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Convergence rate analysis of an iterative algorithm for solving the multiple-sets split equality problem

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Abstract
This paper considers an iterative algorithm of solving the multiple-sets split equality problem (MSSEP) whose step size is independent of the norm of the related operators, and investigates its sublinear and linear convergence rate. In particular, we present a notion of bounded Hölder regularity property for the MSSEP, which is a generalization of the well-known concept of bounded linear regularity property, and give several sufficient conditions to ensure it. Then we use this property to conclude the sublinear and linear convergence rate of the algorithm. In the end, some numerical experiments are provided to verify the validity of our consequences.

Keywords: Convergent rate; Bounded Hölder regularity; Multiple-sets split equality problem

1 Introduction

Set H_1, H_2 and H_3 be three real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be two closed, convex and nonempty sets. And set two operators $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded and linear. Moudafi [1] proposed the split equality problem (SEP) for the first time, which can be formulated as

$$\text{finding } x \in C \text{ and } y \in Q \quad \text{such that } Ax = By. \tag{1.1}$$

This kind of problem attracted many authors' attention because of its widespread applications in many areas of applied mathematics such as intensity-modulated radiation therapy and decomposition methods for partial differential equations. In order to solve the split equality problem, various algorithms were introduced. One of the most significant algorithms is the alternating CQ-algorithm (ACQA), and it was presented by Moudafi [1]. The iterative form of the ACQA is

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases}$$

He also proved that this algorithm converges weakly to a solution of the SEP (1.1).

The ACQA is related to P_C and P_Q. If P_C or P_Q does not have an analytical expression, it might be difficult to implement. Then Moudafi [2] presented the relaxed alternating CQ-algorithm (RACQA) to solve this problem:

$$\begin{cases} x_{k+1} = P_{C_k}(x_k - \gamma A^*(Ax_k - By_k)), \\ y_{k+1} = P_{Q_k}(y_k + \beta B^*(Ax_{k+1} - By_k)). \end{cases}$$

The above algorithm also converges weakly to a solution of the SEP (1.1). Afterwards, for getting a strong convergence result, Shi et al. [3] proposed the following algorithm:

$$\begin{cases} x_{k+1} = P_C[(1-\alpha_k)(x_k - \gamma A^*(Ax_k - By_k))], \\ y_{k+1} = P_Q[(1-\alpha_k)(y_k + \gamma B^*(Ax_k - By_k))]. \end{cases}$$

For more information with respect to the algorithms of solving the split equality problem; see [4, 5] and the references therein. But all these papers did not consider the convergence rate of the algorithms.

In this paper, we think about the multiple-sets split equality problem (MSSEP), which generalizes the split equality problem. It can be characterized mathematically as

$$\text{finding } x \in \bigcap_{i=1}^{t} C_i \quad \text{and} \quad y \in \bigcap_{j=1}^{r} Q_j \quad \text{such that } Ax = By, \tag{1.2}$$

where r and t are two positive integers, $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ are closed, convex and nonempty sets in Hilbert spaces H_1 and H_2, respectively, H_3 is also a Hilbert space, and two operators $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded and linear. Obviously, when $t = r = 1$, the MSSEP (1.2) becomes the SEP (1.1). Without loss of generality, set $t > r$ and take $Q_{r+1} = Q_{r+2} = \cdots = Q_t = H_2$. Set $S_i = C_i \times Q_i \subseteq H = H_1 \times H_2$, $i = 1, 2, \ldots, t$, $S = \bigcap_{i=1}^t S_i$, $G = [A, -B] : H \to H_3$ and G^* be the adjoint operator of G. Then the MSSEP (1.2) can be restated as

$$\text{finding } w = (x, y) \in S \quad \text{such that } Gw = 0. \tag{1.3}$$

To solve the multiple-sets split equality problem, Tian et al. [6] gave the following algorithm and obtained a weak convergence result:

$$\begin{cases} x_{k+1} = x_k + \frac{\rho_{1,k}\sum_{i=1}^t \alpha_i \|P_{C_i}x_k - x_k\|^2}{\|\sum_{i=1}^t \alpha_i (P_{C_i}x_k - x_k)\|^2} \sum_{i=1}^t \alpha_i (P_{C_i}x_k - x_k) \\ \qquad - \frac{\rho_{2,k}\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2} A^*(Ax_k - By_k), \\ y_{k+1} = y_k + \frac{\rho_{1,k}\sum_{i=1}^t \alpha_i \|P_{Q_i}y_k - y_k\|^2}{\|\sum_{i=1}^t \alpha_i (P_{Q_i}y_k - y_k)\|^2} \sum_{i=1}^t \alpha_i (P_{Q_i}y_k - y_k) \\ \qquad + \frac{\rho_{2,k}\|Ax_k - By_k\|^2}{\|B^*(Ax_k - By_k)\|^2} B^*(Ax_k - By_k). \end{cases} \tag{1.4}$$

The step size of the algorithm is split self-adaptive, namely, it does not need any information about the relevant operators, which can save much time for our calculation. The main purpose of this paper is to investigate the sublinear and linear convergence rate of algorithm (1.4).

