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Fractional Hermite–Hadamard type inequalities for interval-valued functions

Xuelong Liu[1], Gouju Ye[1*], Dafang Zhao[1,2] and Wei Liu[1]

*Correspondence: ygjhhu@163.com
[1] College of Science, Hohai University, Nanjing, P.R. China
Full list of author information is available at the end of the article

Abstract

We introduce the concept of interval harmonically convex functions. By using two different classes of convexity, we get some further refinements for interval fractional Hermite–Hadamard type inequalities. Also, some examples are presented.

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1 Introduction

It is well known that convex function and convexity are very important in mathematical economy, probability theory, optimal control theory, and other fields of mathematics. Over the years, classical convexity has been extended and generalized to harmonically convex, h-convex, p-convex, among others. In fact, the concepts of convex function and convexity are founded on inequality, and the important role of inequalities cannot be undermined. Recently, the following Hermite–Hadamard inequality, one of the most important classical inequalities, has gained plenty of attention. Let interval $J^\diamond \subseteq \mathbb{R}$, and $a, b \in J^\diamond$ with $a < b$. If $f : J^\diamond \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The following inequality as the weighted generalization of (1.1) was established by Fejér in [1].

Theorem 1.1 *Let f be a convex function and $\psi(a+b-x) = \psi(x) \ge 0$ holds for all $x \in J^\diamond$, then*

$$f\left(\frac{a+b}{2}\right)\int_a^b \psi(x)\,dx \le \int_a^b f(x)\psi(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b \psi(x)\,dx. \tag{1.2}$$

Due to the difference among the concepts of convexity, integral inequality (1.1) and (1.2) in various forms have also been extensively studied in [2–8]. With the increasing importance of fractional integrals, several authors extend their research by combining

Hermite–Hadamard type inequalities with fractional integrals. In this way, some fractional Hermite–Hadamard type inequalities have been established, see [9–15] and the references therein.

On the other hand, interval analysis was firstly introduced as a significant tool to handle interval uncertainty by Moore in [16]. It has been widely used in various fields [17–20]. Especially, several classical inequalities have been studied with interval-valued functions by Chalco-Cano et al. [21, 22], Costa and Román-Flores. [23], Zhao et al. [24, 25], An et al. [26], and so on. As a further extension, Budak et al. [27] proved the fractional Hermite–Hadamard inequality for interval convex function. Motivated by [9–12, 24, 25, 27], we establish some further refinements for interval fractional Hermite–Hadamard type inequalities. Our results generalize some previous inequalities. In addition, perhaps the results can be recognized as the significant methods to investigate the research of interval-valued differential equations, interval optimization, interval vector spaces, among others.

We give some preliminaries in Sect. 2. In Sect. 3, we introduce the concept of interval harmonically convex functions and prove some interval fractional Hermite–Hadamard type inequalities. Finally, in Sect. 4, some examples are presented.

2 Preliminaries

We begin by using $\mathcal{K}$ denote the space of all intervals of $\mathbb{R}$. Let $D \in \mathcal{K}$,

$$D = [\underline{d}, \overline{d}] = \{x \in \mathbb{R} | \underline{d} \le x \le \overline{d}\}, \quad \underline{d}, \overline{d} \in \mathbb{R}.$$

When $\underline{d} = \overline{d}$, the interval D is said to be degenerate. We call D positive if $\underline{d} > 0$ or negative if $\overline{d} < 0$. We use $\mathcal{K}^+$ and $\mathcal{K}^-$ to represent the sets of all positive intervals and negative intervals. Let $\lambda \in \mathbb{R}$, then

$$\lambda D = \begin{cases} [\lambda \underline{d}, \lambda \overline{d}], & \lambda \ge 0, \\ [\lambda \overline{d}, \lambda \underline{d}], & \lambda < 0. \end{cases}$$

For $D_1, D_2 \in \mathcal{K}$, the addition and Minkowski difference are defined by

$$D_1 + D_2 = [\underline{d_1}, \overline{d_1}] + [\underline{d_2}, \overline{d_2}] = [\underline{d}_1 + \underline{d}_2, \overline{d}_1 + \overline{d}_2]$$

and

$$D_1 - D_2 = [\underline{d_1}, \overline{d}_1] - [\underline{d_2}, \overline{d}_2] = [\underline{d}_1 - \overline{d}_2, \overline{d}_1 - \underline{d}_2],$$

respectively.

