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Two-weight norm inequalities for fractional integral operators with $A_{\lambda,\infty}$ weights

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Abstract

In this paper, we introduce a new class of weights, the $A_{\lambda,\infty}$ weights, which contains the classical A_∞ weights. We prove a mixed $A_{p,q}$–$A_{\lambda,\infty}$ type estimate for fractional integral operators.

MSC: 42B20

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1 Introduction and the main results

Fractional integral operators and the associated maximal functions are very useful tools in harmonic analysis and PDE, especially in the study of differentiability or smoothness properties of functions. Recall that, for $0 < \lambda < n$, the fractional integral operator I_λ of a locally integrable function f defined on $\mathbb{R}^n$ is given by

$$I_\lambda f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\lambda}\, dy.$$

And the fractional maximal function M_λ is defined by

$$M_\lambda f(x) := \sup_{Q \ni x} \frac{1}{|Q|^{\lambda/n}} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the axes. We refer to [1–5] for more results on fractional integral operators.

For $1 < p, q < \infty$, we call a locally integrable positive function $w(x)$ defined on $\mathbb{R}^n$ a weight belongs to $A_{p,q}(\mathbb{R}^n)$ if

$$[w]_{A_{p,q}} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)^q\, dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-p'}\, dx\right)^{q/p'} < \infty.$$

In [6], Muckenhoupt and Wheeden showed that, for $1 < p < n/(n-\lambda)$ and $1/q + 1/p' = \lambda/n$, the fractional integral operator I_λ is bounded from $L^p(w^p)$ to $L^q(w^q)$ if and only if w belongs to $A_{p,q}$. They also proved that the fractional maximal function M_λ is bounded from $L^p(w^p)$ to $L^q(w^q)$ under the same conditions on the weights. Lacey, Moen, Pérez and Torres [7] proved the sharp weighted bound for fractional integral operators. Specifically,

$$\|I_\lambda\|_{L^p(w^p)\to L^q(w^q)} \le C_{n,p}[w]_{A_{p,q}}^{\frac{\lambda}{n}\max\{1,\frac{p'}{q}\}}.$$

And the sharp weighted bound for the fractional maximal function was proved by Pradolini and Salinas [8], i.e.,

$$\|M_\lambda\|_{L^p(w^p)\to L^q(w^q)} \le C_{n,p}[w]_{A_{p,q}}^{\frac{\lambda}{n}\cdot\frac{p'}{q}}. \tag{1.1}$$

Hytönen and Lacey [9] introduced a different approach to improving the sharp A_p estimates for Calderón–Zygmund operators using a mixed A_p–A_∞ condition. Cruz-Uribe and Moen [4] studied the corresponding problem for fractional integral operators. Recall that w is said to be a weight in A_∞ if

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)}\int_Q M(1_Q w)(x)\,dx < \infty,$$

where M is the Hardy–Littlewood maximal function and $w(Q) := \int_Q w(x)\,dx$. There are several equivalent definitions of the A_∞ weights. For example $w \in A'_\infty$ if

$$[w]_{A'_\infty} := \sup_Q \exp\left(\frac{1}{|Q|}\int_Q -\log\bigl(w(x)\bigr)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right) < \infty.$$

In [10], Fujii proved that $w \in A_\infty$ if and only if $w \in A'_\infty$. It is well known that $w \in A_\infty$ if and only if $w \in A_p$ for some $p > 1$, here A_p denotes the class of Muckenhoupt weights for which

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\right)^{p-1} < \infty.$$

Sbordone and Wik [11] showed that

$$[w]_{A'_\infty} = \lim_{p\to\infty}[w]_{A_p}.$$

Hytönen and Pérez [12] showed that $[w]_{A_\infty} \lesssim [w]_{A'_\infty}$, and in fact $[w]_{A_\infty}$ can be substantially smaller.

In this paper, we introduce a new class of weights, called the $A_{\lambda,\infty}$ weights, which is defined with the fractional maximal function.

Definition 1.1 Given $0 < \lambda < n$, $A_{\lambda,\infty}$ consists of all locally integrable functions $w(x)$ on $\mathbb{R}^n$ for which

$$[w]_{A_{\lambda,\infty}} := \sup_Q \frac{1}{w(Q)}\bigl\|M_\lambda(w1_Q)\cdot 1_Q\bigr\|_{n/\lambda} < \infty.$$

We show that A_∞ is a subset of $A_{\lambda,\infty}$. Specifically, we have the following results.

