Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the integer part of the reciprocal of the Riemann zeta function tail at certain rational numbers in the critical strip

WonTae Hwang[1] and Kyunghwan Song[2*]

*Correspondence:
khsong0118@ewha.ac.kr
[2]Institute of Mathematical Sciences, Ewha Womans University, Seoul, Republic of Korea
Full list of author information is available at the end of the article

Abstract

We prove that the integer part of the reciprocal of the tail of $\zeta(s)$ at a rational number $s = \frac{1}{p}$ for any integer with $p \geq 5$ or $s = \frac{2}{p}$ for any odd integer with $p \geq 5$ can be described essentially as the integer part of an explicit quantity corresponding to it. To deal with the case when $s = \frac{2}{p}$, we use a result on the finiteness of integral points of certain curves over $\mathbb{Q}$.

MSC: 11M06; 11B83; 11G30

Keywords: Riemann zeta function tail; Critical strip; Siegel's theorem

1 Introduction

Among various kind of zeta functions in mathematics, one of the most famous and important zeta functions is the Riemann zeta function. For $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$, the Riemann zeta function is defined as the absolutely convergent infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is well known that this function admits an analytic continuation to the whole complex plane $\mathbb{C}$, has an Euler product formula, and satisfies a functional equation.

Recently, Xin [6] initiated the study of a reciprocal sum related to $\zeta(2)$ and $\zeta(3)$, and proved the following two equalities: for any positive integer n, we have

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{k^2}\right)^{-1}\right] = n - 1$$

and

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{k^3}\right)^{-1}\right] = 2n(n - 1),$$

where $[x]$ denotes the greatest integer that is less than or equal to x. One basic observation from this result is that both $n - 1$ and $2n(n - 1)$ are polynomials in the variable n. Xin [6] also proposed a natural problem of determining the existence of an explicit computational formula for $[(\sum_{k=n}^{\infty}\frac{1}{k^s})^{-1}]$ for an integer $s \geq 4$. In an attempt to solve this problem, Xin and Xiaoxue [7] came up with a computational formula for the case $s = 4$, and Xu [8] proved two computational formulas that are related to the Riemann zeta function at $s = 4, 5$, using a slightly different method from that of [7]. Also, the authors [3] introduced an explicit formula for the case $s = 6$, which depends on the residue of n modulo 48.

Now, if we restrict our attention to some rational numbers $0 < s < 1$, then we have the following list of values of the Riemann zeta function:

$$\zeta\left(\frac{1}{2}\right) = -1.46035\ldots, \qquad \zeta\left(\frac{1}{3}\right) = -0.97336\ldots, \qquad \zeta\left(\frac{1}{4}\right) = -0.813278\ldots,$$
$$\zeta\left(\frac{1}{5}\right) = -0.733921\ldots, \qquad \zeta\left(\frac{2}{3}\right) = -2.44758\ldots, \qquad \zeta\left(\frac{2}{5}\right) = -1.1348\ldots.$$

We may ask a similar question for the case when s is a rational number on the critical strip. To this aim, for an integer $n \geq 1$ and a real number s with $0 < s < 1$, we let

$$\zeta_n(s) = \frac{1}{1-2^{1-s}} \cdot \sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{k^s},$$
$$A_{n,s} = \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) + \left(\frac{1}{(n+2)^s} - \frac{1}{(n+3)^s}\right) + \cdots,$$

and

$$B_{n,s} = \left(-\frac{1}{n^s} + \frac{1}{(n+1)^s}\right) + \left(-\frac{1}{(n+2)^s} + \frac{1}{(n+3)^s}\right) + \cdots.$$

Note that we have

$$\zeta_1(s) = \zeta(s)$$

and

$$\zeta_n(s) = \begin{cases} -\frac{1}{1-2^{1-s}} A_{n,s}, & \text{if } n \text{ is even}, \\ -\frac{1}{1-2^{1-s}} B_{n,s}, & \text{if } n \text{ is odd}. \end{cases} \tag{1}$$

Along this line, Kim and Song [4] (resp. Song [5]) proved that

$$\left[\frac{1}{1-2^{1-s}} \cdot \zeta_n(s)^{-1}\right] = \left[(-1)^{n+1} \cdot 2\left(n - \frac{1}{2}\right)^s\right]$$

for every integer $n \geq 1$, at $s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ (resp. $s = \frac{2}{3}$). In this paper, we extend the previous results to the case when either $s = \frac{1}{p}$ for any integer with $p \geq 5$ or $s = \frac{2}{p}$ for any odd integer with $p \geq 5$.