The inclusion "$\subseteq$" is defined by

$$D_1 \subseteq D_2 \quad \Leftrightarrow \quad [\underline{d}_1, \overline{d_1}] \subseteq [\underline{d}_2, \overline{d_2}] \quad \Leftrightarrow \quad \underline{d}_2 \le \underline{d}_1, \overline{d}_1 \le \overline{d}_2.$$

For more basic notations with interval analysis, see [24, 25]. Furthermore, we recall the following results in [20].

Let $\mathcal{F}(x) = [\underline{f}(x), \overline{f}(x)]$, $x \in J^\diamond$. We call $\mathcal{F}(x)$ is Lebesgue integrable if $\underline{f}(x)$ and $\overline{f}(x)$ are measurable and Lebesgue integrable in $J^\diamond$. Moreover, we define $\int_a^b \mathcal{F}(x)\,dx$ as follows:

$$\int_a^b \mathcal{F}(x)\,dx = \left[\int_a^b \underline{f}(x)\,dx, \int_a^b \overline{f}(x)\,dx\right].$$

Let $\mathscr{I}\mathscr{L}_{([a,b])}$ be the collections of all Lebesgue integrable interval-valued functions on $[a,b]$. If $\mathcal{F} \in \mathscr{I}\mathscr{L}_{([a,b])}$, the interval left-sided Riemann–Liouville fractional integral of $\mathcal{F}(x)$ is defined by

$$\mathfrak{J}_{a^+}^{\alpha}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\mu)^{\alpha-1}\mathcal{F}(\mu)\,d\mu, \quad x > a,$$

where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}\,dx$ with $\alpha > 0$.

In [27], the interval right-sided Riemann–Liouville fractional integral of $\mathcal{F}(x)$ is defined by

$$\mathfrak{J}_{b^-}^{\alpha}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\mu-x)^{\alpha-1}\mathcal{F}(\mu)\,d\mu, \quad x < b,$$

where $\Gamma(\alpha)$ is an Euler Gamma function.

It is obvious that $\mathfrak{J}_{a^+}^{\alpha}\mathcal{F}(x) = [J_{a^+}^{\alpha}\underline{f}(x), J_{a^+}^{\alpha}\overline{f}(x)]$, $\mathfrak{J}_{b^-}^{\alpha}\mathcal{F}(x) = [J_{b^-}^{\alpha}\underline{f}(x), J_{b^-}^{\alpha}\overline{f}(x)]$, for all $x \in J^\diamond$.

Definition 2.1 ([6]) $f : J^\diamond \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is called a harmonically convex function if

$$\mathrm{t}f(y) + (1-\mathrm{t})f(x) \geq f\left(\frac{xy}{\mathrm{t}x + (1-\mathrm{t})y}\right)$$

holds for any $x, y \in J^\diamond$ and $\mathrm{t} \in [0,1]$.

Definition 2.2 ([28]) $\mathcal{F} : J^\diamond \to \mathcal{K}^+$ is called an interval convex function if

$$\mathcal{F}\bigl(\mathrm{t}x + (1-\mathrm{t})y\bigr) \supseteq \mathrm{t}\mathcal{F}(x) + (1-\mathrm{t})\mathcal{F}(y)$$

holds for any $x, y \in J^\diamond$ and $\mathrm{t} \in [0,1]$.

3 Main result

First, we give definition of interval harmonically convex functions as follows.

Definition 3.1 $\mathcal{F} : J^\diamond \subseteq \mathbb{R}\backslash\{0\} \to \mathcal{K}^+$ is called an interval harmonically convex function if

$$\mathcal{F}\left(\frac{xy}{\mathrm{t}x + (1-\mathrm{t})y}\right) \supseteq \mathrm{t}\mathcal{F}(y) + (1-\mathrm{t})\mathcal{F}(x)$$

holds for all $x, y \in J^\diamond$ and $\mathrm{t} \in [0,1]$.

Let $\mathcal{FC}(J^\diamond, \mathcal{K}^+)$ and $\mathcal{FHC}(J^\diamond, \mathcal{K}^+)$ denote the family of interval convex and harmonically convex functions in $J^\diamond$, respectively.

In [27], Budak et al. give the following Hermite–Hadamard inequality for the interval convex function.

Theorem 3.2 *Let $\mathcal{F} \in \mathscr{IL}_{([a,b])}$ and $a, b \in J^\diamond$ with $0 \le a < b$. If $\mathcal{F} \in \mathcal{FC}(J^\diamond, \mathcal{K}^+)$, then*

$$\mathcal{F}\left(\frac{a+b}{2}\right) \supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[\mathfrak{I}^\alpha_{a^+}\mathcal{F}(b) + \mathfrak{I}^\alpha_{b^-}\mathcal{F}(a)\right] \supseteq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}. \tag{3.1}$$

Remark 3.3 In Theorem 3.2, if $\underline{f} = \overline{f}$, then we get ([12], Theorem 2).