Theorem 1.2 *For any* $0 < \lambda < n$ *and* $w \in A_\infty$, *we have* $w \in A_{\lambda,\infty}$ *and*

$$[w]_{A_{\lambda,\infty}}^{\frac{n}{\lambda}} \le C_{n,\lambda}[w]_{A_\infty}.$$

With $A_{\lambda,\infty}$ weights, we give a mixed two-weight estimate of fractional integral operators.

Theorem 1.3 *Let* λ, p *and* q *be constants such that* $0 < \lambda < n$ *and* $1/q + 1/p' = \lambda/n$. *For any* $w \in A_{p,q}$, *set* $\mu = w^q$ *and* $\sigma = w^{-p'}$. *Then*

$$\|I_\lambda(\cdot\sigma)\|_{L^p(\sigma)\to L^q(\mu)} \lesssim [w]_{A_{p,q}}^{\frac{1}{q}}\left([\mu]_{A_{\lambda,\infty}}^{\frac{n}{\lambda p'}} + [\sigma]_{A_{\lambda,\infty}}^{\frac{n}{\lambda q}}\right).$$

The above theorem suggests us to generalize the $A_{p,q}$ condition for a pair of weights. Given $1 < p, q < \infty$, we say that a pair of weights (μ,σ) is in the class $A_{p,q}$ if

$$[\mu,\sigma]_{A_{p,q}} := \sup_Q \left(\frac{1}{|Q|}\int_Q \mu(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \sigma(x)\,dx\right)^{q/p'} < \infty.$$

With this notation, we can generalize Theorem 1.3 as follows.

Theorem 1.4 *Let* λ, p *and* q *be constants such that* $0 < \lambda < n$ *and* $1/q + 1/p' = \lambda/n$. *For any pair of weights* $(\mu,\sigma) \in A_{p,q}$ *with* $\mu, \sigma \in A_{\lambda,\infty}$, *we have*

$$\|I_\lambda(\cdot\sigma)\|_{L^p(\sigma)\to L^q(\mu)} \lesssim [\mu,\sigma]_{A_{p,q}}^{\frac{1}{q}}\left([\mu]_{A_{\lambda,\infty}}^{\frac{n}{\lambda p'}} + [\sigma]_{A_{\lambda,\infty}}^{\frac{n}{\lambda q}}\right).$$

The paper is organized as follows. In Sect. 2, we collect some preliminary results. And in Sect. 3, we give proofs for the main results.

2 Preliminaries

In this section, we introduce some preliminary results.

2.1 General dyadic grids

Let $\mathscr{D}$ be a set consisting of cubes in $\mathbb{R}^n$. Recall that $\mathscr{D}$ is said to be a general dyadic grid if it satisfies the following three conditions:

1. for any $Q \in \mathscr{D}$, its side length $l(Q)$ is of the form 2^k for some $k \in \mathbb{Z}$;
2. $Q_1 \cap Q_2 \in \{Q_1, Q_2, \emptyset\}$ for any $Q_1, Q_2 \in \mathscr{D}$;
3. the cubes of a fixed side length 2^k form a partition of $\mathbb{R}^n$.

Given a general dyadic grid $\mathscr{D}$, we call a subset $\mathcal{S} \subset \mathscr{D}$ a sparse family in $\mathscr{D}$ if it satisfies

$$\left|\bigcup_{Q' \in \mathcal{S}, Q' \subsetneq Q} Q'\right| \le \frac{1}{2}|Q|, \quad \forall Q \in \mathcal{S}.$$

For any $Q \in \mathcal{S}$, denote

$$E(Q) := Q \Big\backslash \left(\bigcup_{Q' \in \mathcal{S}, Q' \subsetneq Q} Q'\right).$$

We see from the definition of the sparse family that $|E(Q)| \ge \frac{1}{2}|Q|$ for any $Q \in \mathcal{S}$.

Below we will make extensive use of the dyadic grids

$$\mathscr{D}^\alpha := \{2^{-k}([0,1)^n + m + (-1)^k\alpha) : k \in \mathbb{Z}, m \in Z^n\}, \quad \alpha \in \left\{0, \frac{1}{3}, \frac{2}{3}\right\}^n.$$

Hytönen, Lacey and Pérez [13] proved the following result.