The rest of this paper is organized as follows. In Sect. 2, we recall some definitions and lemmas which are useful for our convergence analysis later. We also introduce a concept of bounded Hölder regularity property for the MSSEP and provide some conditions to guarantee this property. In Sect. 3, under a bounded Hölder regularity assumption, we study the sublinear and linear convergence of algorithm (1.4) and conclude its convergence rate. In Sect. 4, we perform some numerical experiments and clarify the effectiveness of our results.

2 Preliminaries

Set H be a real Hilbert space which has inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For a point $w \in H$ and a set $S \subseteq H$, we denote the classical metric projection of w onto S and the distance of w from S by using $P_S(w)$ and $d_S(w)$, respectively, and they are defined by

$$P_S(w) := \arg\min\bigl\{\|w-v\| : v \in S\bigr\} \quad \text{and} \quad d_S(w) := \inf\bigl\{\|w-v\| : v \in S\bigr\}.$$

Bauschke et al. [7] listed several basic properties of the projection operator. These properties are as follows.

Lemma 2.1 ([7]) *Let S be a closed, convex and nonempty subset of H, then for any $x, y \in H$ and $z \in S$,*

(i) $\langle x - P_S x, z - P_S x\rangle \le 0$;
(ii) $\|P_S x - P_S y\|^2 \le \langle P_S x - P_S y, x - y\rangle$;
(iii) $\|P_S x - z\|^2 \le \|x - z\|^2 - \|P_S x - x\|^2$.

Set operator $G : H \to H_3$ be bounded and linear. We utilize $\ker G = \{x \in H : Gx = 0\}$ to denote the kernel of G. The orthogonal complement of $\ker G$ is represented by $(\ker G)^\perp = \{y \in H : \langle x, y\rangle = 0, \forall x \in \ker G\}$. As is well known, $\ker G$ and $(\ker G)^\perp$ are both closed subspaces of H. Throughout this paper, we denote the solution set of the MSSEP (1.3) by using Γ, which is defined by

$$\Gamma := S \cap \ker G = \{w \in S : Gw = 0\}.$$

We assume that the MSSEP is consistent, then Γ is a closed, convex and nonempty set.

Next, we shift our attention to the bounded Hölder regularity property for a collection of closed and convex subsets of a Hilbert space.

Definition 2.2 ([8]) Let $\{S_i\}_{i\in I}$ be a collection of closed convex subsets in a Hilbert space H and $S = \bigcap_{i\in I} S_i \neq \emptyset$. The collection $\{S_i\}_{i\in I}$ has a bounded Hölder regular intersection if for each bounded set K, there exist an exponent $\gamma \in (0, 1]$ and a scalar $\beta > 0$ such that

$$d_S(w) \le \beta\bigl(\max\bigl\{d_{S_i}(w) : i \in I\bigr\}\bigr)^\gamma, \quad \forall w \in K.$$

Furthermore, if the exponent γ is independent of the set K, we say the collection $\{S_i\}_{i\in I}$ is bounded Hölder regular with uniform exponent γ.

It is obvious that any collection including only a set has a bounded Hölder regular intersection whose uniform exponent γ is equal to 1. The above definition with $\gamma = 1$ is the bounded linear regularity property, which was introduced in [9]. Then we provide a significant notion of bounded Hölder regularity property for the MSSEP (1.3) on the basis of Definition 2.2.

Definition 2.3 The MSSEP is said to satisfy the bounded Hölder regularity property if for each bounded set K, there exist an exponent $\gamma \in (0,1]$ and a scalar $\beta > 0$ such that

$$d_\Gamma(w) \leq \beta\bigl(\max\bigl\{d_S(w), \|Gw\|\bigr\}\bigr)^\gamma, \quad \forall w \in K. \tag{2.1}$$

Furthermore, if the exponent γ is independent of the set K, we say the MSSEP is bounded Hölder regular with uniform exponent γ.

It is worth noting that when $\gamma = 1$, the MSSEP satisfies the bounded linear regularity property [10].

Lemma 2.4 ([11]) *Let $G : H \to H_3$ be a bounded linear operator. Then G is injective and has closed range if and only if G is bounded below, i.e., there exists a positive constant γ such that $\|Gw\| \geq \gamma\|w\|$ for all $w \in H$.*

The following lemma gives some conditions which make the bounded Hölder regularity property for the MSSEP (1.3) hold.