Next, we give some further refinements for interval fractional Hermite–Hadamard type inequalities.

Theorem 3.4 *Let $\mathcal{F} \in \mathscr{IL}_{([a,b])}$, and $a, b \in J^\diamond$ with $0 \le a < b$. If $\mathcal{F} \in \mathcal{FC}(J^\diamond, \mathcal{K}^+)$ and $\psi(a + b - x) = \psi(x) \ge 0$ holds for all $x \in J^\diamond$, then*

$$\begin{aligned}\mathcal{F}\left(\frac{a+b}{2}\right)\left[J^\alpha_{a^+}\psi(b) + J^\alpha_{b^-}\psi(a)\right] &\supseteq \left[\mathfrak{I}^\alpha_{a^+}(\mathcal{F}\psi)(b) + \mathfrak{I}^\alpha_{b^-}(\mathcal{F}\psi)(a)\right] \\ &\supseteq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}\left[J^\alpha_{a^+}\psi(b) + J^\alpha_{b^-}\psi(a)\right].\end{aligned} \tag{3.2}$$

Proof Since $\mathcal{F} \in \mathcal{FC}(J^\diamond, \mathcal{K}^+)$, we have

$$\mathcal{F}\left(\frac{a+b}{2}\right) = \mathcal{F}\left(\frac{\mu a + \nu b + \mu b + \nu a}{2}\right) \supseteq \frac{\mathcal{F}(\mu a + \nu b) + \mathcal{F}(\nu a + \mu b)}{2} \tag{3.3}$$

with $\nu = 1 - \mu$, $\mu \in [0, 1]$.

Multiplying both sides of (3.3) by $2\mu^{\alpha-1}\psi(\mu b + \nu a)$, then

$$2\mu^{\alpha-1}\psi(\mu b + \nu a)\mathcal{F}\left(\frac{a+b}{2}\right) \supseteq \mu^{\alpha-1}\psi(\mu b + \nu a)\left[\mathcal{F}(\mu a + \nu b) + \mathcal{F}(\nu a + \mu b)\right].$$

Consequently,

$$\begin{aligned}&2\mathcal{F}\left(\frac{a+b}{2}\right)\int_0^1 \mu^{\alpha-1}\psi(\mu b + \nu a)\,d\mu \\ &\supseteq \int_0^1 \mu^{\alpha-1}\mathcal{F}(\mu a + \nu b)\psi(\mu b + \nu a)\,d\mu + \int_0^1 \mu^{\alpha-1}\mathcal{F}(\nu a + \mu b)\psi(\mu b + \nu a)\,d\mu \\ &= \left[\int_0^1 \mu^{\alpha-1}\left(\underline{f}(\mu a + \nu b) + \underline{f}(\nu a + \mu b)\right)\psi(\mu b + \nu a)\,d\mu,\right. \\ &\quad \left.\int_0^1 \mu^{\alpha-1}\left(\overline{f}(\mu a + \nu b) + \overline{f}(\nu a + \mu b)\right)\psi(\mu b + \nu a)\,d\mu\right].\end{aligned}$$

