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The John–Nirenberg inequality in ball Banach function spaces and application to characterization of BMO

Mitsuo Izuki[1]*, Takahiro Noi[2] and Yoshihiro Sawano[2]

*Correspondence: izuki@tcu.ac.jp
[1]Faculty of Liberal Arts and Sciences, Tokyo City University, Tokyo, Japan
Full list of author information is available at the end of the article

Abstract

Our goal is to obtain the John–Nirenberg inequality for ball Banach function spaces X, provided that the Hardy–Littlewood maximal operator M is bounded on the associate space X' by using the extrapolation. As an application we characterize BMO, the bounded mean oscillation, via the norm of X.

MSC: Extrapolation; Ball Banach function space; BMO; John–Nirenberg inequality

Keywords: 42B25; 42B35

1 Introduction

The classical BMO semi-norm $\|\cdot\|_{\mathrm{BMO}}$ is defined by

$$\|b\|_{\mathrm{BMO}} := \sup_{Q:\mathrm{cube}} \frac{1}{|Q|} \int_Q \bigl|b(y) - m_Q(b)\bigr|\,dy = \sup_{Q:\mathrm{cube}} m_Q\bigl(\bigl|b - m_Q(b)\bigr|\bigr)$$

for $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Here and below $m_Q(f)$ denotes the average of the locally integrable function f over a cube Q. We follow the standard convention of the usage of the word "cube": By a cube we mean a compact cube whose edges are parallel to the coordinate axes. The BMO space consists of all locally integrable functions b such that $\|b\|_{\mathrm{BMO}} < \infty$. Due to the John–Nirenberg inequality and the L^∞–BMO boundedness of singular integral operators, the BMO space is one of the important function spaces in real analysis. For example, equivalent expressions of the BMO norm $\|\cdot\|_{\mathrm{BMO}}$ are necessary in order to prove boundedness of commutators involving BMO functions on various function spaces.

Given a constant $1 \le p < \infty$, we define

$$\|b\|_{\mathrm{BMO}_{L^p}} := \sup_{Q:\mathrm{cube}} \frac{1}{\|\chi_Q\|_{L^p}} \bigl\|\bigl(b - m_Q(b)\bigr)\chi_Q\bigr\|_{L^p}.$$

It is known that the value $\|b\|_{\mathrm{BMO}_{L^p}}$ is a semi-norm equivalent to $\|b\|_{\mathrm{BMO}}$. The estimate $\|b\|_{\mathrm{BMO}} \le \|b\|_{\mathrm{BMO}_{L^p}}$ is easily obtained by the usual Hölder inequality. On the other hand, the opposite estimate $C\|b\|_{\mathrm{BMO}_{L^p}} \le \|b\|_{\mathrm{BMO}}$ is not obvious. The following is a famous result named the John–Nirenberg inequality [22] which proves the estimate.

Theorem 1.1 *There exist $c_1, c_2 > 0$ such that for all $\lambda > 0$, cubes Q, and $b \in \mathrm{BMO}$,*

$$\left|\left\{x \in Q : \left|b(x) - m_Q(b)\right| > \lambda\right\}\right| \le c_1|Q| \exp\left(-\frac{c_2\lambda}{\|b\|_{\mathrm{BMO}}}\right).$$

We next consider a further generalization of $\|b\|_{\mathrm{BMO}_{L^p}}$ in terms of variable exponent. Replacing the constant p by a measurable function $p(\cdot)$ we define

$$\|b\|_{\mathrm{BMO}_{L^{p(\cdot)}}} := \sup_Q \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}} \left\|\left(b - m_Q(b)\right)\chi_Q\right\|_{L^{p(\cdot)}}.$$

We recall the definition of $\|\cdot\|_{L^{p(\cdot)}}$ later, although it is now well known. The authors have considered the equivalence between $\|\cdot\|_{\mathrm{BMO}}$ and $\|\cdot\|_{\mathrm{BMO}_{L^{p(\cdot)}}}$ and obtained some results:

1. (Izuki [16]) If $p(\cdot) \in \mathcal{P} \cap \mathcal{B}$, then $\|b\|_{\mathrm{BMO}_{L^{p(\cdot)}}}$ and $\|b\|_{\mathrm{BMO}}$ are equivalent.
2. (Izuki–Sawano [19]) If $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ satisfies $p_- = 1$, $p_+ < \infty$ and $p(\cdot) \in LH$, then $\|b\|_{\mathrm{BMO}_{L^{p(\cdot)}}}$ and $\|b\|_{\mathrm{BMO}}$ are equivalent.
3. (Izuki–Sawano–Tsutsui [21]) If $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ satisfies $p_+ < \infty$ and the Hardy–Littlewood maximal operator M is of weak type $(p(\cdot), p(\cdot))$, then $\|b\|_{\mathrm{BMO}_{L^{p(\cdot)}}}$ and $\|b\|_{\mathrm{BMO}}$ are equivalent.

The precise definition of the operator M and the classes $\mathcal{P}$, $\mathcal{B}$, and LH including variable exponent Lebesgue spaces are found in the next section. We note that the result due to Izuki–Sawano–Tsutsui [21] is not included in Theorem 1.2 below.

Finally, we consider the replacement of not only the exponent but also the norm of L^p. Ho [13] has obtained the following result as a byproduct of atomic decomposition via Banach function spaces.

Theorem 1.2 *Suppose that we are given a Banach function space X such that the Hardy–Littlewood maximal operator M is bounded on X'. We define*

$$\|b\|_{\mathrm{BMO}_X} := \sup_Q \frac{1}{\|\chi_Q\|_X} \left\|\left(b - m_Q(b)\right)\chi_Q\right\|_X$$

for $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the norms $\|b\|_{\mathrm{BMO}_X}$ and $\|b\|_{\mathrm{BMO}}$ are equivalent. That is, for some constant $C \ge 1$, we have

$$C^{-1}\|b\|_{\mathrm{BMO}_X} \le \|b\|_{\mathrm{BMO}} \le C\|b\|_{\mathrm{BMO}_X}$$

for any $b \in \mathrm{BMO}$.

The first author [17] has given another simple proof of the theorem by virtue of the Rubio de Francia algorithm [4, 25–27]. The proof in [17] is applicable to the case that X is a ball Banach function space and to characterization of Campanato spaces [20]. In particular, Theorem 1.2 is true for the ball Banach function spaces.

On the other hand, Ho [14] has proved a generalization of the John–Nirenberg inequality to the case of variable exponent.

Theorem 1.3 *Suppose that* $p(\cdot):\mathbb{R}^n \to [1,\infty]$ *satisfies* $p_+ < \infty$ *and* $p(\cdot) \in LH$. *Then there exist* $c_1, c_2 > 0$ *such that for all* $\lambda > 0$, *cubes* Q *and* $b \in \mathrm{BMO}$,

$$\|\chi_{\{x\in Q:|b(x)-m_Q(b)|>\lambda\}}\|_{L^{p(\cdot)}} \le c_1 \|\chi_Q\|_{L^{p(\cdot)}} \exp\left(-\frac{c_2\lambda}{\|b\|_{\mathrm{BMO}}}\right).$$

Our first aim in this paper is to obtain the John–Nirenberg inequality in ball Banach function spaces via an extrapolation theorem. Applying the inequality and the extrapolation again, we will give another proof of Theorem 1.2 in the setting of ball Banach function spaces.

In this paper we use the following notation:

1. Let $E \subset \mathbb{R}^n$ be a measurable set. The symbol $|E|$ denotes the Lebesgue measure and χ_E means the characteristic function.
2. Given a measurable set E such that $|E| > 0$, a measurable function f and a positive constant q, we define

   $$m_E^{(q)}(f) := \left(\frac{1}{|Q|}\int_Q |f(x)|^q\,dx\right)^{1/q}$$

   and $m_E(f) := m_E^{(1)}(f)$.
3. Let w be a locally integrable and positive function defined on $\mathbb{R}^n$. The usual weighted L^1 norm is defined by

   $$\|f\|_{L^1(w)} := \int_{\mathbb{R}^n} |f(x)| w(x)\,dx.$$

   In particular, for a measurable set E, we write

   $$w(E) := \|\chi_E\|_{L^1(w)} = \|w\chi_E\|_{L^1} = \int_E w(x)\,dx.$$
4. The symbol C always denotes a positive constant independent of the main parameters.