Lemma 2.5 *$\{S, \ker G\}$ has a bounded Hölder regular intersection and the range of G is closed, then the MSSEP (1.3) satisfies the bounded Hölder regularity property.*

Proof. $\{S, \ker G\}$ has a bounded Hölder regular intersection, so for any bounded set K, there exist an exponent $\gamma \in (0,1]$ and a scalar $\beta > 0$ such that

$$d_\Gamma(w) = d_{S\cap\ker G}(w) \leq \beta\bigl(\max\bigl\{d_S(w), d_{\ker G}(w)\bigr\}\bigr)^\gamma, \quad \forall w \in K. \tag{2.2}$$

Since G restricted to $(\ker G)^\perp$ is injective and its range is closed, by Lemma 2.4, we know that there exists $\nu > 0$ such that

$$\|Gw_1\| \geq \nu\|w_1\|, \quad \text{for all } w_1 \in (\ker G)^\perp.$$

Hence,

$$d_{\ker G}(w) \leq \frac{1}{\nu}\|Gw\|, \quad \text{for all } w \in H. \tag{2.3}$$

Combining (2.2) and (2.3), we have

$$d_\Gamma(w) \leq \beta\left(\max\left\{d_S(w), \frac{1}{\nu}\|Gw\|\right\}\right)^\gamma, \quad \forall w \in K.$$

Then the proof is split into two cases:

Case 1: when $\frac{1}{\nu} < 1$, we have

$$d_\Gamma(w) \leq \beta\big(\max\big\{d_S(w), \|Gw\|\big\}\big)^\gamma, \quad \text{for all } w \in K.$$

Case 2: when $\frac{1}{\nu} \geq 1$, we have

$$d_\Gamma(w) \leq \frac{\beta}{\nu^\gamma} \max\big\{d_S(w), \|Gw\|\big\}^\gamma, \quad \text{for all } w \in K.$$

The proof is finished.

Lemma 2.6 ([10]) *Let $\{S, \ker G\}$ be boundedly linearly regular and G has closed range. Then the MSSEP (1.3) satisfies the bounded linear regularity property.*

In order to complete the convergence rate analysis of algorithm (1.4), the following definition and lemmas are also essential tools.

Definition 2.7 ([7]) Let C be a nonempty subset of H, and $\{x_k\}$ be a sequence in H. $\{x_k\}$ is called Fejér monotone with respect to C, if

$$\|x_{k+1} - z\| \leq \|x_k - z\|, \quad \forall z \in C.$$

Clearly, a Fejér monotone sequence $\{x_k\}$ is bounded and $\lim_{k\to\infty} \|x_k - z\|$ exists.

Lemma 2.8 ([8]) *Let C be a closed, convex and nonempty set of a Hilbert space H, and s be a positive integer. Suppose that the sequence $\{w_k\}$ is Fejér monotone with respect to C and satisfies*

$$d_C^2(w_{(k+1)s}) \leq d_C^2(w_{ks}) - \delta d_C^{2\theta}(w_{ks}), \quad \forall k \in \mathbf{N},$$

for some $\delta > 0$ and $\theta \geq 1$. Then $w_k \to w^*$ for some $w^* \in C$ and there exist constants $M_1, M_2 \geq 0$ and $r \in [0,1)$ such that

$$\|w_k - w^*\| \leq \begin{cases} M_1 k^{-\frac{1}{2(\theta-1)}}, & \theta > 1, \\ M_2 r^k, & \theta = 1. \end{cases}$$

Furthermore, the constants may be chosen to be

$$\begin{cases} M_1 := 2\max\{(2s)^{\frac{1}{2(\theta-1)}}[(\theta-1)\delta]^{-\frac{1}{2(\theta-1)}}, (2s)^{\frac{1}{2(\theta-1)}} d_C(w_0)\}, \\ M_2 := 2\max\{(\sqrt[4s]{1-\delta})^{-2s} d_C(w_0), \sqrt{d_C(w_0)}\}, \\ r := \sqrt[4s]{1-\delta}, \end{cases}$$

and δ necessarily lies in $(0,1]$ whenever $\theta = 1$.

Lemma 2.9 ([12]) *Let $p > 0$ and $\{\delta_k\}_{k\in\mathbf{N}}$ and $\{\beta_k\}_{k\in\mathbf{N}}$ be two sequences of nonnegative numbers such that*

$$\beta_{k+1} \leq \beta_k\big(1 - \delta_k \beta_k^p\big), \quad \forall k \in \mathbf{N}.$$

Then

$$\beta_k \le \left(\beta_0^{-p} + p\sum_{i=0}^{k-1}\delta_i\right)^{-\frac{1}{p}}, \quad \forall k \in \mathbf{N},$$

where the convention that $\frac{1}{0} = +\infty$ *is adopted.*

Finally, we end this section by reviewing algorithm (1.4) in detail.