Setting $\omega = \mu b + \nu a$, then

$$\begin{aligned}&\frac{2}{(b-a)^\alpha}\mathcal{F}\left(\frac{a+b}{2}\right)\int_a^b (\omega - a)^{\alpha-1}\psi(\omega)\,d\omega \\ &\supseteq \frac{1}{(b-a)^\alpha}\left[\int_a^b (\omega-a)^{\alpha-1}\underline{f}(a+b-\omega)\psi(\omega)\,d\omega + \int_a^b (\omega-a)^{\alpha-1}\underline{f}(\omega)\psi(\omega)\,d\omega,\right. \\ &\quad \left.\int_a^b (\omega-a)^{\alpha-1}\overline{f}(a+b-\omega)\psi(\omega)\,d\omega + \int_a^b (\omega-a)^{\alpha-1}\overline{f}(\omega)\psi(\omega)\,d\omega\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(b-a)^{\alpha}}\left[\int_a^b (b-\omega)^{\alpha-1}\underline{f}(\omega)\psi(a+b-\omega)\,d\omega + \int_a^b (\omega-a)^{\alpha-1}\underline{f}(\omega)\psi(\omega)\,d\omega,\right.\\
&\qquad \left.\int_a^b (b-\omega)^{\alpha-1}\overline{f}(\omega)\psi(a+b-\omega)\,d\omega + \int_a^b (\omega-a)^{\alpha-1}\overline{f}(\omega)\psi(\omega)\,d\omega\right]\\
&= \frac{1}{(b-a)^{\alpha}}\left[\int_a^b (b-\omega)^{\alpha-1}\underline{f}(\omega)\psi(\omega)\,d\omega + \int_a^b (\omega-a)^{\alpha-1}\underline{f}(\omega)\psi(\omega)\,d\omega,\right.\\
&\qquad \left.\int_a^b (b-\omega)^{\alpha-1}\overline{f}(\omega)\psi(\omega)\,d\omega + \int_a^b (\omega-a)^{\alpha-1}\overline{f}(\omega)\psi(\omega)\,d\omega\right]\\
&= \frac{1}{(b-a)^{\alpha}}\left\{\int_a^b (b-\omega)^{\alpha-1}\mathcal{F}(\omega)\psi(\omega)\,d\omega + \int_a^b (\omega-a)^{\alpha-1}\mathcal{F}(\omega)\psi(\omega)\,d\omega\right\}.
\end{aligned}
$$

Therefore

$$
\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\mathcal{F}\left(\frac{a+b}{2}\right)\left[J_{a^+}^{\alpha}\psi(b) + J_{b^-}^{\alpha}\psi(a)\right] \supseteq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[\mathfrak{J}_{a^+}^{\alpha}(\mathcal{F}\psi)(b) + \mathfrak{J}_{b^-}^{\alpha}(\mathcal{F}\psi)(a)\right]. \quad (3.4)
$$

Since $\mathcal{F} \in \mathcal{FC}([a,b],\mathcal{K}^+)$, we have

$$
\mathcal{F}(\mu a + \nu b) \supseteq \mu\mathcal{F}(a) + \nu\mathcal{F}(b)
$$

and

$$
\mathcal{F}(\nu a + \mu b) \supseteq \nu\mathcal{F}(a) + \mu\mathcal{F}(b)
$$

with $\nu = 1-\mu$, $\mu \in [0,1]$. Then

$$
\mathcal{F}(\mu a + \nu b) + \mathcal{F}(\nu a + \mu b) \supseteq \mathcal{F}(a) + \mathcal{F}(b). \quad (3.5)
$$

By multiplying both sides (3.5) with $\mu^{\alpha-1}\psi(\mu b+\nu a)$, and integrating the resulting inequality, we get

$$
\begin{aligned}
&\int_0^1 \mu^{\alpha-1}\mathcal{F}(\mu a + \nu b)\psi(\mu b + \nu a)\,d\mu + \int_0^1 \mu^{\alpha-1}\mathcal{F}(\nu a + \mu b)\psi(\mu b + \nu a)\,d\mu\\
&\quad \supseteq \left[\mathcal{F}(a) + \mathcal{F}(b)\right]\int_0^1 \mu^{\alpha-1}\psi(\mu b + \nu a)\,d\mu, \quad (3.6)
\end{aligned}
$$

and the result follows. □

Remark 3.5 In Theorem 3.4, if $\psi(x) = 1$, inequality (3.2) becomes inequality (3.1) in Theorem 3.2.

If $\underline{f} = \overline{f}$, then we get ([9], Theorem 4).

Theorem 3.6 *Let $\mathcal{F} \in \mathscr{IL}_{([a,b])}$, and $a,b \in J^{\diamond}$ with $0 \le a < b$. If $\mathcal{F} \in \mathcal{FHC}(J^{\diamond},\mathcal{K}^+)$, and $g(x) = \frac{1}{x}, x \in [\frac{1}{b},\frac{1}{a}]$ then*

$$
\mathcal{F}\left(\frac{2ab}{a+b}\right) \supseteq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[\mathfrak{J}_{1/a-}^{\alpha}(\mathcal{F}\circ g)(1/b) + \mathfrak{J}_{1/b+}^{\alpha}(\mathcal{F}\circ g)(1/a)\right]
$$

$$\supseteq \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2}. \tag{3.7}$$

Proof Since $\mathcal{F} \in \mathcal{FHC}(J^\diamond, \mathcal{K}^+)$, we have

$$\mathcal{F}\left(\frac{2xy}{x+y}\right) \supseteq \frac{\mathcal{F}(x) + \mathcal{F}(y)}{2}.$$