2 Preliminaries

2.1 The Muckenhoupt A_p weights

In this subsection we recall the definition of the Muckenhoupt A_p weights and state some fundamental results. For further information on the weights, we refer to [9, 10, 24, 30].

Definition 2.1 Given a locally integrable function f, we define the operator M by

$$Mf(x) := \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|\,dy \quad (x \in \mathbb{R}),$$

where the supremum is taken over all cubes Q containing x. The operator M is said to be the Hardy–Littlewood maximal operator.

Definition 2.2 A weight w is a locally integrable and positive function defined on $\mathbb{R}^n$. Furthermore, a weight w is said to be an A_1 weight if

$$Mw(x) \le Cw(x) \quad \left(x \in \mathbb{R}^n\right)$$

holds. On the other hand, let $1 < p < \infty$ be a constant. A weight w is said to be an A_p weight if w satisfies

$$\sup_Q \frac{1}{|Q|} \| w^{1/p} \chi_Q \|_{L^p} \| w^{-1/p} \chi_Q \|_{L^{p'}} < \infty,$$

where p' is the conjugate exponent of p, namely $1/p + 1/p' = 1$ holds. We denote the set of all A_p weights by A_p for every $1 \le p < \infty$.

Remark 2.3 We can rephrase the definition of A_p without using the Hardy–Littlewood maximal operator M as follows. A weight w is an A_1 weight if and only if

$$[w]_{A_1} := \sup_B \left\{ \frac{1}{|B|} \int_B w(x)\,dx \cdot \| w^{-1} \|_{L^\infty(B)} \right\}$$

is finite. The value $[w]_{A_1}$ is said to be an A_1 constant of w. On the other hand, if $1 < p < \infty$, then the following value

$$[w]_{A_p} := \sup_B \left(\frac{1}{|B|} \| w^{1/p} \chi_B \|_{L^p} \| w^{-1/p} \chi_B \|_{L^{p'}} \right)^p$$

is called an A_p constant of w.

By the Hölder inequality, the Muckenhoupt class is nested; $A_p \subset A_q$ for $1 \le p \le q < \infty$. In view of the relation, we can define the class A_∞ as follows:

Definition 2.4 We define $A_\infty := \bigcup_{1<p<\infty} A_p$, and an A_∞ weight is a weight in the class A_∞.

There are several known definitions equivalent to above; see [11], for example.

Theorem 2.5 *Let w be a weight. Then the following three conditions are equivalent*:

1. $w \in A_\infty$.
2. *There exist two constants $\delta, C > 0$ such that for all cubes Q and $S \subset Q$,*

$$\frac{w(S)}{w(Q)} \le C \left(\frac{|S|}{|Q|} \right)^\delta.$$

3. *The following value, called the A_∞ constant, is finite*:

$$[w]_{A_\infty} := \sup_Q m_Q(w) \exp\bigl(m_Q(\log w^{-1}) \bigr).$$

2.2 Lebesgue spaces with variable exponent

In this subsection we define Lebesgue spaces with variable exponent and some classes of variable exponents.

Definition 2.6 Let $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ be a measurable function. The Lebesgue space $L^{p(\cdot)} = L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ consists of all functions f satisfying $\rho_p(f/\lambda) < \infty$ for some $\lambda > 0$, where

$$\rho_p(f) := \int_{\{p(x)<\infty\}} |f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\{p(x)=\infty\})}. \tag{2.1}$$

Additionally, we can give the norm of $L^{p(\cdot)}$ by

$$\|f\|_{L^{p(\cdot)}} := \inf\bigl\{\lambda > 0 : \rho_p(f/\lambda) \le 1\bigr\}. \tag{2.2}$$

In the statement of variable exponent analysis, we use the following notations.