Our main result is summarized in the following.

Theorem 1 *Let $s = \frac{1}{p}$ for any integer with $p \geq 5$ or $s = \frac{2}{p}$ for any odd integer with $p \geq 5$. Then there exists an integer $N > 0$ such that*

$$\left[\frac{1}{1-2^{1-s}} \cdot \zeta_n(s)^{-1}\right] = \left[(-1)^{n+1} \cdot 2\left(n - \frac{1}{2}\right)^s\right]$$

for every integer $n \geq N$.

For more details, see Corollaries 4 and 5.

This paper is organized as follows: In Sect. 2, we first introduce some properties of $\zeta_n(s)$. Afterwards, we recall a theorem of Siegel on the integral points of a smooth algebraic curve over a number field (see Theorem 4). In Sect. 3, we deal with the case of $s = \frac{1}{p}$, using Theorem 3 below. In Sect. 4, we give a proof of Theorem 1 for the case when $s = \frac{2}{p}$ by invoking a version of the previously introduced theorem of Siegel (see Theorem 5).

2 Preliminaries

2.1 Properties of $\zeta_n(s)$

In this section, we give some useful properties of $\zeta_n(s)$ in terms of the size of its reciprocal. To achieve our goal, we first need the following.

Theorem 2 *Let s be a real number with $0 < s < 1$. Then we have*

$$2\left(n - \frac{1}{2}\right)^s < A_{n,s}^{-1} < 2\left(n - \frac{1}{4}\right)^s$$

for every even integer $n \geq 2$, and

$$-2\left(n - \frac{1}{4}\right)^s < B_{n,s}^{-1} < -2\left(n - \frac{1}{2}\right)^s$$

for every odd integer $n \geq 1$.

Proof For a proof, see [4, Theorem 1]. □

In view of Eq. (1), Theorem 2 has a nice consequence.

Corollary 1 *For any real number s with $0 < s < 1$, we have*

$$-2\left(1 - 2^{1-s}\right)\left(n - \frac{1}{2}\right)^s < \zeta_n(s)^{-1} < -2\left(1 - 2^{1-s}\right)\left(n - \frac{1}{4}\right)^s$$

for every even integer $n \geq 2$, and

$$2\left(1 - 2^{1-s}\right)\left(n - \frac{1}{4}\right)^s < \zeta_n(s)^{-1} < 2\left(1 - 2^{1-s}\right)\left(n - \frac{1}{2}\right)^s$$

for every odd integer $n \geq 1$.

If we do not require the inequality of Theorem 2 to hold for every even or odd integer, as indicated above, then we can obtain a slightly better upper bound of the value of $\zeta_n(s)^{-1}$:

Theorem 3 *Let $\epsilon > 0$ be given. Then, for any real number s with $0 < s < 1$, we have*

$$2\left(n-\frac{1}{2}\right)^s < A_{n,s}^{-1} < 2\left(n-\frac{1}{2}+\epsilon\right)^s$$

for every sufficiently large even integer n, and

$$-2\left(n-\frac{1}{2}+\epsilon\right)^s < B_{n,s}^{-1} \le -2\left(n-\frac{1}{2}\right)^s$$

for every sufficiently large odd integer n.

Proof For a proof, see [4, Theorem 2]. □

As before, combining Eq. (1) and Theorem 3 yields the following.

Corollary 2 *Let $\epsilon > 0$ be given. Then, for any real number s with $0 < s < 1$, we have*

$$-2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}\right)^s < \zeta_n(s)^{-1} < -2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}+\epsilon\right)^s$$

for every sufficiently large even integer n, and

$$2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}+\epsilon\right)^s < \zeta_n(s)^{-1} < 2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}\right)^s$$

for every sufficiently large odd integer n.

2.2 Siegel's theorem on integral points

In this section, we briefly review a theorem of Siegel on the integral points of certain curves that are defined over a number field.

In the sequel, let K be a number field, S a finite set of places of K, and R_S the ring of S-integers in K. Also, let $\overline{K}$ be an algebraic closure of K. Then we have the following fundamental result.