Algorithm 2.10 ([6]) For an arbitrary initial point $w_0 = (x_0, y_0) \in H$, the sequence $\{w_k\}$ is generated by

$$w_{k+1} = w_k + \frac{\rho_{1,k}\sum_{i=1}^t \alpha_i \|P_{S_i}w_k - w_k\|^2}{\|\sum_{i=1}^t \alpha_i (P_{S_i}w_k - w_k)\|^2}\sum_{i=1}^t \alpha_i (P_{S_i}w_k - w_k) - \frac{\rho_{2,k}\|Gw_k\|^2}{\|G^*Gw_k\|^2}G^*Gw_k,$$

or component-wise

$$\begin{cases} x_{k+1} = x_k + \frac{\rho_{1,k}\sum_{i=1}^t \alpha_i\|P_{C_i}x_k - x_k\|^2}{\|\sum_{i=1}^t \alpha_i (P_{C_i}x_k - x_k)\|^2}\sum_{i=1}^t \alpha_i (P_{C_i}x_k - x_k) \\ \qquad - \frac{\rho_{2,k}\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2}A^*(Ax_k - By_k), \\ y_{k+1} = y_k + \frac{\rho_{1,k}\sum_{i=1}^t \alpha_i\|P_{Q_i}y_k - y_k\|^2}{\|\sum_{i=1}^t \alpha_i (P_{Q_i}y_k - y_k)\|^2}\sum_{i=1}^t \alpha_i (P_{Q_i}y_k - y_k) \\ \qquad + \frac{\rho_{2,k}\|Ax_k - By_k\|^2}{\|B^*(Ax_k - By_k)\|^2}B^*(Ax_k - By_k), \end{cases}$$

where $0 < \underline{\rho}_1 \le \rho_{1,k} \le \overline{\rho}_1 < 1$, $0 < \underline{\rho}_2 \le \rho_{2,k} \le \overline{\rho}_2 < 1$ and $\{\alpha_i\}_{i=1}^t > 0$.

3 Main results

In this section, we conclude the sublinear and linear convergence rate of Algorithm 2.10 under a bounded Hölder regularity assumption. Now, we give the most important theorem in this paper and prove it.

Theorem 3.1 *The MSSEP (1.3) satisfies the bounded Hölder regularity property, the sequence* $\{w_k\}$ *is defined by Algorithm 2.10, and* $\{S_i\}_{i=1}^t$ *has a bounded Hölder regular intersection, then* $\{w_k\}$ *converges to a solution* w^* *of the MSSEP (1.3) at least with a sublinear rate* $O(k^{-\iota})$ *for some* $\iota > 0$.

In particular, if the MSSEP satisfies the bounded Hölder regularity property with uniform exponent $q \in (0,1]$ *and* $\{S_i\}_{i=1}^t$ *has a bounded Hölder regular intersection with uniform exponent* $p \in (0,1]$, *then there exist constants* $M_1, M_2, F \ge 0$ *and* $r \in [0,1)$ *such that when* $k \ge F$,

$$\|w_k - w^*\| \le \begin{cases} M_1 k^{-\frac{1}{2(\theta-1)}}, & \theta > 1, \\ M_2 r^k, & \theta = 1, \end{cases}$$

where $\theta = \frac{1}{pq}$.

Proof Set $\beta_k := \frac{\rho_{1,k}\sum_{i=1}^t \alpha_i\|P_{S_i}w_k - w_k\|^2}{\|\sum_{i=1}^t \alpha_i (P_{S_i}w_k - w_k)\|^2}$ and $\gamma_k := \frac{\rho_{2,k}\|Gw_k\|^2}{\|G^*Gw_k\|^2}$. For the first assertion, we will firstly prove that the sequence $\{w_k\}$ is Fejér monotone with respect to Γ.

Since $\Gamma \neq \emptyset$, take $\bar{w} \in \Gamma$, then $G\bar{w} = 0$, and

$$\begin{aligned}
&\|w_{k+1} - \bar{w}\|^2 \\
&\quad = \left\| w_k + \beta_k \sum_{i=1}^{t} \alpha_i (P_{S_i} w_k - w_k) - \gamma_k G^* G w_k - \bar{w} \right\|^2 \\
&\quad = \|w_k - \bar{w}\|^2 + \beta_k^2 \left\| \sum_{i=1}^{t} \alpha_i (P_{S_i} w_k - w_k) \right\|^2 + \gamma_k^2 \left\| G^* G w_k \right\|^2 \\
&\qquad - 2\left\langle \beta_k \sum_{i=1}^{t} \alpha_i (P_{S_i} w_k - w_k), \gamma_k G^* G w_k \right\rangle \\
&\qquad + 2\beta_k \left\langle w_k - \bar{w}, \sum_{i=1}^{t} \alpha_i (P_{S_i} w_k - w_k) \right\rangle - 2\gamma_k \left\langle w_k - \bar{w}, G^* G w_k \right\rangle \\
&\quad \leq \|w_k - \bar{w}\|^2 + 2\beta_k^2 \left\| \sum_{i=1}^{t} \alpha_i (P_{S_i} w_k - w_k) \right\|^2 + 2\gamma_k^2 \left\| G^* G w_k \right\|^2 \\
&\qquad + 2\beta_k \left\langle w_k - \bar{w}, \sum_{i=1}^{t} \alpha_i (P_{S_i} w_k - w_k) \right\rangle - 2\gamma_k \left\langle w_k - \bar{w}, G^* G w_k \right\rangle.
\end{aligned} \tag{3.1}$$