Definition 2.7

1. Given a measurable function $p(\cdot) : \mathbb{R}^n \to [1,\infty]$, we denote the conjugate exponent by $p'(\cdot)$, namely $1/p(\cdot) + 1/p'(\cdot) \equiv 1$ holds. In addition, we define
$$p_+ := \operatorname*{ess.sup}_{x\in\mathbb{R}^n} p(x), \qquad p_- := \operatorname*{ess.inf}_{x\in\mathbb{R}^n} p(x)$$
2. The set $\mathcal{P}$ consists of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ satisfying $1 < p_- \le p_+ < \infty$.
3. The set LH consists of all measurable functions $r(\cdot) : \mathbb{R}^n \to (0,\infty)$ satisfying
$$\bigl|r(x) - r(y)\bigr| \le \frac{C}{-\log(|x-y|)} \quad \bigl(|x-y| \le 1/2\bigr)$$
and
$$\bigl|r(x) - r_\infty\bigr| \le \frac{C}{\log(e + |x|)} \quad \bigl(x \in \mathbb{R}^n\bigr)$$
for some real constant r_∞.
4. The set $\mathcal{B}$ consists of all $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ such that M is bounded on $L^{p(\cdot)}$.

The class LH is established by Cruz-Uribe–Fiorenza–Neugebauer [2, 3] and Diening [6]. Some conditions equivalent to $p(\cdot) \in \mathcal{B}$ are obtained by Diening [7]. For further information, including many properties of function spaces with variable exponent or recent development of the theory of variable exponent analysis, we refer to [5, 8, 18].

2.3 Ball Banach function spaces

Below $\mathcal{M}$ denotes the set of all complex-valued measurable functions defined on $\mathbb{R}^n$. Based on Bennet–Sharpley [1], we define Banach function spaces.

Definition 2.8 Let X be a linear subspace of $\mathcal{M}$. The space X is said to be a Banach function space if there exists a functional $\|\cdot\|_X : X \to [0,\infty)$ satisfying the following conditions for all $f, g, f_k \in \mathcal{M}$ $(k \in \mathbb{N})$:

- (P1) (Norm property)
  - (P1-1) $\|f\|_X = 0$ holds if and only if $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.
  - (P1-2) $\|\lambda f\|_X = |\lambda|\|f\|_X$ for all $\lambda \in \mathbb{C}$.
  - (P1-3) $\|f + g\|_X \le \|f\|_X + \|g\|_X$.
- (P2) (Lattice property) If $0 \le g(x) \le f(x)$ holds for almost every $x \in \mathbb{R}^n$, then we have $\|g\|_X \le \|f\|_X$.
- (P3) (Fatou property) If $0 \le f_1(x) \le f_2(x) \le \cdots$ and $f_k(x) \to f(x)$ $(k \to \infty)$ hold for almost every $x \in \mathbb{R}^n$, then we have $\|f_k\|_X \to \|f\|_X$ $(k \to \infty)$.
- (P4) If a measurable set E satisfies $|E| < \infty$, then we have $\|\chi_E\|_X < \infty$.

(P5) If a measurable set E satisfies $|E| < \infty$, then $\int_E |f(x)|\,dx \le C_E \|f\|_X$ holds, where C_E is a positive constant independent of f.

We next define the associate space and give some fundamental properties.

Definition 2.9 Let X be a Banach function space. The associate space X' consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{X'} := \sup\left\{ \left| \int_{\mathbb{R}^n} f(x)g(x)\,dx \right| : \|g\|_X \le 1 \right\} < \infty.$$

The value $\|\cdot\|_{X'}$ is called the associate norm of X.

Lemma 2.10 *Let X be a Banach function space. Then the following hold*:

1. *The associate space X' is a Banach function space.*
2. (*The Lorentz–Luxemberg theorem*) $(X')' = X$ *holds, in particular, the norm* $\|\cdot\|_X$ *is equivalent to* $\|\cdot\|_{(X')'}$.
3. (*Generalized Hölder's inequality*) *We have that for all* $f \in X$ *and* $g \in X'$,

$$\int_{\mathbb{R}^n} \left|f(x)g(x)\right| dx \le \|f\|_X \|g\|_{X'}.$$

It is known that not only the usual Lebesgue spaces L^p with constant exponent $1 \le p \le \infty$ but also $L^{p(\cdot)}$ are Banach function spaces and that the associate space of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$ (see [23]). Thus we can consider some function spaces, including $L^{p(\cdot)}$, in the context of Banach function spaces. But there exist some examples which do not satisfy the definition of Banach function spaces. In order to treat them, we need a class of generalized function spaces wider than Banach function spaces. Based on Hakim–Sawano [12], we define ball Banach function spaces.