Theorem 4 *Let C be a smooth projective curve of genus g over K, and let $f \in K(C)$ be a nonconstant function. If $g \ge 1$, then the set*

$$\left\{P \in C(K) \mid f(P) \in R_S\right\}$$

is finite.

Proof For a proof, see [2, Theorem D.9.1]. □

In fact, this theorem is more general than what we actually need. We need to use a version of Theorem 4 regarding a hyperelliptic curve, which we describe now: suppose that S includes all the infinite places.

Theorem 5 *Let $f(X) \in K[X]$ be a polynomial of degree at least 3 with distinct roots in $\overline{K}$. Then the equation $Y^2 = f(X)$ has only finitely many solutions $X, Y \in R_S$.*

Proof For a proof, see [2, Theorem D.8.3]. □

Example 1 Let $K = \mathbb{Q}$, $S = \{2, \infty\}$, and let $f(X) = \frac{1}{32}X^5 + \frac{17}{32}$. Then the equation $Y^2 = f(X)$ has only finitely many solutions in $R_S = \mathbb{Z}[\frac{1}{2}]$. This example is closely related to the case of $s = \frac{2}{5}$ and $m = 1$ in Lemma 5 below.

3 The case of $s = \frac{1}{p}$

Throughout this section, let $p \geq 5$ be a fixed integer and let $s = \frac{1}{p}$.

Lemma 1 *Let $n \geq 1$ be an integer. Then there is no integer between $2(n - \frac{1}{2})^s$ and $2(n - \frac{1}{2} + \frac{1}{2^{p+1}})^s$.*

Proof Suppose on the contrary that there is an integer x with $2(n - \frac{1}{2})^s < x < 2(n - \frac{1}{2} + \frac{1}{2^{p+1}})^s$. It follows that

$$2^p n - 2^{p-1} < x^p < 2^p n - 2^{p-1} + \frac{1}{2},$$

which is absurd.

This completes the proof. □

By a similar argument, we can also have the following.

Lemma 2 *Let $n \geq 1$ be an integer. Then there is no integer between $-2(n - \frac{1}{2} + \frac{1}{2^{p+1}})^s$ and $-2(n - \frac{1}{2})^s$.*

Combining all these results, we have the following.

Corollary 3 *There exist integers $n_0, n_1 > 0$ such that*

$$[A_{n,s}^{-1}] = \left[2\left(n - \frac{1}{2}\right)^s\right]$$

for every even integer $n \geq n_0$, and

$$[B_{n,s}^{-1}] = \left[-2\left(n - \frac{1}{2}\right)^s\right]$$

for every odd integer $n \geq n_1$.

Proof Let $\epsilon = \frac{1}{2^{p+1}}$. By Theorem 3, there exist integers $n_0, n_1 > 0$ such that

$$2\left(n - \frac{1}{2}\right)^s < A_{n,s}^{-1} < 2\left(n - \frac{1}{2} + \frac{1}{2^{p+1}}\right)^s$$

for every even integer $n \geq n_0$, and

$$-2\left(n - \frac{1}{2} + \frac{1}{2^{p+1}}\right)^s < B_{n,s}^{-1} < -2\left(n - \frac{1}{2}\right)^s$$

for every odd integer $n \geq n_1$. Then, by Lemmas 1 and 2, it follows that $[A_{n,s}^{-1}] = [2(n-\frac{1}{2})^s]$ for every even integer $n \geq n_0$, and $[B_{n,s}^{-1}] = [-2(n-\frac{1}{2})^s]$ for every odd integer $n \geq n_1$, as desired.

This completes the proof. □

An immediate consequence of Corollary 3 is the following.

Corollary 4 *There exists an integer $N > 0$ such that we have*

$$\left[\frac{1}{1-2^{1-s}} \cdot \zeta_n(s)^{-1}\right] = \left[(-1)^{n+1} \cdot 2\left(n-\frac{1}{2}\right)^s\right]$$

for every integer $n \geq N$.

Proof This follows from Eq. (1) and Corollary 3. □

Remark 1 If $p \in \{2,3,4\}$, then we may take $N = 1$ in view of [4, Corollary 6].

4 The case of $s = \frac{2}{p}$

Throughout this section, let $p \geq 5$ be a fixed odd integer and let $s = \frac{2}{p}$.

To begin with, we introduce one useful inequality.