We get the following formulas by using the properties of the projection operator and the definition of the adjoint operator:

$$\begin{aligned}
&\left\langle w_k - \bar{w}, \sum_{i=1}^{t} \alpha_i (P_{S_i} w_k - w_k) \right\rangle \\
&\quad = \sum_{i=1}^{t} \alpha_i \langle w_k - \bar{w}, P_{S_i} w_k - w_k \rangle \\
&\quad = \sum_{i=1}^{t} \alpha_i \bigl(\langle w_k - P_{S_i} w_k, P_{S_i} w_k - w_k \rangle + \langle P_{S_i} w_k - \bar{w}, P_{S_i} w_k - w_k \rangle \bigr) \\
&\quad = \sum_{i=1}^{t} \alpha_i \bigl(-\|P_{S_i} w_k - w_k\|^2 + \langle P_{S_i} w_k - \bar{w}, P_{S_i} w_k - w_k \rangle \bigr) \\
&\quad = \sum_{i=1}^{t} \alpha_i \bigl(-\|P_{S_i} w_k - w_k\|^2 + \langle P_{S_i} w_k - \bar{w}, P_{S_i} w_k - \bar{w} \rangle \bigr) + \langle P_{S_i} w_k - \bar{w}, \bar{w} - w_k \rangle \\
&\quad \leq \sum_{i=1}^{t} \alpha_i \bigl(-\|P_{S_i} w_k - w_k\|^2 + \bigl(\|P_{S_i} w_k - \bar{w}\|^2 - \|P_{S_i} w_k - \bar{w}\|^2 \bigr) \bigr) \\
&\quad = -\sum_{i=1}^{t} \alpha_i \|P_{S_i} w_k - w_k\|^2
\end{aligned} \tag{3.2}$$

and

$$\left\langle w_k - \bar{w}, G^* G w_k \right\rangle = \langle G w_k - G\bar{w}, G w_k \rangle = \|G w_k\|^2. \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we get

$$\begin{aligned}
&\|w_{k+1}-\bar{w}\|^2 \\
&\quad\leq \|w_k-\bar{w}\|^2 + 2\beta_k^2\left\|\sum_{i=1}^{t}\alpha_i(P_{S_i}w_k-w_k)\right\|^2 + 2\gamma_k^2\left\|G^*Gw_k\right\|^2 \\
&\qquad - 2\beta_k\sum_{i=1}^{t}\alpha_i\|P_{S_i}w_k-w_k\|^2 - 2\gamma_k\|Gw_k\|^2 \\
&\quad= \|w_k-\bar{w}\|^2 - 2\beta_k\left(1-\beta_k\frac{\|\sum_{i=1}^{t}\alpha_i(P_{S_i}w_k-w_k)\|^2}{\sum_{i=1}^{t}\alpha_i\|P_{S_i}w_k-w_k\|^2}\right)\sum_{i=1}^{t}\alpha_i\|P_{S_i}w_k-w_k\|^2 \\
&\qquad - 2\gamma_k\left(1-\gamma_k\frac{\|G^*Gw_k\|^2}{\|Gw_k\|^2}\right)\|Gw_k\|^2. \qquad (3.4)
\end{aligned}$$

According to the assumptions of $\{\rho_{1,k}\}$ and $\{\rho_{2,k}\}$, it follows from (3.4) that

$$\|w_{k+1}-\bar{w}\| \leq \|w_k-\bar{w}\|.$$

That is, the sequence $\{w_k\}$ is Fejér monotone with respect to Γ. Hence, $\{w_k\}$ is bounded and $\lim_{k\to\infty}\|w_k-\bar{w}\|$ exists.

For getting a better conclusion, we need to prove that $d_\Gamma(w_k) < 1$ when k is enough large.

Assume that the following inequality with $\delta > 0$ and $\theta \geq 1$ is true:

$$d_\Gamma^2(w_{k+1}) \leq d_\Gamma^2(w_k) - \delta d_\Gamma^{2\theta}(w_k), \quad \forall k \in \mathbf{N}. \qquad (3.5)$$

And assume that $w_0 \notin \Gamma$ and set $\lambda_k := d_\Gamma^2(w_k)$ and $j := \theta - 1 \geq 0$, then the inequality (3.5) reduces to

$$\lambda_{k+1} \leq \lambda_k\bigl(1-\delta\lambda_k^j\bigr). \qquad (3.6)$$

Then the proof is split into two cases based on the value of θ:

Case 1: when $\theta > 1$, we know that $\frac{1}{\theta-1} > 0$ and by Lemma 2.9, we have

$$\lambda_k \leq \bigl(\lambda_0^{-j} + (\theta-1)\delta k\bigr)^{-\frac{1}{\theta-1}} \leq \bigl((\theta-1)\delta k\bigr)^{-\frac{1}{\theta-1}}, \quad \forall k \in \mathbf{N},$$

that is,

$$d_\Gamma(w_k) \leq \bigl((\theta-1)\delta k\bigr)^{-\frac{1}{2(\theta-1)}}.$$

So we can find a positive integer T_1 such that $d_\Gamma(w_k) < 1$ when $k \geq T_1$.