Lemma 2.1 (Three-lattice lemma) *For any cube $Q \subset \mathbb{R}^n$, there exists a shifted dyadic cube*

$$R \in \mathscr{D}^\alpha = \{2^{-k}([0,1)^n + m + (-1)^k\alpha) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}$$

for some $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, such that $Q \subseteq R$ and $\ell(R) \le 6\ell(Q)$.

2.2 Dyadic lattice

Let Q be any cube in $\mathbb{R}^n$. A dyadic child of Q is any of the 2^n cubes obtained by partitioning Q by n "median hyperplanes" (i.e., the hyperplanes parallel to the faces of Q and dividing each edge into two equal parts).

Passing from Q to its children, then to the children of the children, etc., we obtain a standard dyadic lattice $\mathcal{D}(Q)$ of subcubes of Q.

We refer to Lerner and Nazarov [14] for more properties of the dyadic lattice.

2.3 Dyadic fractional integral operators

Given $0 < \lambda < n$ and a general dyadic grid $\mathscr{D}$ in $\mathbb{R}^n$, we define the dyadic fractional integral operator $I_\lambda^{\mathscr{D}}$ by

$$I_\lambda^{\mathscr{D}} f(x) := \sum_{Q \in \mathscr{D}} \frac{1}{|Q|^{\lambda/n}} \int_Q f(y)\, dy \cdot 1_Q(x).$$

For a sparse family $\mathcal{S} \subseteq \mathscr{D}$, the sparse dyadic fractional integral operators $I_\lambda^{\mathcal{S}}$ are defined similarly. Cruz-Uribe and Moen [4] proved the following two propositions.

Proposition 2.2 *Given $0 < \lambda < n$ and a nonnegative function f, then $I_\lambda f$ is pointwise equivalent to a linear combination of dyadic fractional integral operators, i.e.,*

$$I_\lambda f(x) \simeq \sum_{\alpha \in \{0,1/3,2/3\}^n} I_\lambda^{\mathscr{D}^\alpha} f(x).$$

Proposition 2.3 *Given a bounded, nonnegative function f with compact support and a dyadic grid $\mathscr{D}$, there exists a sparse family $\mathcal{S}$ such that, for all λ with $0 < \lambda < n$, we have*

$$I_\lambda^{\mathscr{D}} f(x) \lesssim I_\lambda^{\mathcal{S}} f(x).$$

2.4 Testing condition

Let λ, p and q be constants such that $0 < \lambda < n$ and $1 < p \le q < \infty$. Lacey, Sawyer and Uriarte-Tuero [15] reduced the proof of the boundedness for $I_\lambda^{\mathcal{S}}(\cdot\sigma)$ from $L^p(\sigma)$ to $L^q(\mu)$ to the boundedness of the testing condition

$$\Theta_{\mu,\sigma}^{\mathscr{D}} := \sup_{R \in \mathscr{D}} \frac{\|1_R I_\lambda^{\mathcal{S}(R)}(\sigma 1_R)\|_{L^q(\mu)}}{\sigma(R)^{1/p}}, \qquad \Theta_{\sigma,\mu}^{\mathscr{D}} := \sup_{R \in \mathscr{D}} \frac{\|1_R I_\lambda^{\mathcal{S}(R)}(\mu 1_R)\|_{L^{p'}(\sigma)}}{\mu(R)^{1/q'}},$$

where the operator $I_\lambda^{\mathcal{S}(R)}$ is defined by

$$I_\lambda^{\mathcal{S}(R)} f(x) := \sum_{Q\in\mathcal{S},Q\subseteq R} \frac{1}{|Q|^{\lambda/n}} \int_Q f\,dy \cdot 1_Q(x).$$

Proposition 2.4 ([15, Theorem 1.11]) *Suppose that* λ, p *and* q *are constants such that* $0<\lambda<n$ *and* $1<p\le q<\infty$. *Let* $\mathscr{D}$ *be a dyadic grid and let* $\mathcal{S}$ *be a sparse subset of* $\mathscr{D}$. *For any pair of weights* (μ,σ), *we have*

$$\big\|I_\lambda^{\mathcal{S}}(\cdot\sigma)\big\|_{L^p(\sigma)\to L^q(\mu)} \simeq \Theta^{\mathscr{D}}_{\mu,\sigma} + \Theta^{\mathscr{D}}_{\sigma,\mu}.$$

3 Proofs of the main results

First, we show that a weight in $A_{p,q}$ is associated with a weight in some $A_{\lambda,\infty}$.