Lemma 3 *For any real number $x \geq 2$, we have*

$$\left(\left(1-\frac{1}{4x}\right)^2 + \frac{1}{8x^2}\right)^{-1/p-1} \cdot \left(1-\frac{1}{4x}\right) + \left(\left(1+\frac{1}{4x}\right)^2 + \frac{1}{8x^2}\right)^{-1/p-1} \cdot \left(1+\frac{1}{4x}\right) < 2.$$

Proof For convenience, let $\alpha = 4x$ and $h(\alpha) = ((1-\frac{1}{\alpha})^2 + \frac{2}{\alpha^2})^{-1/p-1} \cdot (1-\frac{1}{\alpha})$. (Note that $\alpha \geq 8$.) Then it suffices to show that $h(\alpha) + h(-\alpha)$ is a strictly increasing function because we have

$$\lim_{\alpha\to\infty}\left(\left(\left(1-\frac{1}{\alpha}\right)^2 + \frac{2}{\alpha^2}\right)^{-1/p-1} \cdot \left(1-\frac{1}{\alpha}\right) + \left(\left(1+\frac{1}{\alpha}\right)^2 + \frac{2}{\alpha^2}\right)^{-1/p-1} \cdot \left(1+\frac{1}{\alpha}\right)\right) = 2.$$

Indeed, we will use the first derivative test, as follows: note first that we have

$$\begin{aligned}
&\frac{d(h(\alpha)+h(-\alpha))}{d\alpha} \\
&= \frac{d(((1-\frac{1}{\alpha})^2 + \frac{2}{\alpha^2})^{-1/p-1} \cdot (1-\frac{1}{\alpha}) + ((1+\frac{1}{\alpha})^2 + \frac{2}{\alpha^2})^{-1/p-1} \cdot (1+\frac{1}{\alpha}))}{d\alpha} \\
&= \frac{1}{\alpha^4 p}\left(\alpha^2 p\left(\left(\frac{\alpha^2-2\alpha+3}{\alpha^2}\right)^{-1-1/p} - \left(\frac{\alpha^2+2\alpha+3}{\alpha^2}\right)^{-1-1/p}\right)\right. \\
&\quad - 2(p+1)\left((\alpha-1)(\alpha-3)\left(\frac{\alpha^2-2\alpha+3}{\alpha^2}\right)^{-2-1/p}\right. \\
&\quad \left.\left. - (\alpha+1)(\alpha+3)\left(\frac{\alpha^2+2\alpha+3}{\alpha^2}\right)^{-2-1/p}\right)\right).
\end{aligned}$$

Since $\alpha^4 p > 0$, we need to show that

$$\begin{aligned}
&\alpha^2 p\biggl(\biggl(\frac{\alpha^2-2\alpha+3}{\alpha^2}\biggr)^{-1-1/p}-\biggl(\frac{\alpha^2+2\alpha+3}{\alpha^2}\biggr)^{-1-1/p}\biggr)\\
&\quad -2(p+1)\biggl((\alpha-1)(\alpha-3)\biggl(\frac{\alpha^2-2\alpha+3}{\alpha^2}\biggr)^{-2-1/p}\\
&\quad -(\alpha+1)(\alpha+3)\biggl(\frac{\alpha^2+2\alpha+3}{\alpha^2}\biggr)^{-2-1/p}\biggr)\\
&\quad > 0,
\end{aligned}$$

or equivalently (by multiplying $\alpha^{-4-2/p}$ and rearranging)

$$\begin{aligned}
&\bigl(\alpha^2-2\alpha+3\bigr)^{-2-1/p}\bigl(p\bigl(\alpha^2-2\alpha+3\bigr)-2(p+1)(\alpha-1)(\alpha-3)\bigr)\\
&\quad > \bigl(\alpha^2+2\alpha+3\bigr)^{-2-1/p}\bigl(p\bigl(\alpha^2+2\alpha+3\bigr)-2(p+1)(\alpha+1)(\alpha+3)\bigr),
\end{aligned}$$

which is equivalent to

$$\frac{(-p-2)\alpha^2+(6p+8)\alpha+(-3p-6)}{(-p-2)\alpha^2+(-6p-8)\alpha+(-3p-6)} < \biggl(\frac{\alpha^2+2\alpha+3}{\alpha^2-2\alpha+3}\biggr)^{-2-1/p},$$