Case 2: when $\theta = 1$, by (3.6), we have

$$\lambda_{k+1} \leq \lambda_k(1-\delta),$$

where $\delta \in (0,1]$. Then

$$d_\Gamma(w_k) = \sqrt{\lambda_k} \leq \sqrt{\lambda_0}(\sqrt{1-\delta})^k.$$

So we can find a positive integer T_2 such that $d_\Gamma(w_k) < 1$ when $k \geq T_2$.

Set $T := \max\{T_1, T_2\}$, we have $d_\Gamma(w_k) < 1$ when $k \geq T$.

Next, we will prove that the sequence $\{w_k\}$ satisfies the inequality (3.5) for some $\delta > 0$ and $\theta \geq 1$.

Since $\bar{w}$ is arbitrary in Γ, we have

$$\begin{aligned} &d_\Gamma^2(w_{k+1}) \\ &\quad \leq d_\Gamma^2(w_k) - 2\beta_k\left(1 - \beta_k \frac{\|\sum_{i=1}^t \alpha_i(P_{S_i}w_k - w_k)\|^2}{\sum_{i=1}^t \alpha_i\|P_{S_i}w_k - w_k\|^2}\right)\sum_{i=1}^t \alpha_i d_{S_i}^2(w_k) \\ &\qquad - 2\gamma_k\left(1 - \gamma_k \frac{\|G^*Gw_k\|^2}{\|Gw_k\|^2}\right)\|Gw_k\|^2. \end{aligned} \tag{3.7}$$

On the one hand, by the assumptions of $\{\rho_{1,k}\}$ and $\{\rho_{2,k}\}$, we get

$$\lim_{k\to\infty} \inf\left[\beta_k\left(1 - \beta_k \frac{\|\sum_{i=1}^t \alpha_i(P_{S_i}w_k - w_k)\|^2}{\sum_{i=1}^t \alpha_i\|P_{S_i}w_k - w_k\|^2}\right)\right] > 0,$$

and

$$\lim_{k\to\infty} \inf\left[\gamma_k\left(1 - \gamma_k \frac{\|G^*Gw_k\|^2}{\|Gw_k\|^2}\right)\right] > 0.$$

Hence, we can find two positive integers N and M such that

$$a_1 := \inf_{k\geq N}\left[\beta_k\left(1 - \beta_k \frac{\|\sum_{i=1}^t \alpha_i(P_{S_i}w_k - w_k)\|^2}{\sum_{i=1}^t \alpha_i\|P_{S_i}w_k - w_k\|^2}\right)\right] > 0$$

and

$$a_2 := \inf_{k\geq M}\left[\gamma_k\left(1 - \gamma_k \frac{\|G^*Gw_k\|^2}{\|Gw_k\|^2}\right)\right] > 0.$$

Set $L := \max\{N, M\}$, then the inequality (3.7) reduces to

$$d_\Gamma^2(w_{k+1}) \leq d_\Gamma^2(w_k) - 2a_1\sum_{i=1}^t \alpha_i d_{S_i}^2(w_k) - 2a_2\|Gw_k\|^2, \quad \text{for all } k \geq L. \tag{3.8}$$

On the other hand, set K be a bounded set such that $\{w_k : k \in \mathbf{N}\} \subseteq K$, since $\{S_i\}_{i=1}^t$ has a bounded Hölder regular intersection, there exist an exponent $p \in (0, 1]$ and a scalar $\mu > 0$ such that

$$d_S(w_k) \leq \mu\bigl(\max\bigl\{d_{S_i}(w_k), i = 1, 2, \ldots, t\bigr\}\bigr)^p, \quad \forall w_k \in K,$$

that is,

$$\left(\frac{1}{\mu} d_S(w_k)\right)^{\frac{1}{p}} \leq \max\bigl\{d_{S_i}(w_k), i = 1, 2, \ldots, t\bigr\}, \quad \forall w_k \in K. \tag{3.9}$$

