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Finite Morse index solutions of the Hénon Lane–Emden equation

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Abstract

In this paper, we are concerned with Liouville-type theorems of the Hénon Lane–Emden triharmonic equations in whole space. We prove Liouville-type theorems for solutions belonging to one of the following classes: stable solutions and finite Morse index solutions (whether positive or sign-changing). Our proof is based on a combination of the Pohozaev-type identity, monotonicity formula of solutions and a blowing down sequence.

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1 Introduction and main results

The paper is devoted to the study of the following nonlinear sixth-order Hénon type elliptic equation:

$$-\Delta^3 u = |x|^a |u|^{p-1} u, \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $a > 0$, and $p > 1$. We are interested in the Liouville-type theorems—i.e., the nonexistence of the solution u which is stable or of finite Morse index.

The idea of using the Morse index of a solution of a semilinear elliptic equation was first explored by Bahri and Lions [1] to obtain further qualitative properties of the solution. In 2007, Farina [7] made significant progress, and considered the Lane–Emden equation

$$-\Delta u = |u|^{p-1} u, \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $n \geq 2$ and $p > 1$. Farina completely classified finite Morse index solutions (positive or sign-changing) in his seminal paper [7]. His proof makes a delicate application of the classical Moser iteration method. Hereafter, many experts utilized the Moser's iterative method to discuss the stable and finite Morse index solutions of the harmonic and fourth-order elliptic equation and obtained many excellent results. We refer to [4, 5, 9, 16, 17] and the references therein.

However, the classical Moser iterative technique does not completely classify finite Morse index solutions of the biharmonic equation

$$\Delta^2 u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^n. \tag{1.3}$$

Dávila, Dupaigne, Wang and Wei [6] have derived a monotonicity formula for solutions of (1.3) to reduce the nonexistence of nontrivial entire solutions for the problem (1.3), to that of nontrivial homogeneous solutions, and gave a complete classification of stable solutions and those of finite Morse index solutions.

For the triharmonic Lane–Emden equation

$$-\Delta^3 u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^n, \tag{1.4}$$

Harrabi and Rahal [10] proved various Liouville-type theorems for smooth solutions under the assumption that they are stable or stable outside a compact set of $\mathbb{R}^n$. Again, following [6, 9, 17], they established the standard integral estimates via stability property to derive the nonexistence results in the subcritical case by the use of Pohozaev identity. The supercritical case needs more involved analysis, motivated by the monotonicity formula established in [3], they then reduced the nonexistence of nontrivial entire solutions, to that of nontrivial homogeneous solutions similarly to [6]. Through this approach, they gave a complete classification of stable solutions and those which are stable outside a compact set of $\mathbb{R}^n$ possibly unbounded and sign-changing. Inspired by [12], this analysis reveals a new critical exponent called the sixth-order Joseph–Lundgren exponent.

Let us recall that the Liouville-type theorems and properties of the subcritical case has been extensively studied by many authors. Gidas and Spruck have been investigated the optimal Liouville-type theorems in the celebrated paper [8]. Thus, Eq. (1.2) has no positive solution if and only if

$$p < \frac{n+2}{n-2} \quad (= +\infty, \text{ if } n \leq 2).$$

The supercritical case $p > \frac{n+2}{n-2}$ is much less completely understood. Bidaut–Véron and Véron [2] proved the asymptotic behavior of positive solution of (1.2) by the use of the Bochner–Lichnerowicz–Weitzenböck formula in $\mathbb{R}^n$.

On the other hand, understanding of the case $a \neq 0$ is less complete and is more delicate to handle than the case $a = 0$. In [8], Gidas and Spruck concluded that, for $a \leq -2$, the equation

$$-\Delta u = |x|^a u^p, \quad \text{in } \mathbb{R}^n, \tag{1.5}$$

has no positive solution. Recently, Wang and Ye [16] proved some Liouville-type theorems for weak finite Morse index solutions in the low dimensional Euclidean spaces of (1.5) with $a > -2$, $p > 1$ and $n \geq 2$.

The fourth-order Hénon type equation:

$$\Delta^2 u = |x|^a |u|^{p-1}u, \quad \text{in } \mathbb{R}^n \tag{1.6}$$

studied by Hu [11]. He proved Liouville-type theorems for solutions belonging to one of the following classes: stable solutions and finite Morse index solutions (whether positive or sign-changing). His proof is based on a combination of the Pohozaev-type identity, monotonicity formula of solutions and a blowing down sequence.

Inspired by the ideas in [10, 13], our purpose in this paper is to prove the Liouville-type theorems in the class of stable solution and finite Morse index solution.

Thus, for any fixed $a > 0$, we need to recall the following definition.

Definition 1.1 We say that a solution u of (1.1) belonging to $C^6(\mathbb{R}^n)$

- is stable, if for any $\psi \in C_c^3(\mathbb{R}^n)$, we have

$$Q_u(\psi) := \int_{\mathbb{R}^n} \left|\nabla(\Delta\psi)\right|^2 dx - p\int_{\mathbb{R}^n} |x|^a|u|^{p-1}\psi^2\,dx \geq 0;$$

- has Morse index equal to $K \geq 1$ if K is the maximal dimension of a subspace X_K of $C_c^3(\mathbb{R}^n)$ such that $Q_u(\phi) < 0$ for any $\phi \in X_K \setminus \{0\}$;
- is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^n$, if $Q_u(\phi) \geq 0$ for any $\phi \in C_c^3(\mathbb{R}^n \setminus \mathcal{K})$.

Remarks 1.1

1. Clearly, a solution stable if and only if its Morse index is equal to zero.
2. It is well known that any finite Morse index solution u is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^n$. Indeed, there exist $K \geq 1$ and $X_K := \mathrm{Span}\{\phi_1, \ldots, \phi_K\} \subset C_c^3(\mathbb{R}^n)$ such that $Q_u(\phi) < 0$ for any $\phi \in X_K \setminus \{0\}$. Hence, $Q_u(\psi) \geq 0$ for every $\psi \in C_c^3(\mathbb{R}^n \setminus \mathcal{K})$, where $\mathcal{K} := \bigcup_{j=1}^K \mathrm{supp}(\phi_j)$.

Now we can state our main results.

Theorem 1.1 *Let $u \in C^6(\mathbb{R}^n)$ be a stable solution of* (1.1) *and* $1 < p < p_a(n,6)$. *Then* $u \equiv 0$.

Theorem 1.2 *Let $u \in C^6(\mathbb{R}^n)$ be a solution of Eq.* (1.1) *which is stable outside a compact set of $\mathbb{R}^n$.*

- *If $1 < p < p_0(n,6)$ and $p \neq \frac{n+6+2a}{n-6}$, then $u \equiv 0$.*
- *If $p = \frac{n+6+2a}{n-6}$ and $n \geq 7$, then*

$$\int_{\mathbb{R}^n} \left|\nabla(\Delta u)\right|^2 dx = \int_{\mathbb{R}^n} |x|^a|u|^{p+1}\,dx < \infty.$$

Here the representation of $p_a(n,6)$ in Theorem 1.1 is given by (2.2) below and $p_0(n,6)$ in Theorem 1.2 is the sixth-order Joseph–Lundgren exponent which is computed by [10] in the case $a = 0$.

The organization of the rest of the paper is as follows. In Sect. 2, we need to define a critical power of (1.1). In Sect. 3, we construct a monotonicity formula which is a crucial tool to handle the supercritical case, In Sect. 4, we establish some finer integral estimates for the solutions of (1.1). In Sect. 5, we obtain a nonexistence result for the homogeneous stable solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$, where p belongs to $(\frac{n+6+2a}{n-6}, p_a(n,6))$. Then we prove a Liouville-type theorem for the stable solutions of (1.1), this is Theorem 1.1 in Sect. 6. To prove the result, we obtain some estimates of solutions, and we show that the limit of the blowing down sequence $u^\infty(x) = \lim_{\lambda\to\infty} \lambda^{\frac{6+a}{p-1}} u(\lambda x)$ satisfies $E(u,r) \equiv \mathrm{const}$. Here, we use the monotonicity formula of Theorem 3.2. In Sect. 7, we study a Liouville-type theorem of finite Morse index solutions by the use of the Pohozaev-type identity, monotonicity formula and a blowing down sequence.

2 Sixth-order Joseph–Lundgren exponent

The purpose of this section is to provide an implicit existence of the sixth-order Joseph–Lundgren exponent in the supercritical range. For any fixed $a > 0$ and $n \geq 7$, we define

$$J_3 = \alpha(\alpha + 2)(\alpha + 4)(n - 2 - \alpha)(n - 4 - \alpha)(n - 6 - \alpha)$$

and

$$\begin{aligned} F_a(\alpha) = pJ_3 - \frac{(n-6)^2(n-2)^2(n+2)^2}{64} \\ &= (\alpha + 6 + a)(\alpha + 2)(\alpha + 4)(n - 2 - \alpha)(n - 4 - \alpha)(n - 6 - \alpha) \\ &\quad - \frac{(n-6)^2(n-2)^2(n+2)^2}{64}, \end{aligned}$$

where $\alpha = \frac{6+a}{p-1}$. Note that

$$p > \frac{n + 6 + 2a}{n - 6} \quad \Leftrightarrow \quad 0 < \alpha < \frac{n-6}{2}.$$

F_a is increasing on $(0, \frac{n-6}{2})$. A direct computation finds

$$\begin{aligned} F_a\left(\frac{n-6}{2}\right) &= \frac{n + 6 + 2a}{n - 6}\frac{(n-6)^2(n-2)^2(n+2)^2}{64} - \frac{(n-6)^2(n-2)^2(n+2)^2}{64} \\ &= \frac{2(6+a)}{n-6}\frac{(n-6)^2(n-2)^2(n+2)^2}{64} > 0. \end{aligned} \tag{2.1}$$

We have also

$$\begin{aligned} F_a(0) &= \frac{(n-2)(n-6)}{64}\left(-n^4 + 4n^3 + 16n^2 + (3056 + 512a)n - 12{,}336 - 2048a\right) \\ &= \frac{(n-2)(n-6)}{64}E_a(n), \end{aligned}$$

where $E_a(x) = -x^4 + 4x^3 + 16x^2 + (3056 + 512a)x - 12{,}336 - 2048a$.

The function E_a satisfies the following properties:

(1) $E_a(7) > 0$, for all $a > 0$,

(2) $E''_a(x) = -12x^2 + 24x + 32 < 0$ on $[7, +\infty)$,

(3) $\lim_{x\to+\infty} E_a(x) = -\infty$.

Then there exists a unique $x_a \in (7, +\infty)$ such that $E_a(x_a) = 0$ and $E_a(x) > 0$ on $[7, x_a)$.

Note that $n(a)$ is the integer part of x_a.

(i) $\forall n \leq n(a)$, $E_a(n) > 0$. This implies that $F_a(0) > 0$. As a consequence $F_a(\alpha) > 0$, on $(0, \frac{n-6}{2})$.

(ii) $\forall n \geq n(a) + 1$, $E_a(n) < 0$. This implies that $F_a(0) < 0$. Then, there exists a unique $\alpha_a \in (0, \frac{n-6}{2})$ such that $F_a(\alpha_a) = 0$.

For any fixed $a > 0$ and $n \geq 7$, we define

$$p_a(n,6) = \begin{cases} +\infty & \text{if } n \leq n(a), \\ p(n,a) & \text{if } n \geq n(a)+1, \end{cases} \tag{2.2}$$

where $p(n,a) = \frac{6+a}{\alpha_a} + 1$.

Therefore,we find

$$pJ_3 > \frac{(n-6)^2(n+2)^2(n-2)^2}{64},$$

for any $\frac{n+6+2a}{n-6} < p < p_a(n,6)$. In particular, if $a = 0$, then $p_0(n,6)$ in (2.2) is the sixth-order Joseph–Lundgren exponent which is computed by [10].

Notation. Here and in the following, we use $B_r(x)$ to denote the open ball on $\mathbb{R}^n$ central at x with radius r. We also write $B_r = B_r(x)$. C denotes various irrelevant positive constants.