Let $\nu = 1 - \mu$, $\mu \in [0,1]$, setting

$$x = \frac{ab}{\mu b + \nu a}, \qquad y = \frac{ab}{\mu a + \nu b}.$$

By multiplying both sides with $\mu^{\alpha-1}$, and integrating the resulting inequality, we get

$$\begin{aligned}
&\mathcal{F}\left(\frac{2ab}{a+b}\right)\int_0^1 \mu^{\alpha-1}\,d\mu \\
&= \frac{1}{\alpha}\mathcal{F}\left(\frac{2ab}{a+b}\right) \\
&\supseteq \frac{1}{2}\int_0^1 \mu^{\alpha-1}\mathcal{F}\left(\frac{ab}{\mu b + \nu a}\right)d\mu + \int_0^1 \mu^{\alpha-1}\mathcal{F}\left(\frac{ab}{\mu a + \nu b}\right)d\mu \\
&= \frac{1}{2}\left[\int_0^1 \mu^{\alpha-1}\underline{f}\left(\frac{ab}{\mu b + \nu a}\right)d\mu + \int_0^1 \mu^{\alpha-1}\underline{f}\left(\frac{ab}{\mu a + \nu b}\right)d\mu,\right. \\
&\quad \left.\int_0^1 \mu^{\alpha-1}\overline{f}\left(\frac{ab}{\mu b + \nu a}\right)d\mu + \int_0^1 \mu^{\alpha-1}\overline{f}\left(\frac{ab}{\mu a + \nu b}\right)d\mu\right] \\
&= \frac{1}{2}\left(\frac{ab}{b-a}\right)^\alpha\left[\int_{1/b}^{1/a}\left(\mu - \frac{1}{b}\right)^{\alpha-1}\underline{f}\left(\frac{1}{\mu}\right)d\mu + \int_{1/b}^{1/a}\left(\frac{1}{a} - \mu\right)^{\alpha-1}\underline{f}\left(\frac{1}{\mu}\right)d\mu,\right. \\
&\quad \left.\int_{1/b}^{1/a}\left(\mu - \frac{1}{b}\right)^{\alpha-1}\overline{f}\left(\frac{1}{\mu}\right)d\mu + \int_{1/b}^{1/a}\left(\frac{1}{a} - \mu\right)^{\alpha-1}\overline{f}\left(\frac{1}{\mu}\right)d\mu\right] \\
&= \frac{\Gamma(\alpha)}{2}\left(\frac{ab}{b-a}\right)^\alpha\left[J^\alpha_{1/a-}(\underline{f}\circ g)(1/b) + J^\alpha_{1/b+}(\underline{f}\circ g)(1/a),\right. \\
&\quad \left. J^\alpha_{1/a-}(\overline{f}\circ g)(1/b) + J^\alpha_{1/b+}(\overline{f}\circ g)(1/a)\right].
\end{aligned} \tag{3.8}$$

Let $\nu = 1 - \mu$, $\mu \in [0,1]$, then thanks to $\mathcal{F} \in \mathcal{FHC}(J^\diamond, \mathcal{K}^+)$

$$\mathcal{F}\left(\frac{ab}{\mu b + \nu a}\right) \supseteq \mu\mathcal{F}(a) + \nu\mathcal{F}(b)$$

and

$$\mathcal{F}\left(\frac{ab}{\nu b + \mu a}\right) \supseteq \nu\mathcal{F}(a) + \mu\mathcal{F}(b).$$

This implies

$$\mu^{\alpha-1}\left\{\mathcal{F}\left(\frac{ab}{\mu b + \nu a}\right) + \mathcal{F}\left(\frac{ab}{\nu b + \mu a}\right)\right\} \supseteq \mu^{\alpha-1}\left[\mathcal{F}(a) + \mathcal{F}(b)\right].$$

Then

$$\int_0^1 \mu^{\alpha-1}\mathcal{F}\left(\frac{ab}{\mu b+\nu a}\right)d\mu + \int_0^1 \mu^{\alpha-1}\mathcal{F}\left(\frac{ab}{\nu b+\mu a}\right)d\mu \supseteq \left[\mathcal{F}(a)+\mathcal{F}(b)\right]\int_0^1 \mu^{\alpha-1}\,d\mu.$$

Therefore

$$\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[\mathfrak{J}^{\alpha}_{1/a-}(\mathcal{F}\circ g)(1/b)+\mathfrak{J}^{\alpha}_{1/b+}(\mathcal{F}\circ g)(1/a)\right]\supseteq\frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}$$

with $g(x)=\frac{1}{x}$.