Theorem 3.1 *Suppose that* λ, p *and* q *are constants such that* $0<\lambda<n$ *and* $1/q+1/p'=\lambda/n$. *Let* $w\in A_{p,q}$ *and set* $\mu=w^q$. *Then* $\mu\in A_{\lambda,\infty}$ *and*

$$[\mu]_{A_{\lambda,\infty}}^{\frac{n}{\lambda}} \le C_{n,p}[w]_{A_{p,q}}.$$

Proof Since $w\in A_{p,q}$, we have $w^{-1}\in A_{q',p'}$ and

$$\big[w^{-1}\big]_{A_{q',p'}} = \sup_Q \left(\frac{1}{|Q|}\int_Q w^{-p'}(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w^q(x)\,dx\right)^{p'/q} = [w]_{A_{p,q}}^{\frac{p'}{q}}.$$

Using (1.1), this gives us

$$\|M_\lambda\|_{L^{q'}(w^{-q'})\to L^{p'}(w^{-p'})} \le C_{n,p}\big[w^{-1}\big]_{A_{q',p'}}^{\frac{\lambda}{n}\cdot\frac{q}{p'}} = C_{n,p}[w]_{A_{p,q}}^{\frac{\lambda}{n}}.$$

Fix some cube $Q\in\mathbb{R}^n$. Since $\frac{1}{p'}+\frac{1}{q}=\frac{\lambda}{n}$, we see from Hölder's inequality that

$$\begin{aligned}
\big\|M_\lambda(\mu 1_Q)\cdot 1_Q\big\|_{n/\lambda} &= \big\|M_\lambda(\mu 1_Q)\cdot w^{-1}\cdot w1_Q\big\|_{n/\lambda}\\
&\le \big\|M_\lambda(\mu 1_Q)\cdot w^{-1}1_Q\big\|_{p'}\cdot\|w1_Q\|_q\\
&= \big\|M_\lambda(\mu 1_Q)\cdot 1_Q\big\|_{L^{p'}(w^{-p'})}\cdot\mu(Q)^{1/q}\\
&\le C_{n,p}[w]_{A_{p,q}}^{\frac{\lambda}{n}}\|\mu 1_Q\|_{L^{q'}(w^{-q'})}\cdot\mu(Q)^{1/q}\\
&= C_{n,p}[w]_{A_{p,q}}^{\frac{\lambda}{n}}\mu(Q).
\end{aligned}$$

Hence

$$[\mu]_{A_{\lambda,\infty}}^{\frac{n}{\lambda}} \lesssim [w]_{A_{p,q}}.$$

This completes the proof. □

Next we show that A_∞ is contained in $A_{\lambda,\infty}$ for any $0<\lambda<n$.

Proof of Theorem 1.2 Fix some cube $Q \subseteq \mathbb{R}^n$. Let

$$Q^\beta := Q + \ell(Q)\beta, \quad \beta \in \left\{-\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}\right\}^n.$$

By translations and dilations, we see from Lemma 2.1 that for any cube $K \subseteq Q$ there exists some $R \in \mathcal{D}(Q^\beta)$ for some $\beta \in \{-\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}\}^n$ such that $R \in \mathcal{D}(Q^\beta)$ and $\ell(R) \leq 6\ell(K)$. Hence

$$M_\lambda(w1_Q)(x) \cdot 1_Q(x) \leq C_{n,\lambda} \max_{\beta \in \{-\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}\}^n} M_{\lambda,\mathcal{D}(Q^\beta)}(w1_Q)(x) \cdot 1_Q(x),$$

where

$$M_{\lambda,\mathcal{D}(Q^\beta)}(w1_Q)(x) := \sup_{\substack{K \ni x \\ K \in \mathcal{D}(Q^\beta)}} \frac{1}{|K|^{\lambda/n}} \int_K w(x)1_Q(x)\, dx.$$

So it sufficient to estimate

$$\frac{1}{w(Q)^{n/\lambda}} \int_Q \left|M_{\lambda,\mathcal{D}(Q^\beta)}(w1_Q)(x)\right|^{n/\lambda} dx, \quad \beta \in \left\{-\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}\right\}^n.$$

Among each $\mathcal{D}(Q^\beta)$, a subset of principal cubes $\mathscr{P}^\beta = \bigcup_{m=0}^\infty \mathscr{P}_m^\beta$ is constructed as follows: $\mathscr{P}_0^\beta = \{Q^\beta\}$, and then inductively $\mathscr{P}_{m+1}^\beta$ consists of all maximal $P' \in \mathcal{D}(Q^\beta)$ such that

$$\frac{w(P' \cap Q)}{|P'|^{\lambda/n}} > 2^{\lambda/n} \cdot 3^\lambda \frac{w(P \cap Q)}{|P|^{\lambda/n}}$$

for some $P \in \mathscr{P}_m^\beta$ with $P \supset P'$.