3 Monotonicity formula

In this section, we construct a monotonicity formula which plays an important role in dealing to understand supercritical elliptic equations or systems. This approach has been used successfully for the Lane–Emden equation in [6, 10, 11, 13]. We define the functional $E(u,\lambda)$ depending on $\lambda > 0$ and u:

$$\begin{aligned} E(u,\lambda) = {} & \left(\int_{B_1} \frac{1}{2} \left| \nabla \Delta u^\lambda \right|^2 dx - \frac{1}{p+1} \int_{B_1} |x|^a \left| u^\lambda \right|^{p+1} dx \right) \\ & + \int_{\partial B_1} \left(\sum_{0 \leq i,j \leq 4, i+j \leq 5} C^0_{i,j} \lambda^{i+j} \frac{d^i u^\lambda}{d\lambda^i} \frac{d^j u^\lambda}{d\lambda^j} + \sum_{0 \leq i,j \leq 2, i+j \leq 3} C^1_{i,j} \lambda^{i+j} \nabla_\theta \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \right) dx \\ & + \int_{\partial B_1} \left(\sum_{0 \leq i,j \leq 1, i+j \leq 1} C^2_{i,j} \lambda^{i+j} \Delta_\theta \frac{d^i u^\lambda}{d\lambda^i} \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \right) dx. \end{aligned} \tag{3.1}$$

Theorem 3.1 *Let u satisfy Eq. (1.1). Define $u^\lambda(x) = \lambda^\alpha u(\lambda x)$, then*

$$\begin{aligned} \frac{dE(u,\lambda)}{\lambda} = {} & \int_{\partial B_1} \left(3\lambda^5 \left(\frac{d^3 u^\lambda}{d\lambda^3} \right)^2 + A_1 \lambda^3 \left(\frac{d^2 u^\lambda}{d\lambda^2} \right)^2 + A_2 \lambda \left(\frac{du^\lambda}{d\lambda} \right)^2 \right) dx \\ & + \int_{\partial B_1} \left(2\lambda^3 \left(\frac{d^2}{d\lambda^2} \nabla_\theta u^\lambda \right)^2 + A_3 \lambda \left(\frac{d}{d\lambda} \nabla_\theta u^\lambda \right)^2 \right) dx \\ & + 3\lambda \int_{\partial B_1} \left(\frac{\Delta_\theta d u^\lambda}{d\lambda} \right)^2 dx, \end{aligned} \tag{3.2}$$

where $\alpha = \frac{6+a}{p-1}$ and

$$A_1 := 10\delta_1 - 2\delta_2 - 56 + \rho^2 - 2\rho - 2\gamma - 4,$$

$$A_2 := -18\delta_1 + 6\delta_2 - 4\delta_3 + 2\delta_4 + 72 - \rho^2 + \gamma^2 + 2\rho + 2\gamma,$$

$$A_3 := 8\vartheta - 4\beta + 4n - 18 - 2\gamma,$$

$$\beta := \alpha(4 + \alpha - n), \qquad \gamma := \alpha(\alpha - n + 2), \qquad \vartheta := n - 3 - 2\alpha, \qquad \rho := n - 1 - 2\alpha,$$

and

$$\delta_1 = 2(n-1) - 4\alpha, \qquad \delta_2 = 6\alpha(1+\alpha) - 6(n-1)\alpha + (n-1)(n-3),$$
$$\delta_3 = -4\alpha(1+\alpha)(2+\alpha) + 6(n-1)\alpha(1+\alpha) - (n-1)(n-3)(1+2\alpha),$$
$$\delta_4 = (1+\alpha)(2+\alpha)(3+\alpha)\alpha - 2(n-1)(1+\alpha)(2+\alpha)\alpha + (n-1)(n-3)(\alpha+2)\alpha.$$

The proof of Theorem 3.1 is similar to Theorem 2.1 in [13]. Take

$$\overline{E}(u,\lambda) = \lambda^{\frac{6(p+1)+2a}{p-1}-n}\left(\int_{B_\lambda} \frac{1}{2}|\nabla\Delta u|^2 - \frac{1}{p+1}\int_{B_\lambda} |x|^a|u|^{p+1}\right). \tag{3.3}$$

Since the derivation of the derivative for the $E(u,\lambda)$ is complicated, we divide it into several steps. In step 1, we derive $\frac{d}{d\lambda}\overline{E}(u,\lambda)$. In step 2, we calculate the (higher-order) derivatives $\frac{\partial^j}{\partial r^j}u^\lambda$ and $\frac{\partial^i}{\partial\lambda^i}u^\lambda$, $i,j = 1,2,3,4$. In step 3, the operator Δ^2 and its representation will be given. In step 4, we decompose $\frac{d}{d\lambda}\overline{E}(u,\lambda)$. Finally, combining the above four steps, we can obtain the derivative formula, hence get the proof of Theorem 3.1.

Theorem 3.2 *Assume that* $\frac{n+6+2a}{n-6} < p < p_a(n,6)$. *Then* $E(u,\lambda)$, *is a nondecreasing function of* $\lambda > 0$. *Furthermore,*

$$\frac{dE(u,\lambda)}{d\lambda} \ge C(n,p,a)\lambda^{2\alpha-n}\int_{\partial B_\lambda} (\alpha u + \lambda\partial_r u)^2\,dx, \tag{3.4}$$

where $C(n,p,a) > 0$ *is a constant independent of* λ.

Proof The proof follows the main lines of the demonstration of Theorem 2.2 in [13], with small modifications. From Theorem 3.1 we derive

$$\begin{aligned}\frac{dE(u,\lambda)}{d\lambda} &= \int_{\partial B_1}\left(3\lambda^5\left(\frac{d^3u^\lambda}{d\lambda^3}\right)^2 + A_1\lambda^3\left(\frac{d^2u^\lambda}{d\lambda^2}\right)^2 + A_2\lambda\left(\frac{du^\lambda}{d\lambda}\right)^2\right)dx \\ &\quad + \int_{\partial B_1}\left(2\lambda^3\left(\nabla_\theta\frac{d^2u^\lambda}{d\lambda^2}\right)^2 + A_3\lambda\left(\nabla_\theta\frac{du^\lambda}{d\lambda}\right)^2\right)dx \\ &\quad + \int_{\partial B_1}\lambda\left(\Delta_\theta\frac{du^\lambda}{d\lambda}\right)^2 dx,\end{aligned} \tag{3.5}$$

where

$$A_1 = 10\delta_1 - 2\delta_2 - 56 + \rho^2 - 2\rho - 2\gamma - 4, \tag{3.6}$$
$$A_2 = -18\delta_1 + 6\delta_2 - 4\delta_3 + 2\delta_4 + 72 - \rho^2 + \gamma^2 + \rho + 2\gamma, \tag{3.7}$$

and

$$A_3 = 8\vartheta - 4\beta - 2\gamma + 4n - 18. \tag{3.8}$$

By a direct calculation we have

$$\begin{aligned}A_1 &= -10\alpha^2 + (-60+10n)\alpha - n^2 + 24n - 83, \\ A_2 &= 3\alpha^4 + (36-6n)\alpha^3 + \left(3n^2 - 48n + 150\right)\alpha^2 + \left(12n^2 - 114n + 252\right)\alpha \\ &\quad + 9n^2 - 72n + 135,\end{aligned}$$

and

$$A_3 = -6\alpha^2 + (-36 + 6n)\alpha + 12n - 42.$$

Notice that our supercritical condition $p > \frac{n+6+2a}{n-6}$ is equivalent to $0 < \alpha < \frac{n-6}{2}$. Firstly, we have the following lemma, which yields the sign of A_2 and A_3.

Lemma 3.1 *Let $n \geq 7$. If $p > \frac{n+6+2a}{n-6}$, then $A_2 > 0$ and $A_3 > 0$.*

Proof From (3.7), we derive that

$$A_2 = 3(\alpha + 1)(\alpha + 3)\bigl(\alpha - (n-5)\bigr)\bigl(\alpha - (n-3)\bigr), \tag{3.9}$$

and the roots of $A_3 = 0$ are

$$\frac{1}{2}n - 3 - \frac{1}{2}\sqrt{n^2 - 4n + 8}, \qquad \frac{1}{2}n - 3 + \frac{1}{2}\sqrt{n^2 - 4n + 8}.$$

Recall that $p > \frac{n+6+2a}{n-6}$, is equivalent to $0 < \alpha < \frac{n-6}{2}$, we get the conclusion. □

To show monotonicity formula, we proceed to prove the following inequality:

$$\begin{aligned} &3\lambda^5\left(\frac{d^3u^\lambda}{d\lambda^3}\right)^2 + A_1\lambda^3\left(\frac{d^2u^\lambda}{d\lambda^2}\right)^2 + A_2\lambda\left(\frac{du^\lambda}{d\lambda}\right)^2 \\ &\quad \geq \epsilon\lambda\left(\frac{du^\lambda}{d\lambda}\right)^2 + \frac{d}{d\lambda}\left(\sum_{0\leq i,j\leq 2} c_{i,j}\lambda^{i+j}\frac{d^iu^\lambda}{d\lambda^i}\frac{d^ju^\lambda}{d\lambda^j}\right). \end{aligned} \tag{3.10}$$

To deal with the rest of the dimensions, we employ the second idea: we find nonnegative constants d_1, d_2 and constants c_1, c_2 such that we have the following Jordan form decomposition:

$$\begin{aligned} 3\lambda^5\bigl(f'''\bigr)^2 + A_1\lambda^3\bigl(f''\bigr)^2 + A_2\lambda\bigl(f'\bigr)^2 &= 3\lambda\bigl(\lambda^2 f''' + c_1\lambda f''\bigr)^2 + d_1\lambda\bigl(\lambda f'' + c_2 f'\bigr)^2 + d_2\lambda\bigl(f'\bigr)^2 \\ &\quad + \frac{d}{d\lambda}\left(\sum_{i,j} e_{i,j}\lambda^{i+j}f^{(i)}f^{(j)}\right), \end{aligned} \tag{3.11}$$

where the unknown constants are to be determined.

Lemma 3.2 *Let $n \geq 7$. If $p > \frac{n+6+2a}{n-6}$ and A_1 satisfy*

$$A_1 + 12 > 0,$$

then there exist nonnegative numbers d_1, d_2, and real numbers c_1, c_2, $e_{i,j}$ such that the differential inequality (3.11) holds.

Proof Since

$$4\lambda^4 f''' f'' = \frac{d}{d\lambda}\bigl(2\lambda^4\bigl(f''\bigr)^2\bigr) - 8\lambda^3\bigl(f''\bigr)^2$$

and

$$2\lambda^2 f'' f' = \frac{d}{d\lambda}\left(\lambda^2 (f')^2\right) - 2\lambda (f')^2$$

by comparing the coefficients of $\lambda^3 (f'')^2$ and $\lambda (f')^2$, we have

$$d_1 = A_1 - 3c_1^2 + 12c_1, \qquad d_2 = A_2 - \left(c_2^2 - 2c_2\right)\left(A_1 - 3c_1^2 + 12c_1\right).$$

In particular,

$$\max_{c_1} d_1(c_1) = A_1 + 12 \quad \text{and the critical point is } c_1 = 2.$$

Since $A_2 > 0$, we select $c_1 = 2$, $c_2 = 0$. Hence, in this case, by a direct calculation we see that $d_1 = A_1 + 12 > 0$. Then we get the conclusion. □

We conclude from Lemma 3.2 that if $A_1 + 12 > 0$ then (3.10) holds. This implies that when $7 \le n \le 20$, $p > \frac{n+6+2a}{n-6}$ or $n \ge 21$ and

$$\frac{n+6+2a}{n-6} < p < \frac{5n+30-\sqrt{15n^2-60n+190}+10a}{5n-30-\sqrt{15n^2-60n+190}}, \tag{3.12}$$

then (3.10) holds. Combining the idea from the above with the following idea, we can get a better condition to make the monotonicity formula hold. We start from the differential identity (3.11). Recall that the derivative term is a 'good' term since it can be absorbed by other terms.

Let

$$p_{m,a}(n,6) = \begin{cases} +\infty & \text{if } n \le 30, \\ \frac{5n+30-\sqrt{15n^2-60n+190}+10a}{5n-30-\sqrt{15n^2-60n+190}} & \text{if } n \ge 31. \end{cases}$$

Combining all the lemmas of this section, we obtain the following theorem.

