, it follows

$$\begin{aligned}4T_0(r,f_1) &< N_0\left(r,\frac{1}{f_1}\right)+N_0\left(r,\frac{1}{f_1-1}\right)+N_0\left(r,\frac{1}{f_1-a_1}\right)+N_0\left(r,\frac{1}{f_1-a_2}\right)\\ &\quad -N_0\left(r,\frac{1}{F_1}\right)+T_0(r,F_1)+O(1)+S(r,f_1)+S(r,f_2)\\ &< \widetilde{N}_0\left(r,\frac{1}{f_1}\right)+\widetilde{N}_0\left(r,\frac{1}{f_1-1}\right)+\widetilde{N}_0\left(r,\frac{1}{f_1-a_1}\right)+\widetilde{N}_0\left(r,\frac{1}{f_1-a_2}\right)\\ &\quad +T_0(r,F_1)+O(1)+S(r,f_1)+S(r,f_2). \end{aligned} \tag{4.19}$$

In addition, from the definition of $F_1(z)$, we can get

$$m_0(r,F_1) < 2m_0(r,f_1)+S(r,f_1)+S(r,f_2), \tag{4.20}$$

$$N_0(r,F_1) < 2N_0(r,f_1)+\widetilde{N}_0(r,f_1)+S(r,f_1)+S(r,f_2). \tag{4.21}$$

Hence, from (4.19)–(4.21), we can get Lemma 4.1(ii).

Therefore, this completes the proof of Lemma 4.1. □

Lemma 4.2 *Let $f_1(z)$ and $f_2(z)$ be two admissible meromorphic functions in a k-punctured plane Ω, $\alpha_j(z)(\not\equiv 0,1)\in S(f_1)\cap S(f_2)$, $j=1,2,\ldots,5$, be five distinct meromorphic functions in a k-punctured plane Ω, then*

$$2T_0(r,f_s) < \sum_{j=1}^{5}\widetilde{N}_0\left(r,\frac{1}{f_s-\alpha_j}\right)+S(r,f_1)+S(r,f_2),\quad s=1,2. \tag{4.22}$$

Proof Set

$$\begin{aligned} g_s &= \frac{f_s-\alpha_4}{f_s-\alpha_5}\frac{\alpha_3-\alpha_5}{\alpha_3-\alpha_4}\quad (s=1,2),\\ a_j &= \frac{\alpha_j-\alpha_4}{\alpha_j-\alpha_5}\frac{\alpha_3-\alpha_5}{\alpha_3-\alpha_4}\quad (j=1,2). \end{aligned}$$

Then it yields

$$\left|T_0(r,g_s)-T_0(r,f_s)\right| < S(r,f_1)+S(r,f_2),\quad \text{for } s=1,2, \tag{4.23}$$

$$S(r,f_1)+S(r,f_2)=S(r,g_1)+S(r,g_2). \tag{4.24}$$

Here we will consider three cases as follows.

Case 1. If g_1 and g_2 are admissible, then by applying Lemma 4.1 for g_1,g_2,a_1,a_2, we have

$$\begin{aligned}2T_0(r,g_s) &< \widetilde{N}_0(r,g_s)+\widetilde{N}_0\left(r,\frac{1}{g_s}\right)+\widetilde{N}_0\left(r,\frac{1}{g_s-1}\right)+\widetilde{N}_0\left(r,\frac{1}{g_s-a_1}\right)\\ &\quad +\widetilde{N}_0\left(r,\frac{1}{g_s-a_2}\right)+S(r,g_1)+S(r,g_2) \end{aligned} \tag{4.25}$$

for $s = 1, 2$. From the notations of g_1 and g_2 and $\alpha_j \in S(f_1) \cap S(f_2)$, we have

$$\widetilde{N}_0(r, g_s) < N_0\left(r, \frac{1}{f_s - \alpha_5}\right) + O\left(\sum_{j=1}^{5} T_0(r, \alpha_j)\right), \tag{4.26}$$

$$\widetilde{N}_0\left(r, \frac{1}{g_s}\right) < N_0\left(r, \frac{1}{f_s - \alpha_4}\right) + O\left(\sum_{j=1}^{5} T_0(r, \alpha_j)\right), \tag{4.27}$$

and

$$\widetilde{N}_0\left(r, \frac{1}{g_s - a_j}\right) < N_0\left(r, \frac{1}{f_s - \alpha_j}\right) + O\left(\sum_{j=1}^{5} T_0(r, \alpha_j)\right) \tag{4.28}$$

for $s = 1, 2; j = 1, 2$. Substituting (4.26)–(4.28) into (4.25), we can get (4.22) easily.