Since $\lambda/n < 1$, we see from the definition of $\mathscr{P}^\beta$ that, for any $P \in \mathscr{P}_m^\beta$,

$$\begin{aligned}
\left(\sum_{P' \in \mathscr{P}_{m+1}^\beta, P' \subset P} \frac{|P'|}{|P|}\right)^{\lambda/n} &\leq \sum_{P' \in \mathscr{P}_{m+1}^\beta, P' \subset P} \frac{|P'|^{\lambda/n}}{|P|^{\lambda/n}} \\
&\leq \frac{1}{2^{\lambda/n} \cdot 3^\lambda} \sum_{P' \in \mathscr{P}_{m+1}^\beta, P' \subset P} \frac{w(P' \cap Q)}{w(P \cap Q)} \\
&\leq \frac{1}{2^{\lambda/n} \cdot 3^\lambda}.
\end{aligned}$$

That is,

$$\sum_{P' \in \mathscr{P}_{m+1}^\beta, P' \subset P} \left|P'\right| \leq \frac{1}{2 \cdot 3^n} |P|. \tag{3.1}$$

For any $P \in \mathscr{P}^\beta$, we denote

$$E(P) := P \Big\backslash \bigcup_{P' \in \mathscr{P}^\beta, P' \subsetneq P} P'.$$

By (3.1), we have

$$\left|E(P)\right| \geq \left(1 - \frac{1}{2\cdot 3^n}\right)|P|.$$

By the definition of Q^β and $\mathcal{D}(Q^\beta)$, for any $R \in \mathcal{D}(Q^\beta)$ with $R \cap Q \neq \emptyset$, we have $|R \cap Q| \geq \frac{1}{3^n}|R|$. For any $P \in \mathscr{P}^\beta$, it is easy to see that $P \cap Q \neq \emptyset$. Combining with $|E(P)| \geq (1 - \frac{1}{2\cdot 3^n})|P|$, we obtain

$$\left|E(P) \cap Q\right| \geq \frac{1}{2\cdot 3^n}|P|.$$

Hence

$$\begin{aligned}
&\frac{1}{w(Q)^{n/\lambda}} \int_Q \left|M_{\lambda,\mathcal{D}(Q^\beta)}(w1_Q)(x)\right|^{n/\lambda} dx \\
&\quad \leq 2\cdot 3^n \frac{1}{w(Q)^{n/\lambda}} \sum_{P\in\mathscr{P}^\beta} w(P\cap Q)^{n/\lambda} \\
&\quad \leq 2\cdot 3^n \frac{1}{w(Q)^{n/\lambda}} \sum_{P\in\mathscr{P}^\beta} w(P\cap Q)\cdot \omega(Q)^{\frac{n}{\lambda}-1} \\
&\quad \leq 2^2\cdot 3^{2n} \frac{1}{w(Q)} \sum_{P\in\mathscr{P}^\beta} \frac{|E(P)\cap Q|}{|P|}\cdot w(P\cap Q) \\
&\quad \leq 2^2\cdot 3^{2n} \frac{1}{w(Q)} \int_Q M(w1_Q)(x)\,dx \\
&\quad \leq 2^2\cdot 3^{2n}[w]_{A_\infty}.
\end{aligned}$$

Now we get the conclusion as desired. □

Given a locally integrable function f, a Borel measure ν and a cube Q, we denote $\langle f\rangle_Q := \frac{1}{|Q|}\int_Q f(x)\,dx$ and $\langle f\rangle_Q^\nu := \frac{1}{\nu(Q)}\int_Q f(x)\,d\nu(x)$. The following result is used in the proof of Theorem 1.4.