Theorem 3.3 *For $\frac{n+6+2a}{n-6} < p < p_{m,a}(n,6)$, there exists a $C(n,p,a) > 0$ such that*

$$\frac{d}{d\lambda} E(u,\lambda) \ge C(n,p,a) \int_{\partial B_1} \lambda \left(\frac{du^\lambda}{d\lambda}\right)^2 dx.$$

Proof of Theorem 3.2 (Continued) Let $a > 0$ and $n > n(a)$. Recall that $F_a(0) < 0$ and $F_a(\alpha) > 0$ for all $\alpha_a < \alpha < \frac{n-6}{2}$.

We have

$$F_a\left(\frac{1}{2}n - 3 - \frac{1}{10}\sqrt{15n^2-60n+190}\right) < 0 \quad \text{for } n > n(a).$$

Then

$$\frac{1}{2}n - 3 - \frac{1}{10}\sqrt{15n^2-60n+190} < \alpha_a = \frac{6+a}{p(n,a)-1}. \tag{3.13}$$

From (3.13), we get

$$p_a(n,6) < p_{m,a}(n,6), \quad \forall n > n(a).$$

Theorem 3.2 is thus proved. □

4 Integral estimates

The following basic integral estimates for solutions (whether positive or sign-changing) of (1.1) follow from the rescaled test function method.

Lemma 4.1 *Let u be a stable solution of* (1.1), *then there exists a positive constant C such that*

$$\begin{aligned}
&\int_{\mathbb{R}^n} |x|^a|u|^{p+1}\psi^6\,dx + \int_{\mathbb{R}^n} |\nabla\Delta u|^2\psi^6\,dx \\
&\quad\le C\int_{\mathbb{R}^n} |\Delta u|^2\psi^4|\nabla\psi|^2\,dx + C\int_{\mathbb{R}^n} |\nabla u|^2\frac{|\Delta\psi^6|^2}{\psi^6}\,dx + C\int_{\mathbb{R}^n} u^2\frac{|\nabla\Delta\psi^6|^2}{\psi^6}\,dx \\
&\qquad + C\int_{\mathbb{R}^n} |\nabla u|^2\psi^2|\nabla\psi|^4\,dx + C\int_{\mathbb{R}^n} \left|\nabla^2 u\right|^2\psi^4|\nabla\psi|^2\,dx \\
&\qquad + C\int_{\mathbb{R}^n} |\nabla u|^2\psi^4\left|\nabla^2\psi\right|^2\,dx.
\end{aligned} \tag{4.1}$$

Proof Multiplying Eq. (1.1) with $u\psi^6$, where ψ is a test function, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} |x|^a|u|^{p+1}\psi^6\,dx &= \int_{\mathbb{R}^n} -\Delta^3 u.u\psi^6\,dx = \int_{\mathbb{R}^n} \nabla\Delta^2 u\cdot\nabla\left(u\psi^6\right)dx \\
&= -\int_{\mathbb{R}^n} \Delta^2 u\Delta\left(u\psi^6\right)dx = \int_{\mathbb{R}^n} \nabla\Delta u\cdot\nabla\Delta\left(u\psi^6\right)dx.
\end{aligned} \tag{4.2}$$

Since $\Delta(\xi\psi) = \psi\Delta\xi + \xi\Delta\psi + 2\nabla\xi\nabla\psi$, we have

$$\Delta\left(u\psi^6\right) = \psi^6\Delta u + u\Delta\psi^6 + 12\psi^5\nabla u\nabla\psi,$$

therefore,

$$\begin{aligned}
\nabla\Delta\left(u\psi^6\right)\nabla\Delta u &= 6\psi^5\Delta u\nabla\psi\nabla\Delta u + \left(\psi^6\right)(\nabla\Delta u)^2 \\
&\quad + \Delta\psi^6\nabla u\nabla\Delta u + u\nabla\Delta\psi^6\nabla\Delta u + 60\psi^4(\nabla\psi\nabla\Delta u)(\nabla u\nabla\psi) \\
&\quad + 12\psi^5\sum_{i,j}\partial_{ij}u\partial_i\psi\partial_j\Delta u + 12\psi^5\sum_{i,j}\partial_i u\partial_{ij}\psi\partial_j\Delta u,
\end{aligned} \tag{4.3}$$

where ∂_j ($j = 1,\dots,n$) denote the derivatives with respect to $x_1,\dots,x_n$, respectively. A similar method can be applied to dealing with the following term, $|\nabla\Delta(u\psi^3)|^2$. On the other hand, by the stability condition, we have

$$p\int_{\mathbb{R}^n} |x|^a|u|^{p+1}\psi^6\,dx \le \int_{\mathbb{R}^n} \left|\nabla\Delta\left(u\psi^3\right)\right|^2 dx. \tag{4.4}$$

Combining this with (4.2), (4.3) and (4.4), we have

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla\Delta u|^2\psi^6\,dx &\le C\epsilon\int_{\mathbb{R}^n}|\nabla\Delta u|^2\psi^6\,dx + C(\epsilon)\int_{\mathbb{R}^n}|\Delta u|^2\psi^4|\nabla\psi|^2\,dx\\ &\quad + C(\epsilon)\int_{\mathbb{R}^n}|\nabla u|^2\biggl(\frac{|\Delta\psi^6|^2}{\psi^6}+\psi^4\bigl|\nabla^2\psi\bigr|^2\biggr)\,dx\\ &\quad + C(\epsilon)\int_{\mathbb{R}^n}u^2\frac{|\nabla\Delta\psi^6|^2}{\psi^6}\,dx + C(\epsilon)\int_{\mathbb{R}^n}|\nabla u|^2\psi^2|\nabla\psi|^4\,dx\\ &\quad + C(\epsilon)\int_{\mathbb{R}^n}\bigl|\nabla^2 u\bigr|^2\psi^4|\nabla\psi|^2\,dx, \end{aligned}\tag{4.5}$$

we can select ϵ so small that $C\epsilon \le \frac{1}{2}$. Finally, combining with (4.2) and (4.3), we obtain the conclusion of this lemma. □

Proposition 4.1 *Let $u \in C^6(\mathbb{R}^n)$ be a stable solution of* (1.1). *Then there exists a constant $C > 0$ such that*

$$\int_{B_R}\bigl(|x|^a|u|^{p+1}+\bigl|\nabla(\Delta u)\bigr|^2\bigr)\,dx \le CR^{-6}\int_{B_{2R}\setminus B_R}u^2\,dx,\tag{4.6}$$

$$\int_{B_R}\bigl(|x|^a|u|^{p+1}+\bigl|\nabla(\Delta u)\bigr|^2\bigr)\,dx \le CR^{n-\frac{6(p+1)+2a}{p-1}}.\tag{4.7}$$

Proof We let $\psi = \xi^m$ where $m > 1$ in the estimate (4.1), we have

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla\Delta u|^2\xi^{6m}\,dx + \int_{\mathbb{R}^n}|x|^a|u|^{p+1}\xi^{6m}\,dx &\le \int_{\mathbb{R}^n}u^2 g_0(\xi)\,dx + \int_{\mathbb{R}^n}|\nabla u|^2 g_1(\xi)\,dx\\ &\quad + \int_{\mathbb{R}^n}|\Delta u|^2 g_2(\xi)\,dx,\end{aligned}\tag{4.8}$$

where

$$\begin{aligned} g_0(\xi) &= \xi^{6m-6}\sum_{0\le i+j+k+r+s+t\le 6}\bigl|\nabla^i\xi\bigr|\bigl|\nabla^j\xi\bigr|\bigl|\nabla^k\xi\bigr|\bigl|\nabla^r\xi\bigr|\bigl|\nabla^s\xi\bigr|\bigl|\nabla^t\xi\bigr|,\\ g_1(\xi) &= \xi^{6m-4}\sum_{0\le i+j+k+l\le 4}\bigl|\nabla^i\xi\bigr|\bigl|\nabla^j\xi\bigr|\bigl|\nabla^k\xi\bigr|\bigl|\nabla^l\xi\bigr|,\\ g_2(\xi) &= \xi^{6m-2}\sum_{0\le i+j\le 2}\bigl|\nabla^i\xi\bigr|\bigl|\nabla^j\xi\bigr|, \end{aligned}$$

where we define $\nabla^0\xi = \xi$ and notice that $g_m(\xi) \ge 0$ for $m = 0, 1, 2$. Now, we claim that

$$g_1^2(\xi) \le Cg_0(\xi)g_2(\xi), \qquad \bigl|\nabla^2 g_2(\xi)\bigr| \le Cg_1(\xi), \qquad g_2^2(\xi) \le C\xi^{6m}g_1(\xi).\tag{4.9}$$

This claim can be verified by direct calculations and will be used for the following estimates.

Since $|\nabla u|^2 = \frac{1}{2}\Delta(u^2) - u\Delta u$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla u|^2 g_1(\xi)\,dx &= \frac{1}{2}\int_{\mathbb{R}^n}\Delta\bigl(u^2\bigr)g_1(\xi)\,dx - \int_{\mathbb{R}^n}u\Delta u g_1(\xi)\,dx\\ &= \frac{1}{2}\int_{\mathbb{R}^n}u^2\Delta g_1(\xi)\,dx - \int_{\mathbb{R}^n}u\Delta u g_1(\xi)\,dx\end{aligned}$$

$$\le \frac{1}{2}\int_{\mathbb{R}^n} u^2\Delta g_1(\xi)\,dx + \epsilon\int_{\mathbb{R}^n}(\Delta u)^2 g_2(\xi)\,dx + \frac{1}{4\epsilon}\int_{\mathbb{R}^n} u^2 g_0(\xi)\,dx. \tag{4.10}$$

We note the following differential identity

$$(\Delta u)^2 = \sum_{j,k}(u_j u_k)_{jk} - \sum_{j,k}(u_{jk})^2 - \nabla\Delta u.\nabla u.$$

Hence $(\Delta u)^2 \le \sum_{j,k}(u_j u_k)_{jk} - 2\nabla\Delta u.\nabla u$. Therefore we have

$$\begin{aligned}
\int_{\mathbb{R}^n}(\Delta u)^2 g_2(\xi)\,dx &\le \int_{\mathbb{R}^n}\sum_{j,k}(u_j u_k)_{jk} g_2(\xi)\,dx - 2\int_{\mathbb{R}^n}\nabla\Delta u.\nabla u g_2(\xi)\,dx \\
&= \int_{\mathbb{R}^n}\sum_{j,k}(u_j u_k) g_2(\xi)_{jk}\,dx - 2\int_{\mathbb{R}^n}\nabla\Delta u.\nabla u g_2(\xi)\,dx \\
&\le C\int_{\mathbb{R}^n}|\nabla u|^2 g_1(\xi)\,dx + \delta\int_{\mathbb{R}^n}|\nabla\Delta u|^2\xi^{6m}\,dx \\
&\quad + C(\delta)\int_{\mathbb{R}^n}|\nabla u|^2 g_1(\xi)\,dx \\
&\le C\int_{\mathbb{R}^n}|\nabla u|^2 g_1(\xi) + \delta\int_{\mathbb{R}^n}|\nabla\Delta u|^2\xi^{6m}\,dx.
\end{aligned} \tag{4.11}$$

Combining with (4.11) and (4.10), by selecting the positive parameter ϵ small enough, we can obtain

$$\int_{\mathbb{R}^n}|\nabla u|^2 g_1(\xi)\,dx + \int_{\mathbb{R}^n}(\Delta u)^2 g_2(\xi)\,dx \le C\int_{\mathbb{R}^n} u^2 g_0(\xi)\,dx + \delta\int_{\mathbb{R}^n}|\nabla\Delta u|^2\xi^{6m}\,dx.$$

By combining the above inequalities with (4.8) and selecting the positive parameter δ small enough, we have

$$\int_{\mathbb{R}^n}|\nabla\Delta u|^2\xi^{6m}\,dx + \int_{\mathbb{R}^n}|x|^a|u|^{p+1}\xi^{6m}\,dx \le C\int_{\mathbb{R}^n} u^2 g_0(\xi)\,dx. \tag{4.12}$$

This proves (4.6). Further, we let $\xi = 1$ in B_R and $\xi = 0$ in B_{2R}^C, satisfying $|\nabla\xi| \le \frac{C}{R}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}|\nabla\Delta u|^2\xi^{6m}\,dx + \int_{\mathbb{R}^n}|x|^a|u|^{p+1}\xi^{6m}\,dx \\
&\quad\le C\int_{\mathbb{R}^n} u^2 g_0(\xi)\,dx \le C\int_{B_{2R}\backslash B_R}|x|^{\frac{2a}{p+1}}u^2|x|^{\frac{-2a}{p+1}}g_0(\xi)\,dx \\
&\quad\le CR^{-6}\int_{B_{2R}\backslash B_R}|x|^{\frac{2a}{p+1}}|u|^2|x|^{\frac{-2a}{p+1}}\xi^{(6m-6)}\,dx \\
&\quad\le CR^{-6}\left(\int_{B_{2R}\backslash B_R}|x|^a|u|^{p+1}\xi^{(3m-3)(p+1)}\,dx\right)^{\frac{2}{p+1}} R^{(n-\frac{2a}{p-1})\frac{p-1}{p+1}}.
\end{aligned} \tag{4.13}$$

By selecting $m > 1$ and letting m be close to 1, we can make sure that $(3m-3)(p+1) \le 6m$. It follows that (4.7) holds. □

5 Homogeneous solutions

In this section, we obtain a nonexistence result for a homogeneous stable solution of (1.1).