This gives the result. □

Remark 3.7 The function $\mathcal{F}(x)\in\mathcal{FHC}(J^{\diamond},\mathcal{K}^{+})$ if and only if $\mathcal{H}(x)=\mathcal{F}(\frac{ab}{x})\in\mathcal{FC}(J^{\diamond},\mathcal{K}^{+})$. By using inequality (3.1) for $\mathcal{H}(x)$, we obtain inequality (3.7).

Remark 3.8 In Theorem 3.6, if $\underline{f}=\overline{f}$, then we get ([10], Theorem 4).

Theorem 3.9 *Let $\mathcal{F}\in\mathscr{IL}_{([a,b])}$, and $a,b\in J^{\diamond}$ with $0\le a<b$. If $\mathcal{F}\in\mathcal{FHC}(J^{\diamond},\mathcal{K}^{+})$ and $\psi(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}})=\psi(x)\ge 0$ holds for all $x\in J^{\diamond}$, then*

$$\begin{aligned}&\mathcal{F}\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{1/b+}(\psi\circ g)(1/a)+J^{\alpha}_{1/a-}\psi\circ g(1/b)\right]\\&\quad\supseteq\left[\mathfrak{J}^{\alpha}_{1/b+}(\mathcal{F}\psi\circ g)(1/a)+\mathfrak{J}^{\alpha}_{1/a-}(\mathcal{F}\psi\circ g)(1/b)\right]\\&\quad\supseteq\frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}\left[J^{\alpha}_{1/b+}(\psi\circ g)(1/a)+J^{\alpha}_{1/a-}\psi\circ g(1/b)\right]\end{aligned}\tag{3.9}$$

with $g(x)=\frac{1}{x}, x\in[\frac{1}{b},\frac{1}{a}]$.

Proof Since $\mathcal{F}\in\mathcal{FHC}(J^{\diamond},\mathcal{K}^{+})$, we have

$$\mathcal{F}\left(\frac{2ab}{a+b}\right)\supseteq\frac{\mathcal{F}(\frac{ab}{\mu b+\nu a})+\mathcal{F}(\frac{ab}{\mu a+\nu b})}{2}\tag{3.10}$$

with $\nu=1-\mu,\mu\in[0,1]$.

Multiplying both sides of (3.10) by $2\mu^{\alpha-1}\psi(\frac{ab}{\mu b+\nu a})$, we get

$$\begin{aligned}&2\mathcal{F}\left(\frac{2ab}{a+b}\right)\int_0^1\mu^{\alpha-1}\psi\left(\frac{ab}{\mu b+\nu a}\right)d\mu\\&\quad\supseteq\int_0^1\mu^{\alpha-1}\left[\mathcal{F}\left(\frac{ab}{\mu a+\nu b}\right)+\mathcal{F}\left(\frac{ab}{\mu b+\nu a}\right)\right]\psi\left(\frac{ab}{\mu b+\nu a}\right)d\mu\\&\quad=\left[\int_0^1\mu^{\alpha-1}\left\{\underline{f}\left(\frac{ab}{\mu a+\nu b}\right)+\underline{f}\left(\frac{ab}{\mu b+\nu a}\right)\right\}\psi\left(\frac{ab}{\mu b+\nu a}\right)d\mu,\right.\\&\qquad\left.\int_0^1\mu^{\alpha-1}\left\{\overline{f}\left(\frac{ab}{\mu a+\nu b}\right)+\overline{f}\left(\frac{ab}{\mu b+\nu a}\right)\right\}\psi\left(\frac{ab}{\mu b+\nu a}\right)d\mu\right].\end{aligned}$$

Let $\omega = \frac{\mu b+\nu a}{ab}$, then $d\mu = \frac{ab}{b-a}\,d\omega$. One has