Definition 2.11 A ball Banach function space X is defined by replacing (P4), (P5) by the following conditions (P4)′ and (P5)′ respectively, in Definition 2.8:

(P4)′ For all open balls B, we have $\|\chi_B\|_X < \infty$.

(P5)′ For all open balls B, we have $\int_B |f(x)|\,dx \le C_B \|f\|_X$, where C_B is a positive constant independent of f.

The associate space of ball Banach function space can be defined by the same way of the case for Banach function spaces.

We can replace "all open balls" by "all open cubes" or "all compact sets" in (P4)′ and (P5)′. The Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 < q < p < \infty$ do not satisfy (P5) but satisfy (P5)′, that is, the spaces are not Banach function spaces but ball Banach function spaces. This fact is proved by Sawano–Tanaka [29].

We finally note that the norm $\|\cdot\|_X$ has a property similar to that of the Muckenhoupt A_p weights, provided M is bounded on X.

Lemma 2.12 (Izuki [17]) *Let X be a ball Banach function space and suppose that the Hardy–Littlewood maximal operator M is weakly bounded on X, that is,*

$\|\chi_{\{x\in\mathbb{R}^n:Mf(x)>\lambda\}}\|_X \leq C\lambda^{-1}\|f\|_X$ *holds for all* $\lambda > 0$ *and all* $f \in X$. *Then we have that for all cubes* Q,

$$\frac{1}{|Q|}\|\chi_Q\|_X\|\chi_Q\|_{X'} \leq C.$$

Applying the Hölder inequality, we can obtain that the opposite estimate, namely

$$1 \leq \frac{1}{|Q|}\|\chi_Q\|_X\|\chi_Q\|_{X'},$$

is also true.

3 Main results

3.1 The John–Nirenberg inequality

The aim of this note is to prove the following theorem which extends the well-known John–Nirenberg inequality:

Theorem 3.1 *Let* X *be a ball Banach function space such that* M *is bounded on* X' *and write* $B := \|M\|_{X'\to X'}$. *Then for all* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $k \geq 0$,

$$\|\chi_{\{x\in Q:|b(x)-m_Q(b)|>k2^{n+2}\|b\|_{\mathrm{BMO}}\}}\|_X \leq C2^{\frac{-k}{1+2^{n+4}B}}\|\chi_Q\|_X.$$

Remark 3.2 We remark that Theorem 3.1 is significant only when $k \in \mathbb{N}$. That is, if Theorem 3.1 is true for $k \in \mathbb{N}$, then the theorem is valid for general $k \geq 0$.

In fact, for $k \geq 0$ consider the decomposition $k = [k] + (k - [k])$. Then once we show Theorem 3.1 for $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\|\chi_{\{x\in Q:|b(x)-m_Q(b)|>k2^{n+2}\|b\|_{\mathrm{BMO}}\}}\|_X \\
&\quad\leq \|\chi_{\{x\in Q:|b(x)-m_Q(b)|>[k]2^{n+2}\|b\|_{\mathrm{BMO}}\}}\|_X \\
&\quad\leq C2^{\frac{-[k]}{1+2^{n+4}B}}\|\chi_Q\|_X \\
&\quad\leq C2^{\frac{-k}{1+2^{n+4}B}}\|\chi_Q\|_X
\end{aligned}$$

holds. Furthermore, if $k = 0$, then the result is clear. So, one may assume $k \in \mathbb{N}$.

3.2 An extrapolation theorem

The proof of Theorem 3.1 is given by the extrapolation result in [4, Theorem 4.6]. We reexamine the proof of [4, Theorem 4.6] to show the following extrapolation result:

Theorem 3.3 *Let* X *be a ball Banach function space such that* M *is bounded on* X' *and write* $B := \|M\|_{X'\to X'}$. *Define* $\mathfrak{F}$ *to be the set of all pairs* (f,g) *of non-negative measurable functions. Suppose that for every* $w \in A_1$ *satisfying* $[w]_{A_1} \leq 2B$ *the inequality*

$$\|f\|_{L^1(w)} \leq \|g\|_{L^1(w)}$$

holds for all $(f,g) \in \mathfrak{F}$ *such that* $\|f\|_{L^1(w)} < \infty$. *Then we have*

$$\|f\|_X \leq 2\|g\|_X$$

for all $(f,g) \in \mathfrak{F}$ *such that* $\|f\|_X < \infty$.