Lemma 5.1 *Let*

$$\begin{aligned}
J_1 &= (\alpha+4)(n-6-\alpha)+(\alpha+2)(n-4-\alpha)+\alpha(n-2-\alpha),\\
J_2 &= (\alpha+4)(n-6-\alpha)\big[(\alpha+2)(n-4-\alpha)+\alpha(n-2-\alpha)\big]\\
&\quad+\alpha(\alpha+2)(n-2-\alpha)(n-4-\alpha),\\
J_3 &= \alpha(\alpha+2)(\alpha+4)(n-2-\alpha)(n-4-\alpha)(n-6-\alpha).
\end{aligned}$$

If $p \in (\frac{n+6+2a}{n-6}, p_a(n,6))$, *then*

$$J_1 > 0, \qquad J_2 > 0, \qquad J_3 > 0, \qquad pJ_1 > \frac{(n-2)^2}{4} + \frac{(n+2)(n-6)}{2}$$

and

$$pJ_2 > \frac{(n+2)(n-6)(n-2)^2}{8} + \frac{(n+2)^2(n-6)^2}{16}.$$

Proof Since

$$p > \frac{n+6+2a}{n-6} > \frac{n+6}{n-6}, \tag{5.1}$$

we have

$$J_1 > 0, \qquad J_2 > 0 \quad \text{and} \quad J_3 > 0.$$

For $\frac{n+6+2a}{n-6} < p < p_a(n,6)$, we get from the definition of $p_a(n,6)$

$$pJ_3 > \frac{(n-6)^2(n+2)^2(n-2)^2}{4^3}. \tag{5.2}$$

From (5.2), we obtain

$$3^3p^3J_3 > \left(\frac{3}{4}\right)^3(n-2)^6. \tag{5.3}$$

Using the following well-known inequality:

$$\sqrt[3]{xyz} \le \frac{1}{3}(x+y+z), \tag{5.4}$$

where x, y and z are positive real numbers, as follows: $x = (\alpha+4)(n-6-\alpha)$, $y = (\alpha+2) \times (n-4-\alpha)$ and $z = \alpha(n-2-\alpha)$, we derive

$$3^3J_3 < (J_1)^3. \tag{5.5}$$

By the last inequality combined with (5.3), we derive

$$pJ_1 > \frac{3}{4}(n-2)^2. \tag{5.6}$$

Since

$$(n-2)^2 > (n+2)(n-6), \quad \text{for } n \geq 7. \tag{5.7}$$

Inserting the latter into (5.6) we obtain

$$pJ_1 > \frac{(n-2)^2}{4} + \frac{(n+2)(n-6)}{2}. \tag{5.8}$$

Using again the same inequality (5.4) (for $x = (\alpha+4)(n-6-\alpha)(\alpha+2)(n-4-\alpha)$, $y = (\alpha+4)(n-6-\alpha)\alpha(n-2-\alpha)$ and $z = \alpha(\alpha+2)(n-2-\alpha)(n-4-\alpha)$), we derive

$$3^3 J_3^2 < (J_2)^3. \tag{5.9}$$

From (5.1) and (5.2), we deduce that

$$3^3 p^3 J_3^2 > \left(\frac{3}{16}\right)^3 (n+2)^3(n-6)^3(n-2)^6. \tag{5.10}$$

Putting (5.10) into (5.9) gives

$$pJ_2 > \frac{3}{16}(n+2)(n-6)(n-2)^2. \tag{5.11}$$

By (5.7), it follows

$$\frac{3}{16}(n+2)(n-6)(n-2)^2 > \frac{(n+2)(n-6)(n-2)^2}{8} + \frac{(n+2)^2(n-6)^2}{16}.$$

This implies

$$pJ_2 > \frac{(n+2)(n-6)(n-2)^2}{8} + \frac{(n+2)^2(n-6)^2}{16}. \tag{5.12}$$

This finishes the proof of Lemma 5.1. □

Theorem 5.1 *Let $u \in W^{3,2}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ be a homogeneous, stable solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$, $p \in (\frac{n+6+2a}{n-6}, p_a(n,6))$. Assume that $|x|^a|u|^{p+1} \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. Then $u \equiv 0$.*

Proof Let u be a homogeneous solution of (1.1), that is, there exists a $w \in W^{3,2}(\mathbb{S}^{n-1})$ such that in polar coordinates

$$u(r,\theta) = r^{-\frac{6+a}{p-1}} w(\theta).$$

Since $u \in W^{3,2}(B_2 \setminus B_1)$ and $|x|^a|u|^{p+1} \in L^1(B_2 \setminus B_1)$, it implies that $w \in W^{3,2}(\mathbb{S}^{n-1}) \cap L^{p+1}(\mathbb{S}^{n-1})$.

Direct calculations show that

$$-\Delta_\theta^3 w(\theta) + J_1 \Delta_\theta^2 w(\theta) - J_2 \Delta_\theta w(\theta) + J_3 w(\theta) = |w|^{p-1} w, \tag{5.13}$$

where $\alpha = \frac{6+a}{p-1}$,

$$J_1 = (\alpha+4)(n-6-\alpha) + (\alpha+2)(n-4-\alpha) + \alpha(n-2-\alpha),$$
$$J_2 = (\alpha+4)(n-6-\alpha)\big[(\alpha+2)(n-4-\alpha) + \alpha(n-2-\alpha)\big] + \alpha(\alpha+2)(n-2-\alpha)(n-4-\alpha),$$
$$J_3 = \alpha(\alpha+2)(\alpha+4)(n-2-\alpha)(n-4-\alpha)(n-6-\alpha).$$

Because $w \in W^{3,2}(\mathbb{S}^{n-1})$, we can test (5.13) with w, and we get

$$\int_{\mathbb{S}^{n-1}} \left|\nabla_\theta(\Delta_\theta w)\right|^2 + J_1(\Delta_\theta w)^2 + J_2|\nabla_\theta w|^2 + J_3 w^2 d\theta = \int_{\mathbb{S}^{n-1}} |w|^{p+1}\,d\theta. \tag{5.14}$$

As in [10], for any $\epsilon > 0$, choose an $\eta_\epsilon \in C_c^\infty((\frac{\epsilon}{2}, \frac{2}{\epsilon}))$, such that $\eta_\epsilon \equiv 1$ in $(\epsilon, \frac{1}{\epsilon})$, and

$$r\left|\eta_\epsilon'(r)\right| + r^2\left|\eta_\epsilon''(r)\right| + r^3\left|\eta_\epsilon'''(r)\right| \le 64 \quad \text{for all } r > 0.$$

Because $w \in W^{3,2}(\mathbb{S}^{n-1}) \cap L^{p+1}(\mathbb{S}^{n-1})$, $r^{-\frac{n-6}{2}}w(\theta)\eta_\epsilon(r)$ can be approximated by $C_c^\infty(B_{4/\epsilon} \setminus B_{\epsilon/4})$ functions in $W^{3,2}(B_{2/\epsilon} \setminus B_{\epsilon/2}) \cap L^{p+1}(B_{2/\epsilon} \setminus B_{\epsilon/2})$. Hence in the stability condition for u we are allowed to choose a test function of the form $r^{-\frac{n-6}{2}}w(\theta)\eta_\epsilon(r)$.

A simple computation gives

$$\begin{aligned}
\Delta\left(r^{-\frac{n-6}{2}}w(\theta)\eta_\epsilon(r)\right) &= -\frac{(n+2)(n-6)}{4}r^{-\frac{n}{2}+1}w(\theta)\eta_\epsilon(r) + 5r^{-\frac{n}{2}+2}w(\theta)\eta_\epsilon'(r) \\
&\quad + r^{-\frac{n}{2}+3}w(\theta)\eta_\epsilon''(r) + r^{-\frac{n}{2}+1}\Delta_\theta w(\theta)\eta_\epsilon(r),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(\Delta(r^{-\frac{n-6}{2}}w(\theta)\eta_\epsilon(r)))}{\partial r} &= \frac{(n+2)(n-6)(n-2)}{8}w(\theta)r^{-\frac{n}{2}}w(\theta)\eta_\epsilon(r) \\
&\quad + \frac{10(4-n)-(n+2)(n-6)}{4}r^{-\frac{n}{2}+1}w(\theta)\eta_\epsilon'(r) \\
&\quad + \frac{16-n}{2}r^{-\frac{n}{2}+2}w(\theta)\eta_\epsilon''(r) + r^{-\frac{n}{2}+3}w(\theta)\eta_\epsilon'''(r) \\
&\quad + \frac{2-n}{2}r^{-\frac{n}{2}}\Delta_\theta w(\theta)\eta_\epsilon(r) + r^{-\frac{n}{2}+1}\Delta_\theta w(\theta)\eta_\epsilon'(r),
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{r}\nabla_\theta\left(\Delta\left(r^{-\frac{n-6}{2}}w(\theta)\eta_\epsilon(r)\right)\right) &= -\frac{(n+2)(n-6)}{4}r^{-\frac{n}{2}}\nabla_\theta w(\theta)\eta_\epsilon(r) + 5r^{-\frac{n}{2}+1}\nabla_\theta w(\theta)\eta_\epsilon'(r) \\
&\quad + r^{-\frac{n}{2}+2}\nabla_\theta w(\theta)\eta_\epsilon''(r) + r^{-\frac{n}{2}}\nabla_\theta\left(\Delta_\theta w(\theta)\right)\eta_\epsilon(r).
\end{aligned}$$