because $(-p-2)\alpha^2+(-6p-8)\alpha+(-3p-6) < 0$ for any α, p. We prove the last inequality. Indeed, we may assume that $(-p-2)\alpha^2+(6p+8)\alpha+(-3p-6) < 0$ because if $(-p-2)\alpha^2+(6p+8)\alpha+(-3p-6) \geq 0$, then the desired inequality follows trivially. Hence, we have to show that

$$\biggl(\frac{\alpha^2+2\alpha+3}{\alpha^2-2\alpha+3}\biggr)^{2+1/p} < \frac{(p+2)\alpha^2+(6p+8)\alpha+(3p+6)}{(p+2)\alpha^2+(-6p-8)\alpha+(3p+6)}. \tag{2}$$

Since

$$\biggl(\frac{\alpha^2+2\alpha+3}{\alpha^2-2\alpha+3}\biggr)^{2+1/p} < \biggl(\frac{\alpha^2+2\alpha+3}{\alpha^2-2\alpha+3}\biggr)^{7/3}$$

and

$$\begin{aligned}
&\bigl(\alpha^2+2\alpha+3\bigr)^7\bigl((p+2)\alpha^2+(-6p-8)\alpha+(3p+6)\bigr)^3\\
&\quad -\bigl(\alpha^2-2\alpha+3\bigr)^7\bigl((p+2)\alpha^2+(6p+8)\alpha+(3p+6)\bigr)^3\\
&= \bigl(-8p^3-24p^2+32\bigr)\alpha^{19}+\bigl(-88p^3-1032p^2-2304p-1056\bigr)\alpha^{17}\\
&\quad +\bigl(5472p^3+21{,}792p^2+16{,}128p+2304\bigr)\alpha^{15}\\
&\quad +\bigl(-25{,}952p^3+62{,}688p^2+126{,}720p+52{,}224\bigr)\alpha^{13}\\
&\quad +\bigl(-656{,}496p^3-1{,}239{,}120p^2-856{,}320p-217{,}024\bigr)\alpha^{11}\\
&\quad +\bigl(-1{,}969{,}488p^3-3{,}717{,}360p^2-2{,}568{,}960p-651{,}072\bigr)\alpha^9\\
&\quad +\bigl(-700{,}704p^3+1{,}692{,}576p^2+3{,}421{,}440p+1{,}410{,}048\bigr)\alpha^7
\end{aligned}$$

$$
\begin{aligned}
&+ \left(1{,}329{,}696p^3 + 5{,}295{,}456p^2 + 3{,}919{,}104p + 559{,}872\right)\alpha^5 \\
&+ \left(-192{,}456p^3 - 2{,}256{,}984p^2 - 5{,}038{,}848p - 2{,}309{,}472\right)\alpha^3 \\
&+ \left(-157{,}464p^3 - 472{,}392p^2 + 629{,}856\right)\alpha \\
< {} & \left(-8\alpha^4 + 5472\right)p^3\alpha^{15} + \left(-20\alpha^4 + 21{,}792\right)p^2\alpha^{15} + \left(-4p^2\alpha^6 + 32\alpha^6 + 52{,}224\right)\alpha^{13} \\
&+ \left(-2304\alpha^2 + 16{,}128\right)p\alpha^{15} + \left(-1056\alpha^2 + 2304\right)\alpha^{15} \\
&+ \left(-1032p\alpha^4 + 62{,}688p + 126{,}720\right)p\alpha^{13} \\
&+ \left(-656{,}496\alpha^6 + 1{,}329{,}696\right)p^3\alpha^5 + \left(-1{,}239{,}120\alpha^6 + 5{,}295{,}456\right)p^2\alpha^5 \\
&+ \left(-856{,}320\alpha^6 + 3{,}919{,}104\right)p\alpha^5 + \left(-217{,}024\alpha^6 + 559{,}872\right)\alpha^5 \\
&+ \left(-700{,}704 + \frac{1{,}692{,}576}{5} + \frac{3{,}421{,}440}{5^2} + \frac{1{,}410{,}048}{5^3}\right)p^3\alpha^7 \\
&+ \left(-192{,}456p^3 - 2{,}256{,}984p^2 - 5{,}038{,}848p - 2{,}309{,}472\right)\alpha^3 \\
&+ \left(-157{,}464p^3 - 472{,}392p^2 + 629{,}856\right)\alpha \\
< {} & 0
\end{aligned}
$$

for every $p \geq 5$ and $\alpha \geq 8$, we find that the inequality (2) holds for $\alpha \geq 8$, which in turn, implies that $h(\alpha) + h(-\alpha)$ is increasing for $\alpha \geq 8$.