Proof We set $\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{1}{(2B)^k} M^k h(x)$, where it will be understood that M^k denotes the k-fold composition of the Hardy–Littlewood maximal operator and that $M^0 h(x) = |h(x)|$. As in [4, p. 74] or as we can check directly, we have $|h(x)| \leq \mathcal{R}h(x)$, $\|\mathcal{R}h\|_{X'} \leq 2\|h\|_{X'}$, and $[\mathcal{R}h]_{A_1} \leq 2B$. By duality, we have

$$\|f\|_X = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)h(x)\,dx\right| : \|h\|_{X'} \leq 1\right\}.$$

Fix $h \in X'$ such that $\|h\|_{X'} \leq 1$ arbitrarily. If $h = 0$, then $|\int_{\mathbb{R}^n} f(x)h(x)\,dx| \leq 2\|g\|_X$ is obvious. We consider the case $0 < \|h\|_{X'} \leq 1$. Since $[\mathcal{R}h]_{A_1} \leq 2B$, our assumption is applicable. Therefore we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)h(x)\,dx\right| &\leq \int_{\mathbb{R}^n} \bigl|f(x)h(x)\bigr|\,dx \\ &\leq \int_{\mathbb{R}^n} f(x)\mathcal{R}h(x)\,dx \\ &\leq \int_{\mathbb{R}^n} g(x)\mathcal{R}h(x)\,dx \\ &\leq \|\mathcal{R}h\|_{X'}\|g\|_X \\ &\leq 2\|g\|_X.\end{aligned}$$

□

3.3 Proof of Theorem 3.1

For the proof of Theorem 3.1 we will need two additional lemmas: In [28, p. 400], we showed the following local estimates for BMO functions.

Lemma 3.4 *For any* $k \in \mathbb{N} \cup \{0\}$, *a cube* Q *and a nonconstant* $\mathrm{BMO}(\mathbb{R}^n)$*-function* b, *we have*

$$\bigl|\bigl\{x \in Q : \bigl|b(x) - m_Q(b)\bigr| > k2^{n+2}\|b\|_{\mathrm{BMO}}\bigr\}\bigr| \leq 2^{1-k}|Q|.$$

Hytönen and Pérez proved the following quantitative estimate [15, Theorem 2.3].

Lemma 3.5 *Let* $w \in A_\infty$, *and let* $q := 1 + \frac{1}{2^{n+3}[w]_{A_\infty}}$. *Then for all cubes* Q,

$$m_Q^{(q)}(w) \leq 2m_Q(w). \tag{3.1}$$

We complete the proof of Theorem 3.1. Let $w \in A_1$, and write $\varepsilon := \frac{1}{2^{n+3}[w]_{A_\infty}} > 0$. Then we have $m_Q^{(1+\varepsilon)}(w) \leq 2m_Q(w)$ for all cubes Q. Consequently, $\frac{w(E)}{w(Q)} \leq 2(\frac{|E|}{|Q|})^{\frac{\varepsilon}{1+\varepsilon}}$. As a result, we have

$$
\begin{aligned}
w\bigl(\bigl\{x\in Q:\bigl|b(x)-m_Q(b)\bigr|>k2^{n+2}\|b\|_{\mathrm{BMO}}\bigr\}\bigr) &\le 2^{1+\frac{\varepsilon(1-k)}{1+\varepsilon}}w(Q)\\
&\le 2^{1+\frac{1-k}{1+2^{n+3}[w]_{A_1}}}w(Q).
\end{aligned}
$$