$$\begin{aligned}
&2\left(\frac{ab}{b-a}\right)^{\alpha}\mathcal{F}\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\omega-\frac{1}{b}\right)^{\alpha-1}\psi\left(\frac{1}{\omega}\right)d\omega\\
&\supseteq\left(\frac{ab}{b-a}\right)^{\alpha}\left[\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\omega-\frac{1}{b}\right)^{\alpha-1}\underline{f}\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\omega}\right)\psi\left(\frac{1}{\omega}\right)d\omega\right.\\
&\quad+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\omega-\frac{1}{b}\right)^{\alpha-1}\underline{f}\left(\frac{1}{\omega}\right)\psi\left(\frac{1}{\omega}\right)d\omega,\\
&\quad\left.\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\omega-\frac{1}{b}\right)^{\alpha-1}\overline{f}\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\omega}\right)\psi\left(\frac{1}{\omega}\right)d\omega+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\omega-\frac{1}{b}\right)^{\alpha-1}\overline{f}\left(\frac{1}{\omega}\right)\psi\left(\frac{1}{\omega}\right)d\omega\right]\\
&=\left(\frac{ab}{b-a}\right)^{\alpha}\left[\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-\omega\right)^{\alpha-1}\underline{f}\left(\frac{1}{\omega}\right)\psi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\omega}\right)d\omega\right.\\
&\quad+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\omega-\frac{1}{b}\right)^{\alpha-1}\underline{f}\left(\frac{1}{\omega}\right)\psi\left(\frac{1}{\omega}\right)d\omega,\\
&\quad\left.\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\omega-\frac{1}{b}\right)^{\alpha-1}\overline{f}\left(\frac{1}{\omega}\right)\psi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\omega}\right)d\omega+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\omega-\frac{1}{b}\right)^{\alpha-1}\overline{f}\left(\frac{1}{\omega}\right)\psi\left(\frac{1}{\omega}\right)d\omega\right]\\
&=\left(\frac{ab}{b-a}\right)^{\alpha}\left[\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-\omega\right)^{\alpha-1}\underline{f}\left(\frac{1}{\omega}\right)\psi\left(\frac{1}{\omega}\right)d\omega\right.\\
&\quad+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\omega-\frac{1}{b}\right)^{\alpha-1}\underline{f}\left(\frac{1}{\omega}\right)\psi\left(\frac{1}{\omega}\right)d\omega,\\
&\quad\left.\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\omega-\frac{1}{b}\right)^{\alpha-1}\overline{f}\left(\frac{1}{\omega}\right)\psi\left(\frac{1}{\omega}\right)d\omega+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\omega-\frac{1}{b}\right)^{\alpha-1}\overline{f}\left(\frac{1}{\omega}\right)\psi\left(\frac{1}{\omega}\right)d\omega\right].
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\mathcal{F}\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{1/b+}(\psi\circ g)(1/a)+J^{\alpha}_{1/a-}\psi\circ g(1/b)\right]\\
&\supseteq\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\left[\mathfrak{J}^{\alpha}_{1/b+}(\mathcal{F}\psi\circ g)(1/a)+\mathfrak{J}^{\alpha}_{1/a-}(\mathcal{F}\psi\circ g)(1/b)\right].
\end{aligned}\tag{3.11}$$

Similarly, $\mathcal{F}\in\mathcal{FHC}(J^{\diamond},\mathcal{K}^{+})$, then

$$\mathcal{F}\left(\frac{ab}{\mu b+\nu a}\right)+\mathcal{F}\left(\frac{ab}{\nu b+\mu a}\right)\supseteq\mathcal{F}(a)+\mathcal{F}(b).\tag{3.12}$$

Multiplying both sides of (3.12) by $\mu^{\alpha-1}\psi(\frac{ab}{\mu b+\nu a})$, one has

$$\begin{aligned}
&\int_0^1\mu^{\alpha-1}\left[\mathcal{F}\left(\frac{ab}{\mu a+\nu b}\right)+\mathcal{F}\left(\frac{ab}{\mu b+\nu a}\right)\right]\psi\left(\frac{ab}{\mu b+\nu a}\right)d\mu\\
&\supseteq\left[\mathcal{F}(a)+\mathcal{F}(b)\right]\int_0^1\mu^{\alpha-1}\psi\left(\frac{ab}{\mu b+\nu a}\right)d\mu.
\end{aligned}$$

Hence,

$$\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\left[\mathfrak{J}^{\alpha}_{1/b+}(\mathcal{F}\psi\circ g)(1/a)+\mathfrak{J}^{\alpha}_{1/a-}\mathcal{F}\psi\circ g(1/b)\right]$$
$$\supseteq\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}\left[J^{\alpha}_{1/b+}(\psi\circ g)(1/a)+J^{\alpha}_{1/a-}\psi\circ g(1/b)\right], \tag{3.13}$$

and the result follows. □

Remark 3.10 If $\underline{f}=\overline{f}$, then we get ([11], Theorem 5).

If $\psi(x)=1$, inequality (3.9) reduces to inequality (3.7) in Theorem 3.6.