This completes the proof. □

An important consequence of the above lemma is the following.

Lemma 4 *For any even integer $n \geq 4$, we have*

$$2\left(n - \frac{1}{2}\right)^s = 2\left(n^2 - n + \frac{1}{4}\right)^{s/2} < A_{n,s}^{-1} < 2\left(n^2 - n + \frac{3}{4}\right)^{s/2}$$

and for any odd integer $n \geq 3$, we have

$$-2\left(n^2 - n + \frac{3}{4}\right)^{s/2} < B_{n,s}^{-1} < -2\left(n^2 - n + \frac{1}{4}\right)^{s/2} = -2\left(n - \frac{1}{2}\right)^s. \tag{3}$$

Proof Let $n \geq 4$ be an even integer. By Theorem 2, we have

$$2\left(n - \frac{1}{2}\right)^{2/p} < A_{n,2/p}^{-1} < 2\left(n - \frac{1}{4}\right)^{2/p}.$$

Hence, it suffices to show that

$$\frac{1}{n^{2/p}} - \frac{1}{(n+1)^{2/p}} - \frac{1}{2}\left(\left(n - \frac{1}{2}\right)^2 + \frac{1}{2}\right)^{-1/p} + \frac{1}{2}\left(\left(n + \frac{3}{2}\right)^2 + \frac{1}{2}\right)^{-1/p} > 0$$

for any even integer $n \geq 4$. Let $f, g : \mathbb{R}_{\geq 2} \to \mathbb{R}$ be two functions defined by

$$f(x) = \frac{1}{(2x)^{2/p}} - \frac{1}{(2x+1)^{2/p}} - \frac{1}{2}\left(\left(2x - \frac{1}{2}\right)^2 + \frac{1}{2}\right)^{-1/p} + \frac{1}{2}\left(\left(2x + \frac{3}{2}\right)^2 + \frac{1}{2}\right)^{-1/p}$$

and

$$g(x) = (2x)^{-2/p} - \frac{1}{2}\left(\left(2x - \frac{1}{2}\right)^2 + \frac{1}{2}\right)^{-1/p} - \frac{1}{2}\left(\left(2x + \frac{1}{2}\right)^2 + \frac{1}{2}\right)^{-1/p}$$
$$= (2x)^{-2/p} - \frac{1}{2}\left(4x^2 - 2x + \frac{3}{4}\right)^{-1/p} - \frac{1}{2}\left(4x^2 + 2x + \frac{3}{4}\right)^{-1/p}.$$

Then we have $f(x) = g(x) - g(x + \frac{1}{2})$, and hence, we only need to show that $g(x)$ is decreasing. Since

$$g'(x) = \left(-\frac{4}{p}\right) \cdot (2x)^{-2/p-1}$$
$$+ \frac{1}{2p}\left(4x^2 - 2x + \frac{3}{4}\right)^{-1/p-1} \cdot (8x - 2) + \frac{1}{2p}\left(4x^2 + 2x + \frac{3}{4}\right)^{-1/p-1} \cdot (8x + 2),$$

we have to show that

$$(2x)^{-2/p-1} - \left(4x^2 - 2x + \frac{3}{4}\right)^{-1/p-1} \cdot \left(x - \frac{1}{4}\right) - \left(4x^2 + 2x + \frac{3}{4}\right)^{-1/p-1} \cdot \left(x + \frac{1}{4}\right) > 0,$$

which is equivalent to saying that

$$\left(\left(1 - \frac{1}{4x}\right)^2 + \frac{1}{8x^2}\right)^{-1/p-1} \cdot \left(1 - \frac{1}{4x}\right) + \left(\left(1 + \frac{1}{4x}\right)^2 + \frac{1}{8x^2}\right)^{-1/p-1} \cdot \left(1 + \frac{1}{4x}\right) < 2.$$

Then it follows from Lemma 3 that $g(x)$ is decreasing for $x \geq 2$. Also, a similar argument can be used to show that (3) holds for any odd integer $n \geq 3$.