Substituting this into the stability condition for u, we get

$$\begin{aligned}
&p\left(\int_{\mathbb{S}^{n-1}} |w|^{p+1}\,d\theta\right)\left(\int_0^{+\infty} r^{-1}\eta_\epsilon(r)^2\,dr\right) \\
&\quad\le \left[\int_{\mathbb{S}^{n-1}} \left|\nabla_\theta(\Delta_\theta w)\right|^2 + \left(\frac{(n-2)^2}{4} + \frac{(n+2)(n-6)}{2}\right)(\Delta_\theta w)^2\,d\theta\right. \\
&\qquad + \int_{\mathbb{S}^{n-1}} \left(\frac{(n-2)^2(n-6)(n+2)}{8} + \frac{(n-6)^2(n+2)^2}{16}\right)|\nabla_\theta w|^2\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{S}^{n-1}} \frac{(n-6)^2(n+2)^2(n-2)^2}{64} w^2\,d\theta\Big]\left(\int_0^{+\infty} r^{-1}\eta_\epsilon(r)^2\,dr\right)\\
&+O\Big[\int_0^{+\infty}\left(r\eta_\epsilon'(r)^2+r^3\eta_\epsilon''(r)^2+r^5\eta_\epsilon'''(r)^2+r\eta_\epsilon(r)\left|\eta_\epsilon''(r)\right|+r^2\eta_\epsilon(r)\left|\eta_\epsilon'''(r)\right|\right)dr\\
&\times\int_{\mathbb{S}^{n-1}}\left((\Delta_\theta w)^2+|\nabla_\theta w|^2+w^2\right)d\theta\Big].
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_0^{+\infty} r^{-1}\eta_\epsilon(r)^2\,dr\geq|\log\epsilon|,\\
&\int_0^{+\infty}\left(r\eta_\epsilon'(r)^2+r^3\eta_\epsilon''(r)^2+r^5\eta_\epsilon'''(r)^2+r\eta_\epsilon(r)\left|\eta_\epsilon''(r)\right|+r^2\eta_\epsilon(r)\left|\eta_\epsilon'''(r)\right|\right)dr\leq C,
\end{aligned}
$$

for some constant C independent of ϵ. By letting $\epsilon\to 0$, we obtain

$$
\begin{aligned}
p\int_{\mathbb{S}^{n-1}}|w|^{p+1}\,d\theta\leq{}&\int_{\mathbb{S}^{n-1}}\left|\nabla_\theta(\Delta_\theta w)\right|^2+\left(\frac{(n-2)^2}{4}+\frac{(n+2)(n-6)}{2}\right)(\Delta_\theta w)^2\,d\theta\\
&+\int_{\mathbb{S}^{n-1}}\left(\frac{(n-2)^2(n-6)(n+2)}{8}+\frac{(n-6)^2(n+2)^2}{16}\right)|\nabla_\theta w|^2\\
&+\frac{(n-6)^2(n+2)^2(n-2)^2}{64}w^2\,d\theta.
\end{aligned}
$$

Substituting (5.14) into this we get

$$
\begin{aligned}
&\int_{\mathbb{S}^{n-1}}(p-1)\left|\nabla_\theta(\Delta_\theta w)\right|^2+\left(pJ_1-\frac{(n-2)^2}{4}-\frac{(n+2)(n-6)}{2}\right)(\Delta_\theta w)^2\,d\theta\\
&\quad+\int_{\mathbb{S}^{n-1}}\left(pJ_2-\frac{(n-2)^2(n-6)(n+2)}{8}-\frac{(n-6)^2(n+2)^2}{16}\right)|\nabla_\theta w|^2\\
&\quad+\left(pJ_3-\frac{(n-6)^2(n+2)^2(n-2)^2}{64}\right)w^2\,d\theta\\
&\leq 0.
\end{aligned}
\tag{5.15}
$$

Finally, by Lemma 5.1, we observe that $w\equiv 0$. Then

$$
u\equiv 0,\quad \text{in } \mathbb{R}^n.
$$

This finishes the proof of Theorem 5.1. □

Remark 5.1 One can easily check that

$$
u_s(r)=J_3^{\frac{1}{p-1}}r^{-\frac{6+a}{p-1}}
$$

is a singular solution of (1.1) in $\mathbb{R}^n\setminus\{0\}$, where

$$
\alpha=\frac{6+a}{p-1},\qquad J_3=\alpha(\alpha+2)(\alpha+4)(n-2-\alpha)(n-4-\alpha)(n-6-\alpha).
$$

Using the well-known Hardy–Rellich inequality [15] with the best constant

$$\int_{\mathbb{R}^n} \left|\nabla(\Delta\psi)\right|^2 dx \ge \frac{(n-6)^2(n+2)^2(n-2)^2}{64} \int_{\mathbb{R}^n} \frac{\psi^2}{|x|^6} dx, \quad \forall \psi \in H^3(\mathbb{R}^n),$$

we conclude that the singular solution u_s is stable in $\mathbb{R}^n \setminus \{0\}$ if and only if

$$pJ_3 \le \frac{(n-6)^2(n+2)^2(n-2)^2}{64}.$$

6 Classification of stable solutions

For the case, $1 < p \le \frac{n+6+2a}{n-6}$, we apply the integral estimates. For the case, $\frac{n+6+2a}{n-6} < p < p_a(n,6)$, with the energy estimates and the desired monotonicity formula under the condition $\frac{n+6+2a}{n-6} < p < p_a(n,6)$, we can show that the stable solutions must be homogeneous solutions, hence by applying the classification of the homogeneous solutions (see Theorem 5.1), the solutions must be zero.

Proof of Theorem 1.1 *Subcritical case:* $1 < p < \frac{n+6+2a}{n-6}$.

Since $p < \frac{n+6+2a}{n-6}$ implies $n < \frac{6(p+1)+2a}{p-1}$, and combining with (4.7), we find

$$\int_{B_R(x)} \left(\left|\nabla(\Delta u)\right|^2 + |y|^a|u|^{p+1}\right) dy \le CR^{n-\frac{6(p+1)+2a}{p-1}} \to 0, \quad \text{as } R \to +\infty.$$

Consequently, we obtain

$$u \equiv 0.$$

Critical case: $p = \frac{n+6+2a}{n-6}$.

Utilizing the inequality (4.7) once again we find

$$\int_{\mathbb{R}^n} \left(\left|\nabla(\Delta u)\right|^2 + |x|^a|u|^{p+1}\right) dx < +\infty.$$

Then it implies that

$$\lim_{R\to+\infty} \int_{B_{2R}(x)\setminus B_R(x)} \left(\left|\nabla(\Delta u)\right|^2 + |y|^a|u|^{p+1}\right) dy = 0.$$

From (4.6), a direct application of Hölder's inequality leads to

$$\begin{aligned}
\int_{B_R(x)} \left(\left|\nabla(\Delta u)\right|^2 + |y|^a|u|^{p+1}\right) dy &\le CR^{-6} \int_{B_{2R}(x)\setminus B_R(x)} u^2\, dy \\
&\le CR^{-6} \left(\int_{B_{2R}(x)\setminus B_R(x)} |y|^a|u|^{p+1}\, dy\right)^{\frac{2}{p+1}} \\
&\quad \times \left(\int_{B_{2R}(x)\setminus B_R(x)} |y|^{\frac{-2a}{p-1}}\, dy\right)^{\frac{p-1}{p+1}} \\
&\le CR^{(n-\frac{6(p+1)+2a}{p-1})\frac{p-1}{p+1}} \left(\int_{B_{2R}(x)\setminus B_R(x)} |y|^a|u|^{p+1}\, dy\right)^{\frac{2}{p+1}}.
\end{aligned}$$

Since $p = \frac{n+6+2a}{n-6}$, the right side of the above inequality tends to 0 as $R \to +\infty$. So we get

$$u \equiv 0.$$

Supercritical case: $\frac{n+6+2a}{n-6} < p < p_a(n,6)$.

In what follows, we obtain the following three lemmas which play an important role in dealing with the supercritical case. For any $\lambda > 0$, define

$$u^\lambda(x) = \lambda^\alpha u(\lambda x),$$

and u^λ is also a smooth stable solution of (1.1) on $\mathbb{R}^n$. By rescaling (4.7), for all $\lambda > 0$ and balls $B_r(x) \subset \mathbb{R}^n$

$$\int_{B_r(x)} \left(\left|\nabla\left(\Delta u^\lambda\right)\right|^2 + |y|^a\left|u^\lambda\right|^{p+1}\right) dy \le Cr^{n-6-2\alpha}.$$

In particular, u^λ are uniformly bounded in $L^{p+1}_{\mathrm{loc}}(\mathbb{R}^n)$. By elliptic estimates, u^λ are also uniformly bounded in $W^{3,2}_{\mathrm{loc}}(\mathbb{R}^n)$. Hence, up to a subsequence of $\lambda \to +\infty$, we can assume that $u^\lambda \to u^\infty$ weakly in $W^{3,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p+1}_{\mathrm{loc}}(\mathbb{R}^n)$. By compactness embedding, one has $u^\lambda \to u^\infty$ strongly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$. Then, for any ball $B_R(0)$, by interpolation between L^q spaces and noting (4.7), for any $q \in [1, p+1)$, as $\lambda \to +\infty$

$$\left\|u^\lambda - u^\infty\right\|_{L^q(B_R(0))} \le \left\|u^\lambda - u^\infty\right\|^t_{L^1(B_R(0))}\left\|u^\lambda - u^\infty\right\|^{1-t}_{L^{p+1}(B_R(0))} \to 0, \tag{6.1}$$

where $\frac{1}{q} = t + \frac{1-t}{p+1}$. That is, $u^\lambda \to u^\infty$ in $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for any $q \in [1, p+1)$.

For any function $\psi \in C^\infty_c(\mathbb{R}^n)$

$$\begin{aligned}&\int_{\mathbb{R}^n} \left(\nabla\Delta u^\infty \nabla\Delta\psi - |x|^a\left|u^\infty\right|^{p-1}u^\infty\psi\right) dx\\ &\quad = \lim_{\lambda\to+\infty}\int_{\mathbb{R}^n}\left(\nabla\Delta u^\lambda\nabla\Delta\psi - |x|^a\left|u^\lambda\right|^{p-1}u^\lambda\psi\right) dx = 0,\end{aligned}$$

and

$$\int_{\mathbb{R}^n}\left(|\nabla\Delta\psi|^2 - p|x|^a\left|u^\infty\right|^{p-1}\psi^2\right) dx = \lim_{\lambda\to+\infty}\int_{\mathbb{R}^n}\left(|\nabla\Delta\psi|^2 - p|x|^a\left|u^\lambda\right|^{p-1}\psi^2\right) dx \ge 0.$$

Thus $u^\infty \in W^{3,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p+1}_{\mathrm{loc}}(\mathbb{R}^n)$ is a stable solution of (1.1).

Lemma 6.1

$$\lim_{\lambda\to\infty} E(u,\lambda) < \infty.$$

Proof From Theorem 3.2 we know that E is nondecreasing w.r.t. λ, so we only need to show that $E(u,\lambda)$ is bounded. Note that

$$E(u,\lambda) \le \frac{1}{\lambda}\int_\lambda^{2\lambda} E(u,t)\,dt \le \frac{1}{\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda} E(u,\gamma)\,d\gamma\,dt.$$

Since $u^\gamma(x) = \gamma^\alpha u(\gamma x)$, we have the following:

$$\gamma \frac{du^\gamma}{d\gamma} = \gamma^\alpha\big[\alpha u(\gamma x) + r\gamma\partial_r u(\gamma x)\big],$$
$$\gamma^2\frac{d^2u^\gamma}{d\gamma^2} = \gamma^\alpha\big[\alpha(\alpha-1)u(\gamma x) + 2\alpha r\gamma\partial_r u(\gamma x) + r^2\gamma^2\partial_{rr}u(\gamma x)\big]$$

and

$$\begin{aligned}\gamma^3\frac{d^3u^\gamma}{d\gamma^3} &= \gamma^\alpha\big[\alpha(\alpha-1)(\alpha-2)u(\gamma x)\\ &\quad + 3\alpha(\alpha-1)r\gamma\partial_r u(\gamma x) + 3\alpha r^2\gamma^2\partial_{rr}u(\gamma x) + r^3\gamma^3\partial_{rrr}u(\gamma x)\big].\end{aligned}$$

Hence, by scaling we have

$$\begin{aligned}&\int_{B_1}\left(\frac{1}{2}\big|\nabla\Delta u^\gamma\big|^2 - \frac{1}{p+1}|x|^a\big|u^\gamma\big|^{p+1}\right)dx\\ &\quad = \gamma^{6+2\alpha-n}\int_{B_\gamma}\left(\frac{1}{2}|\nabla\Delta u|^2\,dx - \frac{1}{p+1}|x|^a|u|^{p+1}\right)dx.\end{aligned}$$