And since the MSSEP satisfies the bounded Hölder regularity property, there exist an exponent $q \in (0,1]$ and a scalar $\nu > 0$ such that

$$d_\Gamma(w_k) \le \nu\bigl(\max\{d_S(w_k), \|Gw_k\|\}\bigr)^q, \quad \forall w_k \in K,$$

that is,

$$\left(\frac{1}{\nu}d_\Gamma(w_k)\right)^{\frac{1}{q}} \le \max\{d_S(w_k), \|Gw_k\|\}, \quad \forall w_k \in K. \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we get

$$\begin{aligned} d_\Gamma^2(w_{k+1}) &\le d_\Gamma^2(w_k) - 2a_1\alpha\bigl(\max\{d_{S_i}(w_k), i \in \{1,2,\dots,t\}\}\bigr)^2 - 2a_2\|Gw_k\|^2 \\ &\le d_\Gamma^2(w_k) - 2a_1\alpha\mu^{-\frac{2}{p}} d_S^{\frac{2}{p}}(w_k) - 2a_2\|Gw_k\|^2 \\ &\le d_\Gamma^2(w_k) - 2\eta\bigl(d_S^{\frac{2}{p}}(w_k) + \|Gw_k\|^2\bigr), \quad \text{for all } k \ge L, \end{aligned}$$

where $\alpha = \min\{\alpha_i,\ i = 1,2,\dots,t\}$, $\eta = \min\{a_1\alpha\mu^{-\frac{2}{p}}, a_2\}$. Then the proof is split into two cases:

Case 1: when $\max\{d_S(w_k), \|Gw_k\|\} = d_S(w_k)$, we have

$$d_\Gamma^2(w_{k+1}) \le d_\Gamma^2(w_k) - 2\eta d_S^{\frac{2}{p}}(w_k) \le d_\Gamma^2(w_k) - 2\eta\left(\frac{1}{\nu}\right)^{\frac{2}{pq}}\bigl(d_\Gamma(w_k)\bigr)^{\frac{2}{pq}}, \quad \text{for all } k \ge L.$$

So the inequality (3.5) is true with $\delta = 2\eta(\frac{1}{\nu})^{\frac{2}{pq}}$ and $\theta = \frac{1}{pq}$.

Case 2: when $\max\{d_S(w_k), \|Gw_k\|\} = \|Gw_k\|$, we have

$$d_\Gamma^2(w_{k+1}) \le d_\Gamma^2(w_k) - 2\eta\|Gw_k\|^2 \le d_\Gamma^2(w_k) - 2\eta\left(\frac{1}{\nu}\right)^{\frac{2}{q}}\bigl(d_\Gamma(w_k)\bigr)^{\frac{2}{q}}, \quad \text{for all } k \ge L.$$

So the inequality (3.5) is true with $\delta = 2\eta(\frac{1}{\nu})^{\frac{2}{q}}$ and $\theta = \frac{1}{q}$. Set $F := \max\{L, T\}$. When $k \ge F$, we have

$$d_\Gamma^2(w_{k+1}) \le d_\Gamma^2(w_k) - 2\eta\left(\frac{1}{\nu}\right)^{\frac{2}{q}}\bigl(d_\Gamma(w_k)\bigr)^{\frac{2}{pq}}.$$

In conclusion, we get the inequality (3.5) where $\theta = \frac{1}{pq}$ and

$$\delta = \begin{cases} 2\eta(\frac{1}{\nu})^{\frac{2}{pq}}, & \max\{d_S(w_k), \|Gw_k\|\} = d_S(w_k), \\ 2\eta(\frac{1}{\nu})^{\frac{2}{q}}, & \max\{d_S(w_k), \|Gw_k\|\} = \|Gw_k\|. \end{cases}$$

By Lemma 2.8, we see that the first assertion is true.

For the second assertion, the proof is the same as the above proof. And we notice that p and q is independent of K. Then the second assertion can be obtained. The proof is finished. □

The SEP is a special case of the MSSEP. When $t = 1$, Algorithm 2.10 reduces to an iterative algorithm for solving the SEP (1.1) [6]. Thus Theorem 3.1 becomes the following form.

Corollary 3.2 *The SEP* (1.1) *satisfies the bounded Hölder regularity property and the sequence* $\{w_k\}$ *is defined by*

$$w_{k+1} = w_k + \rho_{1,k}(P_S w_k - w_k) - \frac{\rho_{2,k}\|Gw_k\|^2}{\|G^*Gw_k\|^2}G^*Gw_k, \tag{3.11}$$

or component-wise

$$\begin{cases} x_{k+1} = x_k + \rho_{1,k}(P_C x_k - x_k) - \frac{\rho_{2,k}\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2}A^*(Ax_k - By_k), \\ y_{k+1} = y_k + \rho_{1,k}(P_Q y_k - y_k) + \frac{\rho_{2,k}\|Ax_k - By_k\|^2}{\|B^*(Ax_k - By_k)\|^2}B^*(Ax_k - By_k), \end{cases}$$

where $0 < \underline{\rho}_1 \le \rho_{1,k} \le \overline{\rho}_1 < 1$, $0 < \underline{\rho}_2 \le \rho_{2,k} \le \overline{\rho}_2 < 1$, *then* $\{w_k\}$ *converges to a solution* w^* *of the SEP* (1.1) *at least with a sublinear rate* $O(k^{-\iota})$ *for some* $\iota > 0$.

In particular, if the SEP satisfies the bounded Hölder regularity property with uniform exponent $q \in (0,1]$, *then there exist constants* $M_1, M_2 \ge 0$ *and* $r \in [0,1)$ *such that*

$$\|w_k - w^*\| \le \begin{cases} M_1 k^{-\frac{1}{2(\theta-1)}}, & \theta > 1, \\ M_2 r^k, & \theta = 1, \end{cases}$$

where $\theta = \frac{1}{q}$.