Proposition 3.2 ([16, Proposition 2.2]) *Let $1 < s < \infty$, ν be a positive Borel measure and*

$$\phi = \sum_{Q\in\mathscr{D}} \alpha_Q 1_Q, \qquad \phi_Q = \sum_{Q'\subset Q} \alpha_{Q'} 1_{Q'}.$$

Then we have

$$\|\phi\|_{L^s(\nu)} \simeq \left(\sum_{Q\in\mathscr{D}} \alpha_Q \left(\langle\phi_Q\rangle_Q^\nu\right)^{s-1} \nu(Q)\right)^{1/s}.$$

To prove Theorem 1.4, we also need the following lemma.

Lemma 3.3 ([17, Lemma 5.2]) *For all $\gamma \in [0,1)$, we have $\sum_{L:L\subseteq P}\langle w\rangle_L^\gamma |L| \lesssim \langle w\rangle_P^\gamma |P|$.*

We are now ready to give a proof of Theorem 1.4.

Proof of Theorem 1.4 By Propositions 2.2, 2.3 and 2.4, it suffices to show that

$$\Theta_{\mu,\sigma}^{\mathscr{D}} := \sup_{R\in\mathscr{D}} \frac{\|1_R I_\lambda^{\mathcal{S}(R)}(\sigma 1_R)\|_{L^q(\mu)}}{\sigma(R)^{1/p}} \lesssim [\mu,\sigma]_{A_{p,q}}^{\frac{1}{q}} \cdot [\sigma]_{A_{\lambda,\infty}}^{\frac{n}{\lambda q}},$$

$$\Theta_{\sigma,\mu}^{\mathscr{D}} := \sup_{R\in\mathscr{D}} \frac{\|1_R I_\lambda^{\mathcal{S}(R)}(\mu 1_R)\|_{L^{p'}(\sigma)}}{\mu(R)^{1/q'}} \lesssim [\mu,\sigma]_{A_{p,q}}^{\frac{1}{q}} \cdot [\mu]_{A_{\lambda,\infty}}^{\frac{n}{\lambda p'}}.$$

There are two cases.

Case 1: $q \ge 2$. By Proposition 3.2, we have

$$\begin{aligned}
&\left\|1_R I_\lambda^{\mathcal{S}(R)}(\sigma 1_R)\right\|_{L^q(\mu)}^q \\
&\simeq \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \frac{\sigma(Q)}{|Q|^{\frac{\lambda}{n}}} \left(\frac{1}{\mu(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \frac{1}{|Q'|^{\lambda/n}} \sigma(Q')\mu(Q')\right)^{q-1} \cdot \mu(Q) \\
&= \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \frac{\sigma(Q)}{|Q|^{\frac{\lambda}{n}}} \left(\frac{1}{\mu(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\mu\rangle_{Q'}^{\frac{q'}{q}} \langle\sigma\rangle_{Q'}^{\frac{q'}{p'}} \langle\mu\rangle_{Q'}^{1-\frac{q'}{q}} |Q'|^{\frac{1}{q'}} \langle\sigma\rangle_{Q'}^{1-\frac{q'}{p'}} |Q'|^{\frac{1}{p}}\right)^{q-1} \cdot \mu(Q) \\
&\le [\mu,\sigma]_{A_{p,q}}^{\frac{q'}{q}(q-1)} \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \frac{\sigma(Q)}{|Q|^{\frac{\lambda}{n}}} \left(\frac{1}{\mu(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\mu\rangle_{Q'}^{1-\frac{q'}{q}} |Q'|^{\frac{1}{q'}} \langle\sigma\rangle_{Q'}^{1-\frac{q'}{p'}} |Q'|^{\frac{1}{p}}\right)^{q-1} \cdot \mu(Q) \\
&= [\mu,\sigma]_{A_{p,q}} \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \frac{\sigma(Q)}{|Q|^{\frac{\lambda}{n}}} \left(\frac{1}{\mu(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\mu\rangle_{Q'}^{1-\frac{q'}{q}} |Q'|^{\frac{1}{q'}} \cdot \langle\sigma\rangle_{Q'}^{1-\frac{q'}{p'}} |Q'|^{\frac{1}{p}}\right)^{q-1} \cdot \mu(Q) \\
&\le [\mu,\sigma]_{A_{p,q}} \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \frac{\sigma(Q)}{|Q|^{\frac{\lambda}{n}}} \left(\frac{1}{\mu(Q)} \left(\sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\mu\rangle_{Q'}^{p'(1-\frac{q'}{q})} |Q'|^{\frac{p'}{q'}}\right)^{\frac{1}{p'}}\right. \\
&\quad \times \left.\left(\sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\sigma\rangle_{Q'}^{p(1-\frac{q'}{p'})} |Q'|\right)^{\frac{1}{p}}\right)^{q-1} \cdot \mu(Q).
\end{aligned}$$