Thus, if $[w]_{A_1}\le 2B$, then we apply Theorem 3.3 to

$$
\bigl(2^{-1-\frac{1-k}{1+2^{n+4}B}}\chi_{\{x\in Q:|b(x)-m_Q(b)|>k2^{n+2}\|b\|_{\mathrm{BMO}}\}},\chi_Q\bigr)\in\mathfrak{F}
$$

and obtain

$$
\|\chi_{\{x\in Q:|b(x)-m_Q(b)|>k2^{n+2}\|b\|_{\mathrm{BMO}}\}}\|_X\le 2^{2+\frac{1-k}{1+2^{n+3}B}}\|\chi_Q\|_X. \tag{3.2}
$$

3.4 Another proof of Theorem 1.2

Applying Theorem 3.1 (the John–Nirenberg inequality) and Theorem 3.3 (the extrapolation), we can give another proof of Theorem 1.2. We note that we do not use the Rubio de Francia algorithm directly. In this paper we have used the algorithm only to get the extrapolation.

Take $b\in\mathrm{BMO}$ and a cube Q arbitrarily. The estimate $\|b\|_{\mathrm{BMO}}\le C\|b\|_{\mathrm{BMO}_X}$ is easily obtained by Lemmas 2.10 and 2.12. We next prove the opposite inequality. We remark that the norm of the associated space of $L^1(w)$ satisfies

$$
\|f\|_{L^1(w)'}=\bigl\|w^{-1}f\bigr\|_{L^\infty}.
$$

We observe that if $w\in A_1$, then M is bounded on this associate space, that is,

$$
\bigl\|w^{-1}M(fw)\bigr\|_{L^\infty}\le[w]_{A_1}\|f\|_{L^\infty}.
$$

Thus, we are in the position of applying Theorem 3.1 to $X=L^1(w)$ to have

$$
\|\chi_{\{x\in Q:|b(x)-m_Q(b)|>k2^{n+2}\|b\|_{\mathrm{BMO}}\}}\|_{L^1(w)}\le 2^{3+\frac{-k}{1+2^{n+4}[w]_{A_1}}}\|\chi_Q\|_{L^1(w)}
$$

for all $k>0$. Here we have used the precise estimate (3.2) and Remark 3.2 below Theorem 3.1. If we integrate this inequality with respect to $k>0$, then we have

$$
\begin{aligned}
\int_Q\frac{|b(x)-m_Q(b)|}{2^{n+2}\|b\|_{\mathrm{BMO}}}w(x)\,dx &=\biggl\|\int_0^\infty\chi_{\{x\in Q:|b(x)-m_Q(b)|>k2^{n+2}\|b\|_{\mathrm{BMO}}\}}\,dk\biggr\|_{L^1(w)}\\
&=\int_0^\infty\bigl(\|\chi_{\{x\in Q:|b(x)-m_Q(b)|>k2^{n+2}\|b\|_{\mathrm{BMO}}\}}\|_{L^1(w)}\bigr)\,dk\\
&\le\biggl(\int_0^\infty 2^{3+\frac{-k}{1+2^{n+4}[w]_{A_1}}}\,dk\biggr)\|\chi_Q\|_{L^1(w)}\\
&=\frac{8+2^{n+7}[w]_{A_1}}{\log 2}\|\chi_Q\|_{L^1(w)}.
\end{aligned}
$$

Consequently,

$$\int_Q \left|b(x) - m_Q(b)\right| w(x)\,dx \le 2^{2n+11}[w]_{A_1}\|b\|_{\mathrm{BMO}} \int_Q w(x)\,dx.$$

If we use Theorem 3.3, then we have

$$\|b\|_{\mathrm{BMO}_X} \le C\|b\|_{\mathrm{BMO}}.$$

Remark 3.6 In [17, 20] the authors have applied the Rubio de Francia algorithm to get the estimate $\|b\|_{\mathrm{BMO}_X} \le C\|b\|_{\mathrm{BMO}_{L^q}}$ for some $1 < q < \infty$. On the other hand, the proof above has directly yields the estimate $\|b\|_{\mathrm{BMO}_X} \le C\|b\|_{\mathrm{BMO}}$.

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Author details
[1]Faculty of Liberal Arts and Sciences, Tokyo City University, Tokyo, Japan. [2]Department of Mathematics and Information Science, Tokyo Metropolitan University, Tokyo, Japan.

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