This completes the proof. □

Remark 2 The inequalities in the statement of Lemma 4 also hold when $n = 1, 2$. For example, we have

$$B_{1,2/5}^{-1} + 2 \cdot \left(\frac{3}{4}\right)^{1/5} = 0.179457\ldots > 0 \quad \text{and} \quad A_{2,2/5}^{-1} - 2 \cdot \left(\frac{11}{4}\right)^{1/5} = -0.0374817\ldots < 0.$$

Now, we give a result on the finiteness of the integer points of certain affine curves, which will be used later.

Lemma 5 *The affine curve* $C_m : x^p - 2^p y^2 - 2^p y + 2^{p-2} + m = 0$ *defined over* $\mathbb{Q}$ *has only finitely many integer points for each* $1 \leq m \leq 2^{p-1} - 1$.

Proof Let $1 \leq m \leq 2^{p-1} - 1$ be fixed. By completing the square and rearranging, the defining equation of C_m can be written as

$$\left(y + \frac{1}{2}\right)^2 = \frac{1}{2^p} \cdot x^p + \frac{2^{p-1} + m}{2^p}. \tag{4}$$

By letting $Y = y + \frac{1}{2}$, Eq. (4) becomes

$$Y^2 = \frac{1}{2^p} \cdot x^p + \frac{2^{p-1} + m}{2^p}. \tag{5}$$

Let $S = \{2, \infty\}$ be a finite set of places of $\mathbb{Q}$, and let $f(x) = \frac{1}{2^p} \cdot x^p + \frac{2^{p-1}+m}{2^p}$. Then the equation $Y^2 = f(x)$ has only finitely many solutions in $R_S = \mathbb{Z}[\frac{1}{2}]$ by Theorem 5, which in turn, implies that $C_m(\mathbb{Z})$ is also finite. (Note that if (a, b) is an integer point of C_m, then $(a, b + \frac{1}{2})$ is a solution of Eq. (5) in $R_S = \mathbb{Z}[\frac{1}{2}]$.) Since m was arbitrary, the proof is complete. □

In the sequel, let C_m denote the affine curve defined as in Lemma 5 for $1 \le m \le 2^{p-1} - 1$. Using the above finiteness result on integral points, we have the following.

Lemma 6 *There exists an integer $n_0 > 0$ with the property that there is no integer between $2(n^2 - n + \frac{1}{4})^{\frac{s}{2}}$ and $2(n^2 - n + \frac{3}{4})^{\frac{s}{2}}$ for every integer $n \ge n_0$.*

Proof Let S be the set of integers n such that there is an integer between $2(n^2 - n + \frac{1}{4})^{\frac{s}{2}}$ and $2(n^2 - n + \frac{3}{4})^{\frac{s}{2}}$. Let $n \in S$ and suppose that a is an integer with $2(n^2 - n + \frac{1}{4})^{\frac{s}{2}} < a < 2(n^2 - n + \frac{3}{4})^{\frac{s}{2}}$. (Note that, in view of [4, page 9], there is exactly one such a.) It follows that

$$2^p n^2 - 2^p n + 2^{p-2} < a^p < 2^p n^2 - 2^p n + 3 \cdot 2^{p-2}.$$

We may write $a^p = 2^p n^2 - 2^p n + 2^{p-2} + l$ for some $1 \le l \le 2^{p-1} - 1$ so that we have $(a, n) \in C_l(\mathbb{Z})$. (Note also that the uniqueness of a guarantees the uniqueness of such l.) This observation gives rise to a well-defined set map $\varphi : S \to \bigcup_{m=1}^{2^{p-1}-1} C_m(\mathbb{Z})$ given by $\varphi(n) = (a, n)$, where a is uniquely determined as above. It is easy to see that φ is injective, by construction. Now, by Lemma 5, the set $\bigcup_{m=1}^{2^{p-1}-1} C_m(\mathbb{Z})$ is finite, and hence, it follows that S is also finite. Let $n_0 = \max S + 1$. Then, for any integer $n \ge n_0$, we have $n \notin S$, which is the desired result.

This completes the proof. □

By a similar argument, we can also have the following.

Lemma 7 *There exists an integer $n_1 > 0$ with the property that there is no integer between $-2(n^2 - n + \frac{3}{4})^{\frac{s}{2}}$ and $-2(n^2 - n + \frac{1}{4})^{\frac{s}{2}}$ for every integer $n \ge n_1$.*

Using Lemmas 4, 6, and 7, we can prove the following.