Since $\frac{p'}{q'} > 1$, this give us

$$\left(\sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\mu\rangle_{Q'}^{p'(1-\frac{q'}{q})} |Q'|^{\frac{p'}{q'}}\right)^{\frac{1}{p'}} \le \left(\sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\mu\rangle_{Q'}^{q'(1-\frac{q'}{q})} |Q'|\right)^{\frac{1}{q'}}.$$

Note that $q \ge 2$. We have

$$0 \le q'\left(1-\frac{q'}{q}\right) < 1, \qquad 0 \le p\left(1-\frac{q'}{p'}\right) < 1.$$

It follows from Lemma 3.3 that

$$\sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\mu\rangle_{Q'}^{q'(1-\frac{q'}{q})} |Q'| \lesssim \langle\mu\rangle_Q^{q'(1-\frac{q'}{q})} |Q|$$

and

$$\sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\sigma\rangle_{Q'}^{p(1-\frac{q'}{p'})}\left|Q'\right| \lesssim \langle\sigma\rangle_{Q}^{p(1-\frac{q'}{p'})}|Q|.$$

Hence

$$\begin{aligned}
&\left\|1_R I_\lambda^{\mathcal{S}(R)}(\sigma 1_R)\right\|_{L^q(\mu)}^q \\
&\quad\lesssim [\mu,\sigma]_{A_{p,q}} \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \frac{\sigma(Q)}{|Q|^{\frac{\lambda}{n}}}\left(\frac{1}{\mu(Q)}\left(\sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\mu\rangle_{Q'}^{p'(1-\frac{q'}{q})}\left|Q'\right|^{\frac{p'}{q'}}\right)^{\frac{1}{p'}}\right. \\
&\qquad\times\left.\left(\sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}} \langle\sigma\rangle_{Q'}^{p(1-\frac{q'}{p'})}\left|Q'\right|\right)^{\frac{1}{p}}\right)^{q-1}\cdot\mu(Q) \\
&\quad\lesssim [\mu,\sigma]_{A_{p,q}} \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \frac{\sigma(Q)}{|Q|^{\frac{\lambda}{n}}}\left(\frac{1}{\mu(Q)}\left(\langle\mu\rangle_Q^{q'(1-\frac{q'}{q})}|Q|\right)^{\frac{1}{q'}}\left(\langle\sigma\rangle_Q^{p(1-\frac{q'}{p'})}|Q|\right)^{\frac{1}{p}}\right)^{q-1}\cdot\mu(Q) \\
&\quad= [\mu,\sigma]_{A_{p,q}} \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \sigma(Q)^{\frac{q}{p}}.
\end{aligned}$$

Notice that

$$\frac{q}{p}\cdot\frac{\lambda}{n} = \frac{q}{p}\cdot\left(\frac{1}{q}+\frac{1}{p'}\right) = 1+\frac{q-p}{p'p} > 1.$$

This gives us $\frac{q}{p} > \frac{n}{\lambda}$.

Since $\mathcal{S}$ is a sparse family and $|E(Q)| > \frac{1}{2}|Q|$, we have

$$\begin{aligned}
\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \sigma(Q)^{\frac{q}{p}} &\le \sigma(R)^{\frac{q}{p}-\frac{n}{\lambda}} \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \sigma(Q)^{\frac{n}{\lambda}} \\
&\le 2\cdot\sigma(R)^{\frac{q}{p}-\frac{n}{\lambda}} \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \left|E(Q)\right|\cdot\frac{\sigma(Q)^{\frac{n}{\lambda}}}{|Q|} \\
&\le 2\cdot\sigma(R)^{\frac{q}{p}-\frac{n}{\lambda}} \int_R \left(M_\lambda(\sigma 1_R)\right)^{\frac{n}{\lambda}} \\
&\le 2[\sigma]_{A_{\lambda,\infty}}^{\frac{n}{\lambda}}\sigma(R)^{\frac{q}{p}}.
\end{aligned}$$