From Proposition 4.1, we obtain

$$\frac{1}{\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda}\gamma^{6+2\alpha-n}\left(\int_{B_\gamma}\frac{1}{2}|\nabla\Delta u|^2\,dx - \frac{1}{p+1}\int_{B_\gamma}|x|^a|u|^{p+1}\,dx\right)d\gamma\,dt \le C,$$

where $C > 0$ is independent of γ. We have

$$\begin{aligned}&\frac{1}{\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda}\int_{\partial B_1}\gamma^5\frac{d^3u^\gamma}{d\gamma^3}\frac{d^2u^\gamma}{d\gamma^2}\\ &\quad = \frac{1}{\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda}\int_{\partial B_\gamma}\gamma^{2\alpha+1-n}\big[\alpha(\alpha-1)(\alpha-2)u\\ &\qquad + 3\alpha(\alpha-1)\gamma\partial_r u + 3\alpha\gamma^2\partial_{rr}u + \gamma^3\partial_{rrr}u\big]\\ &\qquad \times\big[\alpha(\alpha-1)u + 2\alpha\gamma\partial_r u + \gamma^2\partial_{rr}u\big]\\ &\quad \le C\frac{1}{\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda}t^{2\alpha+1-n}\int_{\partial B_\gamma}\big[u^2 + \gamma^2(\partial_r u)^2 + \gamma^4(\partial_{rr}u)^2 + \gamma^6(\partial_{rrr}u)^2\big]\\ &\quad \le C\frac{1}{\lambda^2}\int_\lambda^{2\lambda}t^{2\alpha+1-n}\int_{\partial B_{3\lambda}}\big[u^2 + \gamma^2(\partial_r u)^2 + \gamma^4(\partial_{rr}u)^2 + \gamma^6(\partial_{rrr}u)^2\big]\\ &\quad \le C\lambda^{n-2\alpha}\frac{1}{\lambda^2}\int_\lambda^{2\lambda}t^{2\alpha+1-n}\,dt \le C\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}&\left|\frac{1}{\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda}\int_{\partial B_1}\gamma\frac{d}{d\gamma}\big[\Delta_\theta u^\gamma\big]^2\right|\\ &\quad = \left|\frac{1}{\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda}\int_{\partial B_\gamma}\gamma^{2\alpha+2-n}\frac{d}{d\gamma}\big(\gamma^2\Delta u - \gamma^2\partial_{rr}u - (n-1)\gamma\partial_r u\big)^2\right|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\lambda^2}\int_\lambda^{2\lambda} t^{2\alpha+1-n}\int_t^{t+\lambda}\int_{\partial B_\gamma}\big|\big[2\gamma^2\Delta u-2\gamma^2\partial_{rr}u-(n-1)\gamma\partial_r u\big]\\
&\quad\times\big[\gamma^2\Delta u-\gamma^2\partial_{rr}u-(n-1)\gamma\partial_r u\big]\big|\\
&\le \frac{1}{\lambda^2}\int_\lambda^{2\lambda} t^{2\alpha+1-n}\int_{B_{3\lambda}}\big|\big[2\gamma^2\Delta u-2\gamma^2\partial_{rr}u-(n-1)\gamma\partial_r u\big]\\
&\quad\times\big[\gamma^2\Delta u-\gamma^2\partial_{rr}u-(n-1)\gamma\partial_r u\big]\big|\\
&\le C\frac{1}{\lambda^2}\int_\lambda^{2\lambda} t^{2\alpha+1-n}\int_{B_{3\lambda}}\big[\gamma^4(\Delta u)^2+\gamma^4(\partial_{rr}u)^2+\gamma^2\partial_r u\big]\\
&\le C\lambda^{n-2\alpha}\frac{1}{\lambda^2}\int_\lambda^{2\lambda} t^{2\alpha+1-n}\,dt\le C.
\end{aligned}
\tag{6.3}
$$

The remaining terms can be treated similarly as the estimate (6.2) or (6.3). □

Lemma 6.2 *u^∞ is homogeneous.*

Proof Due to the scaling invariance of the functional E (i.e., $E(u,R\lambda)=E(u^\lambda,R)$) and the monotonicity formula, for any given $R_2>R_1>0$, we see that

$$
\begin{aligned}
0 &= \lim_{i\to\infty}\big(E(u,R_2\lambda_i)-E(u,R_1\lambda_i)\big)\\
&= \lim_{i\to\infty}\big(E\big(u^{\lambda_i},R_2\big)-E\big(u^{\lambda_i},R_1\big)\big)\\
&\ge C(n,p)\liminf_{i\to\infty}\int_{B_{R_2}\setminus B_{R_1}} r^{2\alpha-n}\left(\alpha u^{\lambda_i}+r\frac{\partial u^{\lambda_i}}{\partial r}\right)^2 dx\\
&\ge C(n,p)\int_{B_{R_2}\setminus B_{R_1}} r^{2\alpha-n}\left(\alpha u^{\infty}+r\frac{\partial u^{\infty}}{\partial r}\right)^2 dx.
\end{aligned}
\tag{6.4}
$$

In the last inequality we have used the weak convergence of the sequence (u^{λ_i}) to the function u^∞ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ as $i\to\infty$. This implies that

$$
\alpha\frac{u^\infty}{r}+\frac{\partial u^\infty}{\partial r}=0,\quad \text{a.e. in } \mathbb{R}^n,
$$

integrating over r shows that

$$
u^\infty(x)=|x|^{-\alpha}u^\infty\left(\frac{x}{|x|}\right).
$$

That is, u^∞ is homogeneous. □

Lemma 6.3

$$
\lim_{r\to\infty}E(u,r)=0.
$$

Proof From Lemma 3.2, it implies that u^∞ is a homogeneous, stable solution of (1.1). Therefore, from Theorem 5.1, we have

$$
u^\infty\equiv 0.
$$

Combining with (6.1), we find that

$$\lim_{\lambda\to+\infty} u^\lambda = 0, \quad \text{strongly in } L^2\big(B_4(0)\big)$$

implies

$$\lim_{\lambda\to+\infty} \int_{B_4(0)} \big(u^\lambda\big)^2\,dx = 0.$$

By (4.7)

$$\lim_{\lambda\to+\infty} \int_{B_3(0)} \big(\big|\nabla\big(\Delta u^\lambda\big)\big|^2 + |x|^a\big|u^\lambda\big|^{p+1}\big)\,dx \le C \lim_{\lambda\to+\infty} \int_{B_4(0)} \big(u^\lambda\big)^2\,dx = 0. \tag{6.5}$$

By the interior L^2 estimate, we get

$$\lim_{\lambda\to+\infty} \int_{B_2(0)} \sum_{k\le 3} \big|\nabla^k u^\lambda\big|^2\,dx = 0.$$

In particular, we can choose a sequence $\lambda_i \to +\infty$ such that

$$\int_{B_2(0)} \sum_{k\le 3} \big|\nabla^k u^{\lambda_i}\big|^2\,dx \le 2^{-i}.$$

By this choice we have

$$\int_1^2 \sum_{i=1}^{\infty} \int_{\partial B_r} \sum_{k\le 3} \big|\nabla^k u^{\lambda_i}\big|^2\,dr \le \sum_{i=1}^{\infty} \int_1^2 \int_{\partial B_r} \sum_{k\le 3} \big|\nabla^k u^{\lambda_i}\big|^2\,dr \le 1,$$

that is, the function

$$g(r) := \sum_{i=1}^{\infty} \int_{\partial B_r} \sum_{k\le 3} \big|\nabla^k u^{\lambda_i}\big|^2 \in L^1\big((1,2)\big).$$

There exists an $r_0 \in (1,2)$ such that $g(r_0) < +\infty$. From this we get

$$\lim_{i\to+\infty} \big\|u^{\lambda_i}\big\|_{W^{3,2}(\partial B_{r_0})} = 0.$$

Combining this with (6.5) and the scaling invariance of $E(u,r)$, we get

$$\lim_{i\to+\infty} E(u,\lambda_i r_0) = \lim_{i\to+\infty} E\big(u^{\lambda_i}, r_0\big) = 0.$$

Since $\lambda_i r_0 \to +\infty$ and $E(u,r)$ is nondecreasing in r, we get

$$\lim_{r\to+\infty} E(u,r) = 0. \qquad \square$$

The smoothness of u implies that

$$\lim_{r\to 0} E(u,r) = 0.$$

From the monotonicity of $E(u,r)$ and Lemma 6.3, it implies that

$$E(u,r) = 0, \quad \text{for all } r > 0.$$

Therefore, by the monotonicity formula we know that u is homogeneous, then $u \equiv 0$ by Theorem 5.1. □

7 Classification of the finite Morse index solutions

We proceed based on a Pohozaev-type identity, the decay estimates from the doubling lemma [14], the monotonicity formula and the classification of the homogeneous solutions and stable solutions we obtained before.

7.1 Subcritical and critical case

We need the following Pohozaev identity.

Lemma 7.1 *Let $u \in C^6(\mathbb{R}^n)$ be a solution of (1.1) and $\psi \in C_c^3(B_{2R})$. Then*

$$\begin{aligned}
&\frac{2(n+a)}{(n-6)(p+1)} \int_{B_{2R}} |x|^a |u|^{p+1} \psi \, dx - \int_{B_{2R}} \left|\nabla(\Delta u)\right|^2 \psi \, dx \\
&= -\frac{1}{n-6} \int_{B_{2R}} \left|\nabla(\Delta u)\right|^2 (\nabla\psi \cdot x) \, dx - \frac{4}{n-6} \int_{B_{2R}} \nabla(\Delta u) \nabla\psi \, \Delta u \, dx \\
&\quad - \frac{4}{n-6} \int_{B_{2R}} \nabla(\Delta u) \nabla\psi \nabla^2 (\nabla u \cdot x) \, dx - \frac{4}{n-6} \int_{B_{2R}} \nabla(\Delta u) \nabla(\nabla u \cdot x) \nabla^2 \psi \, dx \\
&\quad - \frac{2}{n-6} \int_{B_{2R}} \nabla(\Delta u) \nabla(\Delta\psi)(\nabla u \cdot x) \, dx - \frac{2}{n-6} \int_{B_{2R}} \left(\nabla(\Delta u) \cdot x\right) \nabla^2 u \Delta\psi \, dx \\
&\quad - \frac{2}{n-6} \int_{B_{2R}} \nabla(\Delta u) \nabla u \Delta\psi \, dx - \frac{2}{(n-6)(p+1)} \int_{B_{2R}} |x|^a |u|^{p+1} (x \cdot \nabla\psi) \, dx. \qquad (7.1)
\end{aligned}$$

Proof Multiplying Eq. (1.1) by $(\nabla u \cdot x)\psi$ and integrating in B_{2R}, we get

$$\begin{aligned}
\int_{B_{2R}} |x|^a |u|^{p-1} u (\nabla u \cdot x) \psi \, dx &= \int_{B_{2R}} -\Delta^3 u (\nabla u \cdot x) \psi \, dx \\
&= -\int_{B_{2R}} \Delta^2 u (\nabla u \cdot x) \Delta\psi \, dx \\
&\quad - 2 \int_{B_{2R}} \Delta^2 u \nabla(\nabla u \cdot x) \nabla\psi \, dx \\
&\quad - \int_{B_{2R}} \Delta^2 u \Delta(\nabla u \cdot x) \psi \, dx \\
&:= J_1 + J_2 + J_3. \qquad (7.2)
\end{aligned}$$

By integrating by parts, we get

$$\begin{aligned}
J_1 &= -\int_{B_{2R}} \Delta^2 u(\nabla u\cdot x)\Delta\psi\,dx \\
&= \int_{B_{2R}} \nabla(\Delta u)\nabla\Delta\psi(\nabla u\cdot x)\,dx + \int_{B_{2R}} \nabla(\Delta u)\nabla(\nabla u\cdot x)\Delta\psi\,dx \\
&= \int_{B_{2R}} \nabla(\Delta u)\nabla\Delta\psi(\nabla u\cdot x)\,dx + \int_{B_{2R}} \bigl(\nabla(\Delta u)\cdot x\bigr)\nabla^2 u\Delta\psi\,dx \\
&\quad + \int_{B_{2R}} \nabla(\Delta u)\nabla u\Delta\psi\,dx, \qquad (7.3)\\
J_2 &= -2\int_{B_{2R}} \Delta^2 u\nabla(\nabla u\cdot x)\nabla\psi\,dx \\
&= 2\int_{B_{2R}} \nabla(\Delta u)\nabla^2(\nabla u\cdot x)\nabla\psi\,dx + 2\int_{B_{2R}} \nabla(\Delta u)\nabla(\nabla u\cdot x)\nabla^2\psi\,dx, \qquad (7.4)
\end{aligned}$$

and

$$\begin{aligned}
J_3 &= -\int_{B_{2R}} \Delta^2 u\Delta(\nabla u\cdot x)\psi\,dx \\
&= -\int_{B_{2R}} \Delta^2 u\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\left(\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}x_j\right)\psi\,dx \\
&= -\int_{B_{2R}} \Delta^2 u\bigl(\nabla(\Delta u)\cdot x\bigr)\psi - 2\int_{B_R} \Delta^2 u\Delta u\psi\,dx \\
&= \frac{6-n}{2}\int_{B_{2R}} \bigl|\nabla(\Delta u)\bigr|^2\psi\,dx + \frac{1}{2}\int_{B_{2R}} \bigl|\nabla(\Delta u)\bigr|^2(x\cdot\nabla\psi)\,dx \\
&\quad + 2\int_{B_{2R}} \nabla(\Delta u)\nabla\psi\,\Delta u\,dx. \qquad (7.5)
\end{aligned}$$