Theorem 6 *There exist integers $n_0, n_1 > 0$ such that*

$$[A_{n,s}^{-1}] = \left[2\left(n - \frac{1}{2}\right)^s\right]$$

for every even integer $n \ge n_0$, and

$$[B_{n,s}^{-1}] = \left[-2\left(n - \frac{1}{2}\right)^s\right]$$

for every odd integer $n \ge n_1$.

Proof By Lemmas 4 and 6, there exists an integer $n_0 > 0$ such that

$$2\left(n - \frac{1}{2}\right)^s < A_{n,s}^{-1} < 2\left(n^2 - n + \frac{3}{4}\right)^{\frac{s}{2}}$$

and there is no integer between $2(n-\frac{1}{2})^s$ and $2(n^2-n+\frac{3}{4})^{\frac{s}{2}}$ for every even integer $n \geq n_0$. Then it follows that $[A_{n,s}^{-1}] = [2(n-\frac{1}{2})^s]$ for every even integer $n \geq n_0$. Similarly, by Lemmas 4 and 7, there exists an integer $n_1 > 0$ such that

$$-2\left(n^2-n+\frac{3}{4}\right)^{\frac{s}{2}} < B_{n,s}^{-1} < -2\left(n-\frac{1}{2}\right)^s$$

and there is no integer between $-2(n^2-n+\frac{3}{4})^{\frac{s}{2}}$ and $-2(n-\frac{1}{2})^s$ for every odd integer $n \geq n_1$. Then it follows that $[B_{n,s}^{-1}] = [-2(n-\frac{1}{2})^s]$ for every odd integer $n \geq n_1$.

This completes the proof. □

As an immediate consequence of Theorem 6, we have the following.

Corollary 5 *There exists an integer $N > 0$ such that we have*

$$\left[\frac{1}{1-2^{1-s}} \cdot \zeta_n(s)^{-1}\right] = \left[(-1)^{n+1} \cdot 2\left(n-\frac{1}{2}\right)^s\right]$$

for every integer $n \geq N$.

Proof This follows from Eq. (1) and Theorem 6. □

Remark 3 If $p = 3$, then we may take $N = 1$ in view of [5, Theorem 3.12].

We conclude this paper with the following.

Remark 4 In light of [2, Remark D.9.5], it might be possible to find such a suitable $N > 0$ in Corollaries 4 and 5 by adopting a theorem of Baker (see [1]).

Acknowledgements
The authors would like to thank the editors and anonymous referees who reviewed the manuscript. Kyunghwan Song was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (Grant No. 2019R1A6A1A11051177).

Funding
We do not receive any funding.

Availability of data and materials
We declare that the materials described in the manuscript, including all relevant raw data, will be freely available to any scientist wishing to use them for non-commercial purposes, without breaching participant confidentiality.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors equally contributed to the manuscript. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics, Korea Institute for Advanced Study, Seoul, Republic of Korea. [2]Institute of Mathematical Sciences, Ewha Womans University, Seoul, Republic of Korea.

Publisher's Note

Received: 18 June 2019 Accepted: 17 October 2019 Published online: 23 October 2019

References

1. Baker, A.: Transcendental Number Theory. Cambridge University Press, Cambridge (1975)
2. Hindry, M., Silverman, J.: Diophantine Geometry: An Introduction. Springer, Berlin (2000)
3. Hwang, W., Song, K.: A reciprocal sum related to the Riemann zeta function at $s = 6$. arXiv:1709.07994
4. Kim, D., Song, K.: The inverses of tails of the Riemann zeta function. J. Inequal. Appl. **2018**(1), 157 (2018)
5. Song, K.: The inverses of tails of the Riemann zeta function for some real and natural numbers. Ph.D. thesis (2019)
6. Xin, L.: Some identities related to Riemann zeta-function. J. Inequal. Appl. **2016**(1), 32 (2016)
7. Xin, L., Xiaoxue, L.: A reciprocal sum related to the Riemann ζ-function. J. Math. Inequal. **11**(1), 209–215 (2017)
8. Xu, H.: Some computational formulas related to the Riemann zeta-function tails. J. Inequal. Appl. **2016**(1), 132 (2016)