4 Examples

Example 4.1 Let $\mathcal{F}(x)=[-\sqrt{x}+2,\sqrt{x}+2]$, $x\in[0,2]$, and $\alpha=\frac{1}{2}$. Then $\mathcal{F}\in\mathcal{FC}([0,2],\mathcal{K}^{+})$, and we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathfrak{J}^{\alpha}_{a+}\mathcal{F}(b)+\mathfrak{J}^{\alpha}_{b-}\mathcal{F}(a)\right]\\
&\quad=\frac{\Gamma(3/2)}{2\sqrt{2}}\left\{\frac{1}{\sqrt{\pi}}\int_0^2(2-s)^{-\frac{1}{2}}[-\sqrt{s}+2,\sqrt{s}+2]\,ds\right.\\
&\quad\quad\left.+\frac{1}{\sqrt{\pi}}\int_0^2 s^{-\frac{1}{2}}[-\sqrt{s}+2,\sqrt{s}+2]\,ds\right\}\\
&\quad=\frac{1}{4\sqrt{2}}\left\{[-\pi+4\sqrt{2},\pi+4\sqrt{2}]+[-2+4\sqrt{2},2+4\sqrt{2}]\right\}\\
&\quad=\left[\frac{8\sqrt{2}-\pi-2}{4\sqrt{2}},\frac{8\sqrt{2}+\pi+2}{4\sqrt{2}}\right].
\end{aligned}$$

On the other hand,

$$\mathcal{F}\left(\frac{a+b}{2}\right)=\mathcal{F}\left(\frac{0+2}{2}\right)=\mathcal{F}(1)=[1,3]$$

and

$$\frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}=\left[2-\frac{\sqrt{2}}{2},2+\frac{\sqrt{2}}{2}\right].$$

Thus,

$$[1,3]\supseteq\left[\frac{8\sqrt{2}-\pi-2}{4\sqrt{2}},\frac{8\sqrt{2}+\pi+2}{4\sqrt{2}}\right]\supseteq\left[2-\frac{\sqrt{2}}{2},2+\frac{\sqrt{2}}{2}\right].$$

Consequently, Theorem 3.2 is verified.

Example 4.2 Let $\mathcal{F}:[0,2]\to\mathcal{K}$ is defined as the above example, and

$$\psi(x)=\begin{cases}\sqrt{x}, & x\in[0,1],\\ \sqrt{2-x}, & x\in(1,2],\end{cases}$$

then $\psi(2-x)=\psi(x)\geq 0$ for all $x\in[0,2]$. Let $\alpha=\frac{1}{2}$, we obtain

$$\begin{aligned}&\left[\mathfrak{J}^{\alpha}_{a^+}(\mathcal{F}\psi)(b)+\mathfrak{J}^{\alpha}_{b^-}(\mathcal{F}\psi)(a)\right]\\&=\frac{1}{\sqrt{\pi}}\left\{\int_0^2(2-s)^{-\frac{1}{2}}\psi(s)[-\sqrt{s}+2,\sqrt{s}+2]\,ds+\int_0^2 s^{-\frac{1}{2}}\psi(s)[-\sqrt{s}+2,\sqrt{s}+2]\,ds\right\}\\&=\frac{1}{\sqrt{\pi}}\left\{\left[\frac{8-8\sqrt{2}}{3}+\pi,\frac{8\sqrt{2}-8}{3}+\pi\right]+\left[-\frac{4}{3}+\pi,\frac{4}{3}+\pi\right]\right\}\\&=\frac{1}{\sqrt{\pi}}\left[\frac{4-8\sqrt{2}}{3}+2\pi,\frac{8\sqrt{2}-4}{3}+2\pi\right].\end{aligned}$$

Furthermore, by Example 4.1, we have

$$[\sqrt{\pi},3\sqrt{\pi}]\supseteq\frac{1}{\sqrt{\pi}}\left[\frac{4-8\sqrt{2}}{3}+2\pi,\frac{8\sqrt{2}-4}{3}+2\pi\right]\supseteq\sqrt{\pi}\left[2-\frac{\sqrt{2}}{2},2+\frac{\sqrt{2}}{2}\right].$$

Consequently, Theorem 3.4 is verified.

5 Conclusions

In this research, we get a new extension of interval harmonically convex functions and some further refinements for interval fractional Hermite–Hadamard type inequalities. The results obtained in this work are the promotions of those given in previous research. Moreover, our results can be recognized as significant methods in the fields of mathematics. At a further research direction, we will investigate the integral inequalities with a new class of fractional integral.

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Abbreviations

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Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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Author details

[1]College of Science, Hohai University, Nanjing, P.R. China. [2]School of Mathematics and Statistics, Hubei Normal University, Huangshi, P.R. China.

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