Now, we calculate the left hand side of Eq. (7.2). A direct calculation shows that

$$\begin{aligned}
J_4 &= \int_{B_{2R}} |x|^a|u|^{p-1}u(\nabla u\cdot x)\psi\,dx \\
&= -\frac{n+a}{p+1}\int_{B_{2R}} |x|^a|u|^{p+1}\psi\,dx - \frac{1}{p+1}\int_{B_{2R}} |x|^a|u|^{p+1}(x\cdot\nabla\psi)\,dx. \qquad (7.6)
\end{aligned}$$

From the identities, (7.3)–(7.6), we obtain the identity (7.1). □

Lemma 7.2 *Let $u \in C^6(\mathbb{R}^n)$ be a solution of* (1.1) *which is stable outside a compact set of $\mathbb{R}^n$. If $p \in (1, \frac{n+6+2a}{n-6})$, then*

(a)

$$\int_{\mathbb{R}^n} \bigl|\nabla(\Delta u)\bigr|^2\,dx = \frac{2(n+a)}{(n-6)(p+1)}\int_{\mathbb{R}^n} |x|^a|u|^{p+1}\,dx;$$

(b)

$$\int_{\mathbb{R}^n} |\nabla\Delta u|^2\,dx = \int_{\mathbb{R}^n} |x|^a|u|^{p+1}\,dx < +\infty.$$

Proof Let $u \in C^6(\mathbb{R}^n)$ be a solution of Eq. (1.1) which is stable outside a compact set of $\mathbb{R}^n$. Proposition 4.1 still holds if the support of ψ is outside B_{R_0}. Take $\phi \in C_0^\infty(B_{2R} \setminus B_{2R_0})$ such that $\phi \equiv 1$ in $B_R \setminus B_{3R_0}$ and $\sum_{k\le 5} |x|^k|\nabla^k\phi| \le C$. Then, by choosing $\psi = \phi^m$, where m is bigger than 1, we get $|x|^{\frac{a}{p+1}} u \in L^{p+1}(\mathbb{R}^n)$ and $\nabla(\Delta u) \in L^2(\mathbb{R}^n)$, $\forall p \in (1, \frac{n+6+2a}{n-6})$.

So,

$$\int_{B_{2R}\setminus B_R} \left(|x|^a|u|^{p+1} + \left|\nabla(\Delta u)\right|^2\right) \to 0, \quad \text{as } R \to +\infty. \tag{7.7}$$

Replace ψ by $\psi_R^4 \in C_c^3(\mathbb{R}^n)$ in Lemma 7.1 where $\psi_R(x) = 1$ on B_R and $\psi_R(x) = 0$ on $\mathbb{R}^n \setminus B_{2R}$. First, observe that, since p is subcritical,

$$R^{-6} \int_{B_{2R}\setminus B_R} u^2 \to 0, \quad \text{as } R \to +\infty. \tag{7.8}$$

In fact, by Hölder's inequality, we get

$$R^{-6} \int_{B_{2R}\setminus B_R} u^2 \le CR^{(n-\frac{6(p+1)+2a}{p-1})\frac{p-1}{p+1}} \left(\int_{B_{2R}\setminus B_R} |x|^a|u|^{p+1}\,dy\right)^{\frac{2}{p+1}}.$$

Now, to prove Lemma 7.2, we will show that any terms on the right hand side of (7.1) tend to 0 as $R \to +\infty$. For the first and second terms on the right hand side of (7.1), applying Hölder's inequality, we derive

$$\begin{aligned}\int_{B_{2R}} \left|\nabla(\Delta u)\right|^2\left(x\cdot\nabla\left(\psi_R^4\right)\right)dx &= \int_{B_{2R}\setminus B_R} \left|\nabla(\Delta u)\right|^2\left(x\cdot\nabla\left(\psi_R^4\right)\right)dx \\ &\le C\int_{B_{2R}\setminus B_R} \left|\nabla(\Delta u)\right|^2 dx\end{aligned}$$

and

$$\begin{aligned}\int_{B_{2R}} \nabla(\Delta u)\nabla\left(\psi_R^4\right)\Delta u\,dx &= \int_{B_{2R}\setminus B_R} \nabla(\Delta u)\nabla\left(\psi_R^4\right)\Delta u\,dx \\ &\le C\left(\int_{B_{2R}\setminus B_R} \left|\nabla(\Delta u)\right|^2 dx\right)^{\frac12} \\ &\quad\times\left(\int_{B_{2R}\setminus B_R} \left|\nabla(\Delta u)\right|^2 dx + R^{-6}\int_{B_{2R}\setminus B_R} u^2\,dx\right)^{\frac12}.\end{aligned}$$

Taking into account that p is subcritical, (7.7) and (7.8), we derive that the above terms tend to 0 as $R \to +\infty$. Except the third term, the remaining terms on the right hand side of (7.1) can be treated similarly as above. The third term needs more analysis. By an application of Hölder's inequality and using Proposition 4.1, we obtain

$$\begin{aligned}\int_{B_{2R}} \nabla(\Delta u)\nabla\left(\psi_R^4\right)\nabla^2(\nabla u\cdot x)\,dx &\le C\left(\int_{B_{2R}\setminus B_R} \left|\nabla(\Delta u)\right|^2 dx\right)^{\frac12} \\ &\quad\times\left(\int_{B_{2R}} \left|\nabla(\Delta u)\right|^2 dx + R^{-6}\int_{B_{2R}\setminus B_R} u^2\,dx\right)^{\frac12}\end{aligned}$$

(for more details see [10]). As above the third term on the right hand side of (7.1) tends to 0 as $R \to +\infty$.

Finally, we deduce that

$$\int_{\mathbb{R}^n} \left|\nabla(\Delta u)\right|^2 dx = \frac{2(n+a)}{(n-6)(p+1)} \int_{\mathbb{R}^n} |x|^a |u|^{p+1} dx.$$

By the interior elliptic estimates and Hölder's inequality, we have

$$\begin{aligned}
R^{-4} \int_{B_{2R}\backslash B_R} |\nabla u|^2 dx &\le C \int_{B_{3R}\backslash B_{R/2}} |\nabla \Delta u|^2 dx + C\biggl(\int_{B_{3R}\backslash B_{R/2}} |x|^a |u|^{p+1} dx\biggr)^{\frac{2}{p+1}}, \\
R^{-2} \int_{B_{2R}\backslash B_R} |\Delta u|^2 dx &\le C \int_{B_{3R}\backslash B_{R/2}} |\nabla \Delta u|^2 dx + C\biggl(\int_{B_{3R}\backslash B_{R/2}} |x|^a |u|^{p+1} dx\biggr)^{\frac{2}{p+1}}, \\
R^{-6} \int_{B_{2R}\backslash B_R} |u|^2 dx &\le C \int_{B_{3R}\backslash B_{R/2}} |\nabla \Delta u|^2 dx + C\biggl(\int_{B_{3R}\backslash B_{R/2}} |x|^a |u|^{p+1} dx\biggr)^{\frac{2}{p+1}}.
\end{aligned}$$

Therefore, we have

$$\max\biggl(R^{-4} \int_{B_{2R}\backslash B_R} |\nabla u|^2, R^{-2} \int_{B_{2R}\backslash B_R} |\Delta u|^2, R^{-6} \int_{B_{2R}\backslash B_R} u^2\biggr) \to 0, \quad \text{as } R \to +\infty.$$

On the other hand, testing (1.1) with a compact support function ψ^2, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \bigl(|\nabla \Delta u|^2 \psi^2 - |x|^a |u|^{p+1} \psi^2\bigr) dx = -\biggl(\int_{\mathbb{R}^n} \bigl(\nabla \Delta u.\Delta u \nabla \psi^2 + \nabla \Delta u \cdot \nabla \Delta \psi^2.u \\
+ \nabla \Delta u \nabla u \Delta \psi^2 + \nabla \Delta u \nabla\bigl(2\nabla u \nabla \psi^2\bigr)\bigr) dx\biggr).
\end{aligned}$$

By selecting $\psi(x) = \zeta(\frac{x}{R})^{3m}$, $m > 1$ and $\zeta \in C_c^\infty(B_2)$ and $\zeta \equiv 1$ in B_1 and $\sum_{k\le 3} |\nabla^k \zeta| \le C$, we get

$$\begin{aligned}
&\biggl|\int_{\mathbb{R}^n} \biggl(|\nabla \Delta u|^2 \zeta\biggl(\frac{x}{R}\biggr)^{6m} - |x|^a |u|^{p+1} \zeta\biggl(\frac{x}{R}\biggr)^{6m}\biggr) dx\biggr| \\
&\quad \le C\biggl(R^{-4} \int_{B_{2R}\backslash B_R} |\nabla u|^2 dx + R^{-2} \int_{B_{2R}\backslash B_R} |\Delta u|^2 dx + R^{-6} \int_{B_{2R}\backslash B_R} u^2 dx\biggr).
\end{aligned}$$

Now letting $R \to +\infty$, we obtain

$$\int_{\mathbb{R}^n} \bigl(|\nabla \Delta u|^2 - |x|^a |u|^{p+1}\bigr) dx = 0.$$

Therefore, we obtain the conclusions. □

Proof of Theorem 1.2 Let u be a solution to (1.1), which is stable outside a compact set of $\mathbb{R}^n$.

Subcritical case: $1 < p < \frac{n+6+2a}{n-6}$.

By Lemma 7.2, we have

$$\left(1 - \frac{2(n+a)}{(n-6)(p+1)}\right) \int_{B_R} |x|^a |u|^{p+1}\, dx = 0. \tag{7.9}$$

Since $1 - \frac{2(n+a)}{(n-6)(p+1)} \neq 0$, $\forall p \in (1, \frac{n+6+2a}{n-6})$, then

$$\int_{\mathbb{R}^n} |x|^a |u|^{p+1} = 0,$$

which yields $u \equiv 0$ in $\mathbb{R}^n$.

Critical case: $p = \frac{n+6+2a}{n-6}$.

We can proceed as in the proof of Item (b) of Lemma 7.2, to derive that

$$\int_{\mathbb{R}^n} |\nabla \Delta u|^2\, dx = \int_{\mathbb{R}^n} |x|^a |u|^{p+1}\, dx < +\infty. \qquad \square$$

7.2 Supercritical case

To classify finite Morse index solutions in the supercritical case, applying the doubling lemma in [14], we get the following estimates.