So

$$\left\|1_R I_\lambda^{\mathcal{S}(R)}(\sigma 1_R)\right\|_{L^q(\mu)}^q \lesssim [\mu,\sigma]_{A_{p,q}}\cdot[\sigma]_{A_{\lambda,\infty}}^{\frac{n}{\lambda}}\sigma(R)^{\frac{q}{p}}.$$

Taking the supreme over all cubes $R\in\mathscr{D}$, we obtain

$$\Theta_{\mu,\sigma}^{\mathscr{D}} \lesssim [\mu,\sigma]_{A_{p,q}}^{\frac{1}{q}}\cdot[\sigma]_{A_{\lambda,\infty}}^{\frac{n}{\lambda q}}.$$

Case 2: $1 < q < 2$. In this case, we have $0 \leq 1 - \frac{q}{p'} < 1$. Using Proposition 3.2 and Lemma 3.3, we get

$$\begin{aligned}
&\big\| 1_R I_\lambda^{\mathcal{S}(R)}(\sigma 1_R)\big\|_{L^q(\mu)}^q \\
&\simeq \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \frac{1}{|Q|^{\frac{\lambda}{n}}}\sigma(Q)\bigg(\frac{1}{\mu(Q)}\sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}}\frac{1}{|Q'|^{\lambda/n}}\sigma(Q')\mu(Q')\bigg)^{q-1}\cdot\mu(Q) \\
&= \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \frac{1}{|Q|^{\frac{\lambda}{n}}}\sigma(Q)\bigg(\frac{1}{\mu(Q)}\sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}}\langle\mu\rangle_{Q'}\langle\sigma\rangle_{Q'}^{\frac{q}{p'}}\cdot\langle\sigma\rangle_{Q'}^{1-\frac{q}{p'}}\big|Q'\big|^{2-\frac{\lambda}{n}}\bigg)^{q-1}\cdot\mu(Q) \\
&\leq [\mu,\sigma]_{A_{p,q}}^{q-1}\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \frac{1}{|Q|^{\frac{\lambda}{n}}}\sigma(Q)\bigg(\frac{|Q|^{1-\frac{\lambda}{n}}}{\mu(Q)}\sum_{\substack{Q'\in\mathcal{S}\\ Q'\subset Q}}\langle\sigma\rangle_{Q'}^{1-\frac{q}{p'}}\big|Q'\big|\bigg)^{q-1}\cdot\mu(Q) \\
&\lesssim [\mu,\sigma]_{A_{p,q}}^{q-1}\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \frac{1}{|Q|^{\frac{\lambda}{n}}}\sigma(Q)\bigg(\frac{|Q|^{1-\frac{\lambda}{n}}}{\mu(Q)}\langle\sigma\rangle_{Q}^{1-\frac{q}{p'}}|Q|\bigg)^{q-1}\cdot\mu(Q) \\
&= [\mu,\sigma]_{A_{p,q}}^{q-1}\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}}\langle\mu\rangle_Q^{2-q}\langle\sigma\rangle_Q^{\frac{q}{p'}(2-q)}\cdot\sigma(Q)^{\frac{q}{p}} \\
&\lesssim [\mu,\sigma]_{A_{p,q}}\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}}\sigma(Q)^{\frac{q}{p}}.
\end{aligned}$$

We see from the arguments in Case 1 that

$$\sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}}\sigma(Q)^{\frac{q}{p}} \leq 2[\sigma]_{A_{\lambda,\infty}}^{\frac{n}{\lambda}}\sigma(R)^{\frac{q}{p}}.$$

Hence

$$\big\| 1_R I_\lambda^{\mathcal{S}(R)}(\sigma 1_R)\big\|_{L^q(\mu)}^q \lesssim [\mu,\sigma]_{A_{p,q}}\cdot[\sigma]_{A_{\lambda,\infty}}^{\frac{n}{\lambda}}\sigma(R)^{\frac{q}{p}}.$$

Taking the supreme over all cubes $R\in\mathscr{D}$, we obtain

$$\Theta_{\mu,\sigma}^{\mathscr{D}} \lesssim [\mu,\sigma]_{A_{p,q}}^{\frac{1}{q}}\cdot[\sigma]_{A_{\lambda,\infty}}^{\frac{n}{\lambda q}}.$$

The estimates of $\Theta_{\sigma,\mu}^{\mathscr{D}}$ can be proved with the symmetry. This completes the proof. □

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