Case 2. If g_1 and g_2 are rational, then from Remark 2.1 we have $T_0(r, g_s) = O(\log r) = S(r, f_1) + S(r, f_2)$ for $s = 1, 2$. Thus, combining (4.23) and (4.24), it yields $T_0(r, f_s) = S(r, f_1) + S(r, f_2)$ for $s = 1, 2$. Hence the conclusions hold.

Case 3. If one of g_1, g_2 is rational, without loss of generality assume that g_1 is rational and g_2 is admissible. From Case 2 and Case 1, we have $T_0(r, f_1) = S(r, f_1) + S(r, f_2)$ and

$$T_0(r, f_2) < \sum_{j=1}^{5} \widetilde{N}_0\left(r, \frac{1}{f_2 - \alpha_j}\right) + S(r, f_1) + S(r, f_2).$$

Then the conclusion holds.

From Cases 1–3, this completes the proof of Lemma 4.2. □

Proof of Theorem 2.2 Take any distinct $s_1, \ldots, s_5 \in \{1, 2, \ldots, l\}$, and in view of Lemma 4.2, it follows

$$2T_0(r, f_i) < \sum_{j=1}^{5} \widetilde{N}_0\left(r, \frac{1}{f_i - \alpha_{s_j}}\right) + S(r), \quad i = 1, 2. \tag{4.29}$$

Thus, we can conclude

$$2\binom{l}{5} T_0(r, f_i) \le \frac{5}{l}\binom{l}{5} \sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_i - \alpha_j}\right) + S(r), \quad i = 1, 2,$$

that is,

$$T_0(r, f_i) \le \frac{5}{2l} \sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_i - \alpha_j}\right) + S(r), \quad i = 1, 2. \tag{4.30}$$

Suppose that $f_1 \not\equiv f_2$. In view of $\widetilde{E}(\alpha_j, \Omega, f_1) \subseteq \widetilde{E}(\alpha_j, \Omega, f_2)$ for all $1 \le j \le l$, we have

$$\sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_1 - \alpha_j}\right) \le \widetilde{N}_0\left(r, \frac{1}{f_1 - f_2}\right) \le T_0(r, f_1) + T_0(r, f_2) + O(1). \tag{4.31}$$

In view of (4.30) and (4.31), we can deduce

$$\sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_1-\alpha_j}\right) \leq \left(\frac{5}{2l}+o(1)\right)\sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_1-\alpha_j}\right) + \left(\frac{5}{2l}+o(1)\right)\sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_2-\alpha_j}\right)$$

for $r \notin E$, which implies

$$\left(\frac{2l-5}{2l}+o(1)\right)\sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_1-\alpha_j}\right) \leq \left(\frac{5}{2l}+o(1)\right)\sum_{j=1}^{l} \widetilde{N}_0\left(r, \frac{1}{f_2-\alpha_j}\right)$$

for $r \notin E$. This leads to

$$\liminf_{r\to+\infty} \frac{\sum_{j=1}^{l} \widetilde{N}_0(r, \frac{1}{f_1-\alpha_j})}{\sum_{j=1}^{l} \widetilde{N}_0(r, \frac{1}{f_2-\alpha_j})} \leq \frac{5}{2l-5},$$

which is a contradiction with the assumption of Theorem 2.2. Thus, $f_1 \equiv f_2$.

Therefore, this completes the proof of Theorem 2.2. □

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The main idea of this paper was proposed by HYX. HYX prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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