Lemma 7.3 *Let $n \geq 1$, $1 < p < p_0(n,6)$ and $\tau \in (0,1]$. Let $c \in C^\tau(\overline{B}_1)$ satisfy*

$$\|c\|_{C^\tau(\overline{B}_1)} \leq C_1 \quad \text{and} \quad c(x) \geq C_2, \quad x \in \overline{B}_1, \tag{7.10}$$

for some constants $C_1, C_2 > 0$. There exists a constant C, depending only on α, C_1, C_2, p, n, such that, for any classical solution u of

$$-\Delta^3 u = c(x)|u|^{p-1}u, \quad x \in B_1, \tag{7.11}$$

u satisfies

$$\left|u(x)\right|^{\frac{p-1}{6}} \leq C\left(1 + \operatorname{dist}^{-1}(x, \partial B_1)\right).$$

Proof Arguing by contradiction, we suppose that there exist sequences c_k, u_k verifying (7.10), (7.11) and points y_k, such that the functions

$$M_k = |u_k|^{\frac{p-1}{6}}$$

satisfy

$$M_k(y_k) > 2k\left(1 + \operatorname{dist}^{-1}(y_k, \partial B_1)\right) \geq 2k \operatorname{dist}^{-1}(y_k, \partial B_1).$$

By the doubling lemma in [14], there exists x_k such that

$$M_k(x_k) \geq M_k(y_k), \qquad M_k(x_k) \geq 2k \operatorname{dist}^{-1}(x_k, \partial B_1),$$

and

$$M_k(z) \le 2M_k(x_k), \quad \text{for all } z \text{ such that } |z - x_k| \le kM_k^{-1}(x_k). \tag{7.12}$$

We have

$$\lambda_k = M_k^{-1}(x_k) \to 0, \quad k \to \infty, \tag{7.13}$$

due to $M_k(x_k) \ge M_k(y_k) > 2k$.

Next we let

$$v_k = \lambda_k^{\frac{6}{p-1}} u_k(x_k + \lambda_k y), \qquad \tilde{c}_k(y) = c_k(x_k + \lambda_k y).$$

We note that $|v_k|^{\frac{p-1}{6}}(0) = 1$,

$$|v_k|^{\frac{p-1}{6}}(y) \le 2, \quad |y| \le k, \tag{7.14}$$

due to (7.12), and we see that v_k satisfies

$$-\Delta^3 v_k = \tilde{c}_k(y)|v_k|^{p-1}v_k, \quad |y| \le k. \tag{7.15}$$

On the other hand, due to (7.10), we have $C_2 \le \tilde{c}_k \le C_1$ and, for each $R > 0$ and $k \ge k_0(R)$ large enough,

$$\left|\tilde{c}_k(y) - \tilde{c}_k(z)\right| \le C_1\left|\lambda_k(y - z)\right|^\tau \le C_1|y - z|^\tau, \quad |y|, |z| \le R. \tag{7.16}$$

Therefore, by Ascoli's theorem, there exists $\tilde{c}$ in $C(\mathbb{R}^n)$, with $\tilde{c} \ge C_2$ such that, after extracting a subsequence, $\tilde{c}_k \to \tilde{c}$ in $C_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, (7.16) and (7.12) imply that $|\tilde{c}_k(y) - \tilde{c}_k(z)| \to 0$ as $k \to \infty$, so that the function $\tilde{c}$ is actually a constant $C > 0$. Now, for each $R > 0$ and $1 < q < \infty$, by (7.14), (7.13) and interior elliptic L^q estimates, the sequence v_k is uniformly bounded in $W^{3,q}(B_R)$. Using standard embeddings and interior elliptic Schauder estimates, after extracting a subsequence, we may assume that $v_k \to v$ in $C^6_{\mathrm{loc}}(\mathbb{R}^n)$. It follows that v is a classical solution of

$$-\Delta^3 v = C|v|^{p-1}v, \quad y \in \mathbb{R}^n,$$

and $|v|^{\frac{p-1}{6}}(0) = 1$. This contradicts the Liouville-type result [10] and concludes the proof. □

Proposition 7.1 *Let u be a (positive or sign-changing) solution to (1.1) which is stable outside a compact set of $\mathbb{R}^n$. There exist constants C and R_0 such that*

$$\left|u(x)\right| \le C|x|^{-\alpha}, \quad \textit{for all } x \in B_{R_0}(0)^c, \tag{7.17}$$

$$\sum_{k \le 5} |x|^{\alpha+k}\left|\nabla^k u(x)\right| \le C, \quad \textit{for all } x \in B_{3R_0}(0)^c. \tag{7.18}$$

Proof Assume that u is stable outside B_{R_0} and $|x_0| > 2R_0$. We denote

$$R = \frac{1}{2}|x_0|$$

and observe that, for all $y \in B_1$, $\frac{|x_0|}{2} < |x_0 + Ry| < \frac{3|x_0|}{2}$, so that $x_0 + Ry \in B_{R_0}(0)^c$. Let us thus define

$$U(y) = R^\alpha u(x_0 + Ry).$$

Then U is a solution of

$$-\Delta^3 U = c(y)|U|^{p-1}U, \quad y \in B_1, \text{ with } c(y) = \left|y + \frac{x_0}{R}\right|^a.$$

Notice that $|y + \frac{x_0}{R}| \in [1,3]$ for all $y \in \overline{B}_1$. Moreover $\|c\|_{C^1(\overline{B}_1)} \le C(a)$. Then applying Lemma 7.3, we have $|U(0)| \le C$, hence

$$\left|u(x_0)\right| \le CR^{-\alpha},$$

which yields the inequality (7.17).

Next, we only prove the inequality (7.18). For any x_0 with $|x_0| > 3R_0$, take $\lambda = \frac{|x_0|}{2}$ and define

$$\overline{u}(x) = \lambda^\alpha u(x_0 + \lambda x).$$

From (7.17), $|\overline{u}| \le C_0$ in $B_1(0)$. Standard elliptic estimates give

$$\sum_{k\le 5}\left|\nabla^k \overline{u}(0)\right| \le C.$$

Rescaling back we get (7.18). □

Proof of Theorem 1.2. Supercritical case: $p > \frac{n+6+2a}{n-6}$ and $p < p_0(n,6)$.

Lemma 7.4 *There exists a constant C such that, for all $r > 3R_0$, $E(u,r) \le C$.*

Proof From the monotonicity formula, combining the derivative estimates (7.18), we have the following estimates:

$$\begin{aligned} E(u,r) &\le r^{2\alpha+6-n}\left(\int_{B_r}\left(|\nabla\Delta u|^2 + |x|^a|u|^{p+1}\right)dx\right) \\ &\quad + \sum_{j,k\le 4, j+k\le 5} r^{2\alpha+1-n+j+k}\int_{\partial B_r}\left|\nabla^j u\right|\left|\nabla^k u\right| d\sigma \\ &\le C. \end{aligned} \tag{7.19}$$

This constant only depends on the constant in (7.18). □

As a consequence, we have the following.

Corollary 7.1

$$\int_{B^c_{3R_0}} \frac{(\alpha|x|^{-1}u(x) + \frac{\partial u}{\partial r}(x))^2}{|x|^{n-2\alpha}}\,dx < +\infty. \tag{7.20}$$

As before, we define a blowing down sequence

$$u^\lambda(x) = \lambda^\alpha u(\lambda x).$$

By Proposition 7.1, u^λ are uniformly bounded in $C^7(B_r(0) \setminus B_{1/r}(0))$ for any fixed $r > 1$. u^λ is stable outside $B_{R_0/\lambda}(0)$. There exists a function $u^\infty \in C^6(\mathbb{R}^n \setminus \{0\})$, such that up to a subsequence of $\lambda \to +\infty$, u^λ converges to $u^\infty \in C^6_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. u^∞ is a stable solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$.

For any $r > 1$, we get from (7.20)

$$\begin{aligned}\int_{B_r\setminus B_{1/r}} \frac{(\alpha|x|^{-1}u^\infty(x) + \frac{\partial u^\infty}{\partial r}(x))^2}{|x|^{n-2\alpha}}\,dx &= \lim_{\lambda\to+\infty}\int_{B_r\setminus B_{1/r}} \frac{(\alpha|x|^{-1}u^\lambda(x) + \frac{\partial u^\lambda}{\partial r}(x))^2}{|x|^{n-2\alpha}}\,dx \\ &= \lim_{\lambda\to+\infty}\int_{B_{\lambda r}\setminus B_{\lambda/r}} \frac{(\alpha|x|^{-1}u(x) + \frac{\partial u}{\partial r}(x))^2}{|x|^{n-2\alpha}}\,dx \\ &= 0.\end{aligned}$$

Hence, u^∞ is homogeneous, and from Theorem 5.1, $u^\infty \equiv 0$. This holds for every limit of u^λ as $\lambda \to +\infty$, thus we have

$$\lim_{|x|\to+\infty} |x|^\alpha \left|u(x)\right| = 0.$$

From (7.18), we get

$$\lim_{|x|\to+\infty} \sum_{k\le 6} |x|^{\alpha+k}\left|\nabla^k u(x)\right| = 0.$$

For $\varepsilon > 0$, take an R such that, for $|x| > R$,

$$\sum_{k\le 6} |x|^{\alpha+k}\left|\nabla^k u(x)\right| \le \varepsilon.$$

Then, for $r \gg R$,

$$\begin{aligned}E(u,r) &\le Cr^{2\alpha+6-n}\int_{B_R(0)} \left(\left|\nabla(\Delta u)\right|^2 + |x|^a|u|^{p+1}\right)dx \\ &\quad + C\epsilon r^{2\alpha+6-n}\int_{B_r(0)\setminus B_R(0)} |x|^{-2\alpha-6}\,dx + C\epsilon r^{2\alpha+7-n}\int_{\partial B_r(0)} |x|^{-2\alpha-6}\,d\sigma \\ &\le C(R)\left(r^{2\alpha+6-n} + \varepsilon\right).\end{aligned}$$

Since $2\alpha + 6 - n < 0$ and ε can be arbitrarily small, we derive $\lim_{r\to+\infty} E(u,r) = 0$. Because $\lim_{r\to 0} E(r,u) = 0$ (by the smoothness of u), the same argument for stable solutions implies that $u \equiv 0$. $\square$

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References
1. Bahri, A., Lions, P.L.: Solutions of superlinear elliptic equations and their Morse indices. Commun. Pure Appl. Math. **45**, 1205–1215 (1992)
2. Bidaut-Véron, M., Véron, L.: Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations. Invent. Math. **106**, 489–539 (1991)
3. Blatt, S.: Monotonicity formulas for extrinsic triharmonic maps and the triharmonic Lane–Emden equation. J. Differ. Equ. (2017). https://doi.org/10.1016/j.jde.2017.01.025
4. Cowan, C.: Liouville theorems for stable Lane–Emden systems and biharmonic problems. Nonlinearity **26**, 2357–2371 (2013)
5. Dancer, E.N., Du, Y.H., Guo, Z.M.: Finite Morse index solutions of an elliptic equation with supercritical exponent. J. Differ. Equ. **250**, 3281–3310 (2011)
6. Dávila, J., Dupaigne, L., Wang, K., Wei, J.: A monotonicity formula and a Liouville-type theorem for a fourth order supercritical problem. Adv. Math. **258**, 240–285 (2014)
7. Farina, A.: On the classification of solutions of the Lane–Emden equation on unbounded domains of $\mathbb{R}^n$. J. Math. Pures Appl. **87**, 537–561 (2007)
8. Gidas, B., Spruck, J.: Global and local behavior of positive solutions of nonlinear elliptic equations. Commun. Pure Appl. Math. **34**, 525–598 (1981)
9. Hajlaoui, H., Harrabi, A., Ye, D.: On stable solutions of biharmonic problem with polynomial growth. Pac. J. Math. **270**, 79–93 (2014)
10. Harrabi, A., Rahal, B.: On the sixth-order Joseph–Lundgren exponent. Ann. Henri Poincaré **18**(3), 1055–1094 (2017)
11. Hu, L.: Liouville-type theorems for the fourth order nonlinear elliptic equation. J. Differ. Equ. **256**, 1817–1846 (2014)
12. Karageorgis, P.: Stability and intersection properties of solutions to the nonlinear biharmonic equation. Nonlinearity **22**, 1653–1661 (2009)
13. Luo, S., Wei, J., Zou, W.: On the triharmonic Lane–Emden equation. Preprint (2016). arXiv:1607.04719
14. Polácik, P., Quittner, P., Souplet, P.: Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. Duke Math. J. **139**(3), 555–579 (2007)
15. Rellich, F.: Perturbation Theory of Eigenvalue Problems. Gordon & Breach, New York (1969)
16. Wang, C., Ye, D.: Some Liouville theorems for Hénon type elliptic equations. J. Funct. Anal. **262**(4), 1705–1727 (2012)
17. Wei, J.C., Ye, D.: Liouville theorems for finite Morse index solutions of biharmonic problem. Math. Ann. **356**, 1599–1612 (2013)