Its proof is similar to the proof of Theorem 3.1.

4 Numerical experiments

Set $H_1 = R$, $H_2 = R^2$ and $H_3 = R^3$. We consider the SEP which has two subsets $C = \{x \in H_1 : \|x\| \le 15\}$ and $Q = \{x \in H_2 : \|x\| \le 15\}$. The two operators $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are defined by

$$A(x) = (x, 0, 0) \quad \text{and} \quad B(y, z) = (y, z, 0), \quad \text{for all } x, y, z \in R,$$

respectively. Set $S = C \times Q \subseteq H_3$ and $G = [A, -B] : H_3 \to H_3$. G is defined by

$$G(x, y, z) = (x - y, -z, 0), \quad \text{for all } (x, y, z) \in R^3.$$

Then $\ker G = \{(x, x, 0), x \in R\} \neq \emptyset$, the range of G is closed and the solution set of this SEP is $\Gamma = S \cap \ker G = \{(x, x, 0), x \in C\}$. It is easy to know that the SEP satisfies the bounded linear regularity property by Lemma 2.6, namely, it satisfies the bounded Hölder regularity property.

Take $w_0 = (x_0, y_0, z_0) \in S$. In consideration of algorithm (3.11), we have

$$\begin{cases} x_{k+1} = x_k - \frac{\rho_{2,k}[(x_k - y_k)^2 + z_k^2]}{[2(x_k - y_k)^2 + z_k^2]}(x_k - y_k), \\ y_{k+1} = y_k - \frac{\rho_{2,k}[(x_k - y_k)^2 + z_k^2]}{[2(x_k - y_k)^2 + z_k^2]}(-x_k + y_k), \\ z_{k+1} = z_k - \frac{\rho_{2,k}[(x_k - y_k)^2 + z_k^2]}{[2(x_k - y_k)^2 + z_k^2]}z_k. \end{cases}$$

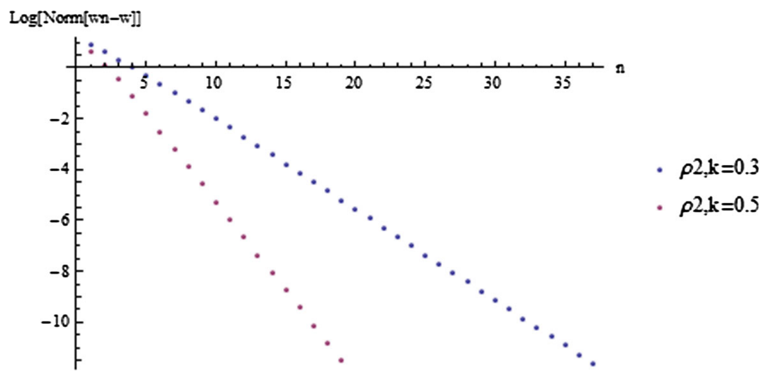

Figure 1 Error = 10^{-5}, $w_1 = (6,10,2)$, $w = (8,8,0)$

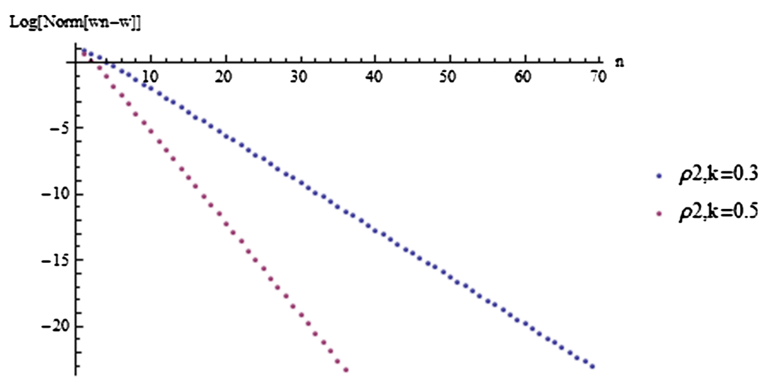

Figure 2 Error = 10^{-10}, $w_1 = (6,10,2)$, $w = (8,8,0)$

In this algorithm, we take $\rho_{2,k} = 0.3, 0.5$, respectively. Then we get some numerical experiments which were run on a personal Dell computer with Intel(R)Core(TM)i5-4210U CPU 1.70 GHz and RAM 4.00 GB. And we wrote all the programs in Wolfram Mathematica (version 9.0).

We take the initial value $w_0 = (6,10,2)$. Set the error to be 10^{-5}, 10^{-10}, respectively. Note that we denote the number of iterations and the logarithm of the error by using the x-coordinate and the y-coordinate of the figures, respectively.

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Authors' contributions
The main idea of this paper was proposed by SS, MF and LS prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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